Math 8201 Graduate abstract algebra- Fall 2001, Vic Reiner
Midterm exam 1- Due Friday October 12, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Each of the 7 problems is worth approximately the same number of points.

1. For which triples (a, b, c) of positive integers satisfying $b|ac$ is the homomorphism

$$\mathbb{Z}/a\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/b\mathbb{Z}$$
$$\overline{n} \text{ } mod \text{ } a \mapsto \overline{cn} \text{ } mod \text{ } b$$

(a) surjective ?
(b) injective ?
(c) an isomorphism?

2. (a) Show that the abelianization $G^{ab} := G/G'$ is the "largest" abelian quotient of G by proving the following: If $N \triangleleft G$ and G/N is abelian, then $G' < N$ and there is a surjective homomorphism $G^{ab} \twoheadrightarrow G/N$.
(b) Show that if $A, B < G$ and G/A, G/B are abelian, then $G/(A \cap B)$ is abelian.

3. Let $N \triangleleft G$ with $[G : N] = 2$. Show that for any subgroup $H < G$, the intersection $H \cap N$ is normal in H and has $[H : H \cap N] \leq 2$.

4. Let $N \triangleleft G$ with $|G| = 24, |N| = 8$. Show that any g in $G - N$ has order of g divisible by 3.

5. Prove or disprove:
(i) $S_4 \cong D_{24}$
(ii) $D_{12} \cong \mathbb{Z}/2\mathbb{Z} \times D_6$
(iii) $D_{16} \cong \mathbb{Z}/2\mathbb{Z} \times D_8$

6. Show that for any two permutations σ, τ in S_n, the two permutations $\sigma\tau$ and $\tau\sigma$ have the same cycle sizes (counted with multiplicities).

7. Let p be the smallest prime number dividing the order $|G|$ of a finite group G, and let $H \triangleleft G$ with $|H| = p$. Show that $H < Z(G)$.