Math 8202 Extra Lecture on Ideal-Variety Correspondence

Refs:
- Dummit & Foote Chap. 15
- Cox, Little & O'Shea "Ideals, Varieties, and Algorithms" (Chapters 4, 5)

Much intuition about ideals I in polynomial rings $R = \Bbbk[x_1, \ldots, x_n]$ for $\Bbbk$ a field comes from this 2-way correspondence:

$$\begin{tikzcd} \left\{\begin{array}{c}\text{subsets}\\ I \subset \Bbbk[x_1,\ldots,x_n]\\ \text{of polynomials}\end{array}\right\} \arrow[r, shift left=2, "Z(-)"] & \left\{\begin{array}{c}\text{subsets } Z \subset \Bbbk^n\\ \text{of points}\end{array}\right\} \arrow[l, shift left=2, mapsto, "I(-)"] \end{tikzcd}$$

$$I \longmapsto Z(I) = \text{zero locus of } I$$

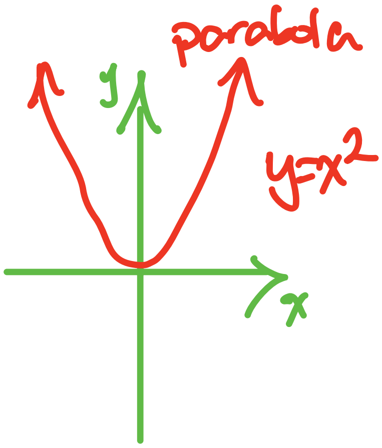
$$:= \{\underset{(z_1,\ldots,z_n)}{\underline{z}} \in \Bbbk^n : f(z_1,\ldots,z_n)=0 \;\; \forall f(\underline{x}) \in I\}$$

$$\text{vanishing ideal of } Z = I(Z) \longleftarrow\!\!\shortmid \; Z$$

$$:= \{f(\underline{x}) \in \Bbbk[x_1,\ldots,x_n] : f(\underline{z})=0 \;\; \forall \underline{z} \in Z\}$$

EXAMPLES In $\mathbb{R}[x,y]$ and $\mathbb{R}^2$,

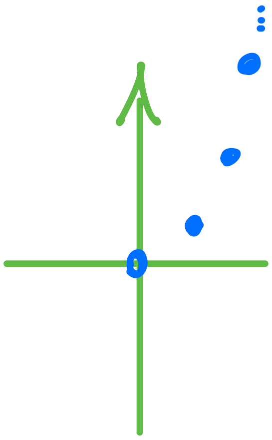
$$I = \{(y-x^2)^2\} \subset \mathbb{R}[x,y] \longmapsto Z(I) =$$



$$I(Z) = ((y-x^2)) \longleftarrow\!\shortmid \; Z := \{(t,t^2) : t = 0,1,2,\dots\}$$

$$= (y-x^2)\,\mathbb{R}[x,y]$$

a principal ideal



(EASY) FACTS:

- $I(Z)$ is always an ideal of $\Bbbk[x_1,\dots,x_n]$, and hence always finitely generated by Hilbert Basis Thm:

$$I(Z) = (f_1(\underline{x}), f_2(\underline{x}), \dots, f_t(\underline{x})) \text{ for some } \{f_i(\underline{x})\}_{i=1,\dots,t}$$

- In fact, it's a radical ideal $I(Z) = \mathrm{rad}\, I(Z)$:

$$f(\underline{x})^N \in I(Z) \Rightarrow f(\underline{x}) \in I(Z), \quad \text{since } \underset{\forall \underline{z} \in Z}{f(\underline{z})^N = 0} \Rightarrow f(\underline{z}) = 0.$$

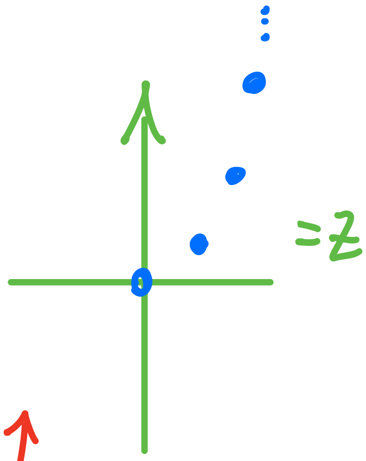
- Sets of the form $Z(I) \subset k^n$ are called (affine) algebraic sets or Zariski-closed sets, and they are closed in the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n$

- $I \subset J \quad \Rightarrow \quad Z(I) \supset Z(J)$

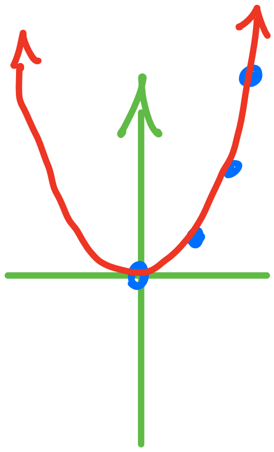
- $Z_1 \subset Z_2 \quad \Rightarrow \quad I(Z_1) \supset I(Z_2)$

- $Z(I(Z)) \supseteq Z$

  sometimes strict, e.g.



has $Z(I(Z)) =$



- $I(Z(I)) \supseteq I$

  sometimes strict, e.g.

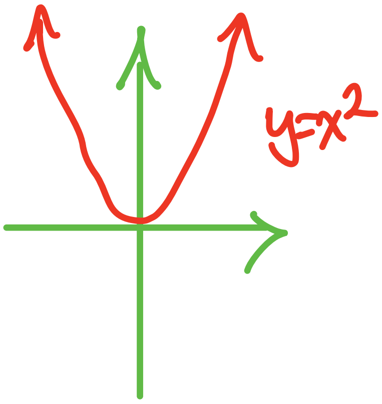
  $I = ((y-x^2)^2)$ has $I(Z(I)) = (y-x^2)$

- $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$
- $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2) = Z(I_1 \cdot I_2)$
- $I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2)$
- $Z(I_1 + I_2) = Z(I_1) \cap Z(I_2)$

EXAMPLE:

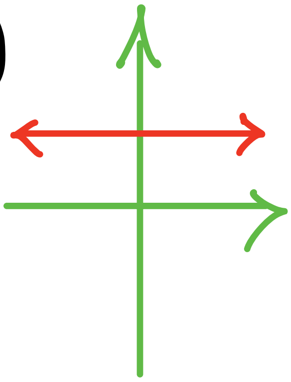
$I_1 = (y-x^2)$

$Z(I_1) =$



$I_2 = (y-1)$

$Z(I_2) =$

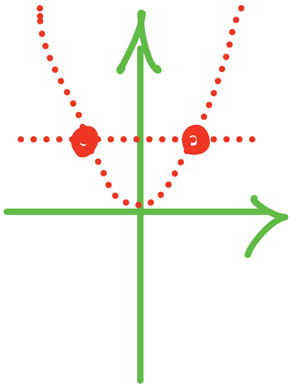


$I_1 + I_2 = (y-x^2, y-1)$

$= (y-1, x^2-1)$

$Z(I_1+I_2) =$

$Z(I_1) \cap Z(I_2)$



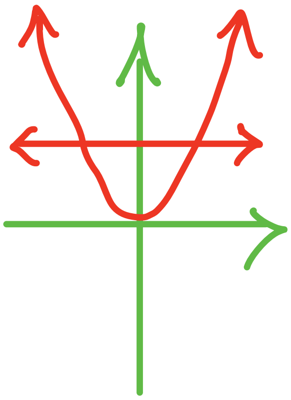
$I_1 \cap I_2 = I_1 I_2$

$= ((y-x^2)(y-1))$

$Z(I_1 \cap I_2) = Z(I_1 I_2) =$

$= Z(I_1) \cup Z(I_2)$

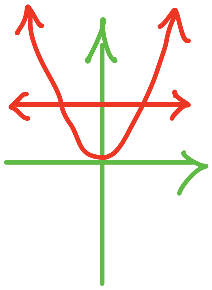
reducible



THEOREM (Hilbert's Nullstellensatz) When $\mathbb{k}$ is algebraically closed (e.g. $\mathbb{k}=\mathbb{C}$) the maps $I \mapsto Z(I)$, $Z \mapsto I(Z)$ give bijections

- $\left\{ \begin{array}{c} \text{radical ideals} \\ I \subset \mathbb{k}[x_1, \dots, x_n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{algebraic subsets} \\ Z \subset \mathbb{k}^n \end{array} \right\}$

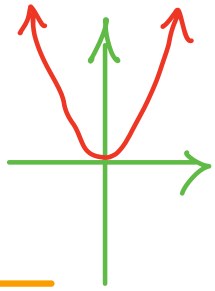
e.g. $I = \left((y-1)(y-x^2) \right) \longleftrightarrow$



- $\left\{ \begin{array}{c} \text{prime ideals} \\ I \subset \mathbb{k}[x_1, \dots, x_n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{algebraic subsets} \\ Z \subset \mathbb{k}^n \end{array} \right\}$

$Z \neq Z_1 \cup Z_2$ with $Z_i \subsetneq Z$ algebraic sets

e.g. $I = \left((y-x^2) \right) \longleftrightarrow$

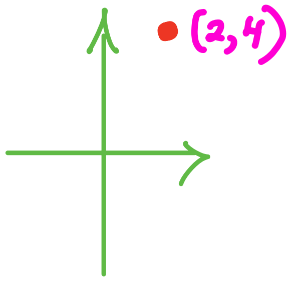


- $\left\{ \begin{array}{c} \text{maximal ideals} \\ M \subset \mathbb{k}[x_1, \dots, x_n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{points} \\ z \in \mathbb{k}^n \end{array} \right\}$

$M = (x_1 - z_1, \dots, x_n - z_n)$

$z = (z_1, \dots, z_n)$

e.g. $I = (x-2, y-4) \longleftrightarrow$



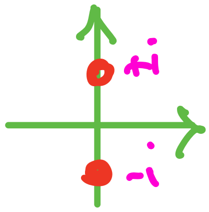
REMARK: $\Bbbk$ algebraically closed is important to make the above maps bijective.

e.g.

$$\begin{tikzcd} \underset{\underset{(x^2+1)}{\shortparallel}}{I} \subset \mathbb{R}[x] \arrow[r, mapsto] & Z(I)=\emptyset \subset \mathbb{R}^1 \arrow[d, equal] \\ J=(1)=\mathbb{R}[x] \arrow[r, mapsto] & Z(J)=\emptyset \end{tikzcd}$$

[but $I \subset \mathbb{C}[x] \longmapsto Z(I)=\{+i,-i\}\subset \mathbb{C}^1$

$$I = (x^2+1) = ((x+i)(x-i))$$



]

Affine algebraic sets $Z \subset \Bbbk^n$ are sometimes called affine varieties, and when irreducible, called irreducible (affine) varieties.

One can define their dimension $\dim(Z)$ as the number of steps in the longest chain of subvarieties

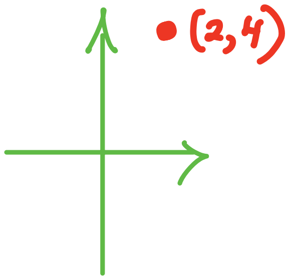
$$(\emptyset \neq) Z_0 \subsetneq Z_1 \subsetneq \dots \subset Z_{\ell-1} \subsetneq Z_\ell =: Z$$

with each Z_i irreducible, or equivalently, of prime ideals

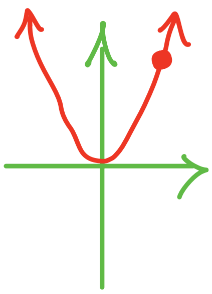
$$P_0 \supsetneq P_1 \supsetneq \dots \supsetneq P_{\ell-1} \supsetneq P_\ell =: I(Z)$$

EXAMPLE:

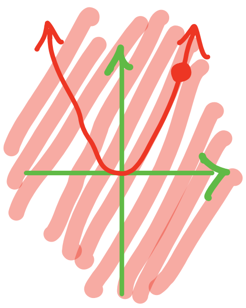
dim 0



dim 1



dim 2



$$\begin{array}{ccccc}
Z_0 = \{(2,4)\} & \subsetneq & Z_1 = \text{parabola } y=x^2 & \subsetneq & Z_2 = \mathbb{R}^2 \\
 & & & & \\
P_0 & \supsetneq & P_1 & \supsetneq & P_2 \\
\| & & \| & & \| \\
(x-2, y-4) & & (y-x^2) & & (0)
\end{array}$$

DEFINITION: The quotient ring $\mathbb{k}[x_1, \dots, x_n]/I(Z)$ is called the coordinate ring of the affine variety Z, sometimes denote $\mathbb{k}[Z]$.

So when Z is irreducible, $\mathbb{k}[Z]$ is a domain and there is another way to characterize its dimension $\dim(Z)$, in terms of the fraction field

$$\mathbb{k}(Z) := \operatorname{Frac} \mathbb{k}[Z]$$

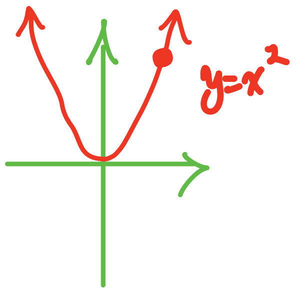
called the (rational) function field of Z.

$\dim(Z) = \mathrm{trdeg}_{\mathbb{k}}\, \mathbb{k}(Z)$, the transcendence degree of the field $\mathbb{k}(Z)$ over the subfield $\mathbb{k}$

$:= \max\{ d :$ $\mathbb{k}(Z)$ contains a subfield isomorphic to the rational functions $f(x_1, \ldots, x_d)$ $\}$

EXAMPLE

$Z =$



in $\mathbb{R}^2$

has coordinate ring $\mathbb{R}[Z] = \mathbb{R}[x,y]/(y-x^2) \cong \mathbb{R}[x]$

so function field $\mathbb{R}(Z) = \mathrm{Frac}\, \mathbb{R}[Z] \cong \mathbb{R}(x)$

and $\dim(Z) = \mathrm{trdeg}_{\mathbb{R}}\, \mathbb{R}(x) = 1$.

A ring map $\mathbb{k}[Z_1] \xrightarrow{f} \mathbb{k}[Z_2]$ gives rise to a map on maximal ideals in the opposite direction:

$$\begin{tikzcd} \text{also a maximal ideal} & f^{-1}(M) \arrow[d, phantom, "\cap"] & M \arrow[l, squiggly] \arrow[d, phantom, "\cap"] & \text{a maximal ideal} \\ & \mathbb{k}[Z_1] & \mathbb{k}[Z_2] & \end{tikzcd}$$

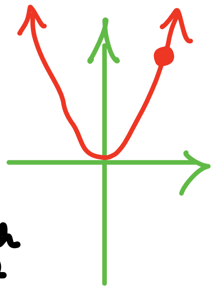
So at least when $\mathbb{k}$ is algebraically closed, this gives a geometric map on the points

$$Z_1 \xleftarrow{f^*} Z_2$$

called a regular map or morphism of varieties

EXAMPLES:

① The inclusion $Z =$



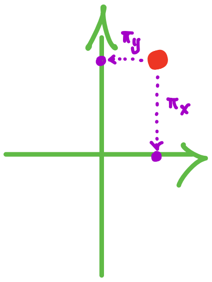
$\hookrightarrow \mathbb{R}^2$

corresponds to the ring map

$$\mathbb{R}[Z] = \mathbb{R}[x,y]/(y-x^2) \twoheadleftarrow \mathbb{R}[x,y] = \mathbb{R}[\mathbb{R}^2]$$

---

② One has two coord. projection maps

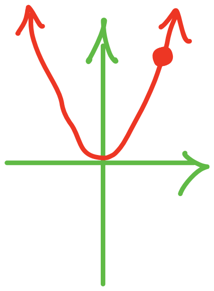


$$\begin{tikzcd} & & & y \\ (x,y) \arrow[urrr, mapsto] \arrow[ddrrr, mapsto] & \mathbb{R}^2 \arrow[r, "\pi_y"] \arrow[dr, bend left, "\pi_x"'] & \mathbb{R} & \\ & & \mathbb{R} & \\ & & & x \end{tikzcd}$$

corresponding to

$$\begin{tikzcd} & \mathbb{R}[y] \arrow[dl] \\ \mathbb{R}[x,y] & \\ & \mathbb{R}[x] \arrow[ul] \end{tikzcd}$$

③ The two composites



$$\begin{tikzcd}
 & & \mathbb{R} \\
Z \arrow[r, hook] \arrow[rru, bend left] \arrow[rrd, bend right] & \mathbb{R}^2 \arrow[ru, "\pi_y"] \arrow[rd, "\pi_x"'] & \\
 & & \mathbb{R}
\end{tikzcd}$$

correspond to these two composite ring maps

$$\begin{tikzcd}
 & & \mathbb{R}[y] \arrow[dl] \arrow[dll, bend right, "f_y"'] \\
\mathbb{R}[x,y]/(y-x^2) & \mathbb{R}[x,y] \arrow[l, two heads] & \\
 & & \mathbb{R}[x] \arrow[ul, hook] \arrow[ull, bend left, "f_x"]
\end{tikzcd}$$

One can check

$$\begin{aligned} f_x : \mathbb{R}[x] &\longrightarrow \mathbb{R}[x,y]/(y-x^2) \\ x &\longmapsto \bar{x} \end{aligned}$$

is actually a ring isomorphism,

showing projection

$$\begin{aligned} \mathbb{R}^2 &\xrightarrow{\pi_x} \mathbb{R} = \{x\text{-axis}\} \\ (x,y) &\longmapsto x \end{aligned}$$

restricts to an isomorphism of the varieties $Z \cong \mathbb{R}$.

However $f_y : \mathbb{R}[y] \longrightarrow \mathbb{R}[x,y]/(y-x^2)$

$$y \longmapsto \bar{y}$$

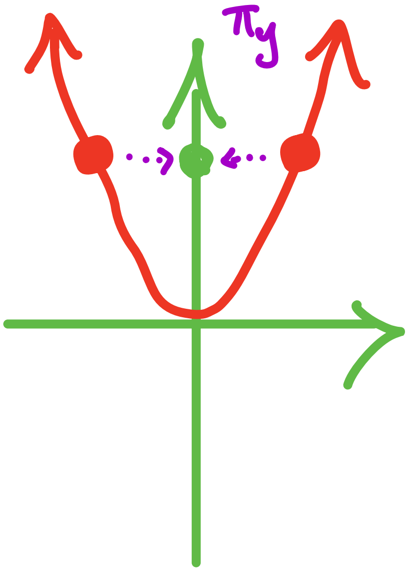
is not a ring isomorphism

e.g. $\bar{x} \notin \mathrm{im}(f_y) = \mathrm{span}_{\Bbbk}\{1, \bar{x}^2, \bar{x}^4, \bar{x}^6, \dots\}$

This is partly reflecting the fact that

$$\mathbb{R}^2 \xrightarrow{\pi_y} \mathbb{R} = \{y\text{-axis}\}$$
$$(x,y) \longmapsto y$$

when restricted to a map $Z \longrightarrow \mathbb{R}$ is not an isomorphism between the two varieties, rather a generically 2-to-1 map.



e.g. $\pi_y^{-1}(4) = \{(+2,4), (-2,4)\}$

$\pi_y^{-1}(0) = \{(0,0)\}$

$$\pi_y^{-1}(-1) = \begin{cases} \emptyset & \text{if } \Bbbk = \mathbb{R} \\ \{(+i,-1), (-i,-1)\} & \text{if } \Bbbk = \mathbb{C} \end{cases}$$