Math 8201 Graduate abstract algebra- Fall 2010, Vic Reiner
Group theory and linear algebra practice problems
from old prelim exams

Diagonalizability and triangularizability

1. Let S, T be linear transformations $V \to V$ for a finite-dimensional vector space V over an algebraically closed field k.

Assuming $ST = TS$, show that S, T have a simultaneous eigenvector, that is, a nonzero vector v such that $Sv = \lambda v$ and $Tv = \mu v$ for some λ, μ in k.

Some variations on this problem:

(a) Replace S, T by a commutative ring whose elements are linear endomorphisms of V (and still show that there is a simultaneous eigenvector for every element of the ring).
(b) Assume further that S, T are both *diagonalizable*, and show that they are simultaneously diagonalizable.
(c) Replace S, T by a finite *abelian group* of linear automorphisms of a *complex* vector space V and show that the group is simultaneously diagonalizable.

2. Suppose that a finite-dimensional vector space V over a field k has a basis of eigenvectors for a linear map $T : V \to V$. Let W be a T-stable subspace of V (that is, $TW \subset W$). Show that W has a basis of eigenvectors for T.

3. Let T be a complex $n \times n$ matrix with $T^* = T$ where $*$ denotes conjugate-transpose. Show that there is an n-by-n matrix U with $U^*U = 1$ such that U^*TU is diagonal.

Sylow-type questions from group theory

4. Classify up to isomorphism all groups of order pq where p, q are primes and $q \equiv 1 \bmod p$.

5. Classify up to isomorphism the groups of order $2p$ where p is an odd prime.

6. Show that a group of order 15 is necessarily cyclic.

7. Exhibit a nonabelian group of order 21.

8. Let $p \neq q$ be odd primes. Show that any group of order $2pq$ is solvable.

9. Show that if G is any group of order $385 = 5 \cdot 7 \cdot 11$ then the center has order divisible by 7.

10. Let G be a group of order 72 for which the center $Z(G)$ has order divisible by 8. Show that G is abelian.

11. Show that a group of order $3 \cdot 5 \cdot 17$ has a normal subgrop of order 17.

Other group theory questions

12. Let G be a group of order 105. Suppose that G acts transitively on a set X. What are the possible cardinalities of X?

13. Let G be a cyclic group of order 12. Show that the equation $x^5 = g$ is solvable for every g in G, and that the solution x is unique (for a given g).

14. Let p be the smallest prime dividing the order of a finite group G. Show that a subgroup H of index p is necessarily normal.

15. Exhibit four mutually non-isomorphic groups of order 8, and prove they are not isomorphic.

16. Let G be the group of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}/3\mathbb{Z}, a \neq 0 \neq c \right\}$$

Determine whether G is isomorphic to any of the groups A_4 (alternating group), D_{12} (dihedral group), $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (product of cyclic groups).

17. Let a finite group G act on a finite set S, with $|S| \geq 2$. Suppose that G acts transitively. Show that there is an element of g in G which does not have a fixed point on S.

18. Classify abelian groups of order 48.