Math 8201 Graduate abstract algebra- Fall 2001, Vic Reiner
Midterm exam 2- Due Friday October 16, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Each of the 7 problems is worth approximately the same number of points.

1. Let G be a finite group and n an integer with $\gcd(n, |G|) = 1$. Show that every g in G has a unique n^{th} root x in G, i.e. there is a unique x in G with $x^n = g$. (Hint: consider the set map $G \to G$ given by $x \mapsto x^n$).

2. Give an example of a finite group G and three *normal* subgroups N_1, N_2, N_3 satisfying

$$N_1, N_2, N_3 \triangleleft G$$
$$G = N_1 N_2 N_3$$
$$N_1 \cap N_2 = N_1 \cap N_3 = N_2 \cap N_3 = \{1\}$$

but for which

$$G \ncong N_1 \times N_2 \times N_3.$$

3. What is $|\mathrm{Aut}(\underbrace{\mathbb{Z}/p\mathbb{Z} \times \cdots \mathbb{Z}/p\mathbb{Z}}_{n \text{ times}})|$ for a prime p?

4. Let A, N be solvable subgroups of a group G with $N \triangleleft G$. Show that AN is solvable.

5. Show that a group G cannot be written as a union $G = H_1 \cup H_2$ for two *proper* subgroups $H_1, H_2 \lneqq G$.

6. (a) Prove that the subgroup $\langle a^2, b^3 \rangle$ of the free group $F(\{a, b\})$ is isomorphic to the free group $F(\{x, y\})$.
(b) Prove that the subgroup $\langle a^2, ab, b^2 \rangle$ of the free group $F(\{a, b\})$ is isomorphic to the free group $F(\{x, y, z\})$.

7. Let G be a finite group with $|G| = pqr$ for 3 primes $p < q < r$ with $q \nmid r - 1$. Assume G has a normal subgroup $N \triangleleft G$ of order $|N| = p$. Show G is cyclic.