Math 8202 Grad Algebra Fall 2025

Syllabus items:

- office hour
- text: Dummit & Foote
- prerequisites
- exams & grading

Intro, Definitions & Examples (see D&F §7.1, 7.2, 7.3)

The class starts with some advanced linear algebra, and then focuses on rings, fields, modules.

DEF'N: A ring (= algebra) (= ring with 1) R is a set with two (binary) operations $+, \cdot$

- making $R^+ = (R,+)$ an abelian group with (additive) identity 0
- with $(R, \cdot)$ associative $(ab)c = a(bc)$ and having a multiplicative identity 1, but not necessarily commutative, and not every $r \in R$ has a multiplicative inverse.
- $\cdot$ distributes over $+$: $a(b+c) = ab + ac$
  $(b+c)a = ba + ca$

Call $u \in R$ a unit if it has a 2-sided mult. inverse
i.e. $\exists\, u^{-1} \in R$ with $u \cdot u^{-1} = 1 = u^{-1} \cdot u$

One calls $R^{\times} := \{\text{units } u \in R\}$ with $\cdot$
the group of units of R

$$\begin{tikzcd}
 & \{\text{rings}\} & \\
\{\text{commutative rings}\} \arrow[ur, phantom, "\subset" sloped] & & \{\text{division rings } (= \text{skew fields})\} \arrow[ul, phantom, "\supset" sloped] \\
 & \{\text{fields} = \text{commutative division rings}\} \arrow[ul, phantom, "\supset" sloped] \arrow[ur, phantom, "\subset" sloped] &
\end{tikzcd}$$

i.e. $ab = ba$

$1 \neq 0$ and $R^{\times} = R - \{0\}$
i.e. all nonzero $u \neq 0$ in R are units

EXAMPLES:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathbb{Q} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathbb{R} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathbb{R}[x] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\cap"] & \mathbb{R}[x,y] \arrow[r, phantom, "\subset"] & \cdots \\
\mathbb{Z}[i] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathbb{Q}[i] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathbb{C} \arrow[d, phantom, "\cap"] & \mathbb{R}(x) & \mathbb{R}[[x]] & \\
\mathbb{H}_{\mathbb{Z}} \arrow[r, phantom, "\subset"] & \mathbb{H}_{\mathbb{Q}} \arrow[r, phantom, "\subset"] & \mathbb{H} & & &
\end{tikzcd}$$

integers

rationals fields reals

— polynomials ⟶

formal power series

Gaussian integers

complexes

rational functions

commutative rings ↑

quaternions

division rings

$\mathbb{Z}[i] := \{a+bi : a,b \in \mathbb{Z}\}$

$\mathbb{Q}[i] := \{a+bi : a,b \in \mathbb{Q}\}$

$\mathbb{R}[[x]] := \{f = \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{R}\}$

$\mathbb{H}_{\mathbb{Z}} := \{a+bi+cj+dk : a,b,c,d \in \mathbb{Z}\}$

$\mathbb{H}_{\mathbb{Q}} := \{a+bi+cj+dk : a,b,c,d \in \mathbb{Q}\}$

$\mathbb{H} := \{a+bi+cj+dk : a,b,c,d \in \mathbb{R}\}$

$i^2 = j^2 = k^2 = -1$, $\quad ij = k = -ji$
$jk = i = -kj$
$ki = j = -ik$

EXERCISE: Why is $\mathbb{Q}[i]$ a field?
Who is $(a+bi)^{-1}$ in $\mathbb{Q}[i]$ if a,b are not both 0?

EXERCISE: Who is $(a+bi+cj+dk)^{-1}$ in $\mathbb{H}$
if a,b,c,d are not all 0?

EXERCISE: Prove
$$\begin{array}{ccl} \mathbb{Z}^\times & = & \{\pm 1\} \\ \cap & & \\ \mathbb{Z}[i]^\times & = & \{\pm 1, \pm i\} \\ \cap & & \\ \mathbb{H}_{\mathbb{Z}}^\times & = & \{\pm 1, \pm i, \pm j, \pm k\} \end{array}$$
$\left(= Q_8\right.$ quaternion group$\left.\right)$

EXERCISE: Prove $\mathbb{R}[x]^\times = \mathbb{R}[x,y]^\times = \mathbb{R}^\times = \mathbb{R}-\{0\}$

EXERCISE: Prove $\mathbb{R}[[x]]^\times = \{ f = a_0 + a_1 x + a_2 x^2 + \ldots : a_0 \in \mathbb{R}^\times \}$

REMARK: $\mathbb{Z}, \mathbb{Z}[i], \mathbb{R}[x], \mathbb{R}[[x]]$
are all principal ideal domains, making their module theory simpler, almost as nice as fields.

Before more examples...

DEF'N: A ring (algebra) homomorphism $R \xrightarrow{f} S$

is an abelian group homomorphism $R^+ \xrightarrow{f} S^+$

(so $f(0)=0$
$f(a+b)=f(a)+f(b)$)

that also respects 1 and ×, meaning $f(1)=1$
$f(ab)=f(a)f(b)$.

Call it a monomorphism (injection) if injective, i.e. $\ker(f)=0$.
an epimorphism (surjection) if surjective, i.e. $\mathrm{im}(f)=S$.
an isomorphism (bijection) if bijective.

A subring (subalgebra) $Q \subseteq R$ is an additive subgroup $Q^+ \subseteq R^+$ forming a ring with the same $+, \times, 1$.

A {(2-sided) ideal / left ideal / right ideal} $I \subseteq R$ is an additive subgroup $I^+ \subseteq R^+$

also closed under 2-sided multiplication by any $r \in R$ on

{both sides : $\forall i \in I, r \in R$ $ri, ir \in I$
left $ri \in I$
right $ir \in I$}

For any ring homomorphism

$$\begin{tikzcd} R \arrow[r, "f"] & S \\ \ker(f) \arrow[u, phantom, "\cup"] & \operatorname{im}(f) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

= a (2-sided) ideal of R

= a subalgebra of S

2-sided ideals $I \subset R$ are what let's one define

quotient rings $R/I := R^+/I^+ = \overset{\text{additive cosets}}{\{r+I : r \in R\}}$

with × defined by $(r_1+I)(r_2+I) := r_1 r_2 + I$

This is just like the situation for groups:

- group homomorphisms $G \xrightarrow{f} H$

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \ker(f) \arrow[u, phantom, "\cup"] & \operatorname{im}(f) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

have ker(f) = a normal subgroup of G, im(f) = a subgroup of H

- normal subgroups $K \trianglelefteq G$ let one define quotient groups $G/K = \overset{\text{cosets}}{\{gK : g \in G\}}$

$$g_1K \cdot g_2K := g_1 g_2 K$$

MORE EXAMPLES

① integers modulo m

$$\begin{tikzcd} \mathbb{Z} \arrow[r, two heads, "f", "\text{reduction mod } m"'] \arrow[d, phantom, "\cup"] & \mathbb{Z}/m\mathbb{Z} \\ m\mathbb{Z} & \end{tikzcd}$$

$= \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{m-1}\}$
with $+, \times$ followed by taking remainder mod m

$m\mathbb{Z} = \ker(f)$

$\mathbb{Z}/m\mathbb{Z} = \mathrm{im}(f)$

EXERCISE: $\mathbb{Z}/m\mathbb{Z}$ is a field $\Leftrightarrow$ $m=p$ is prime
(and then we'll call it $\mathbb{F}_p = \mathbb{Z}/p$)

We'll later discuss the ring injections
$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^d}$ $(\neq \mathbb{Z}/p^d\mathbb{Z}\ !)$
constructing all finite fields.

② Matrix rings $\mathrm{Mat}_{n\times n}(\mathbb{k})$ over a field $\mathbb{k}$
$:= \{ \text{all } A \in \mathbb{k}^{n\times n} \}$
with matrix $+$ and $\times$.

$$0 = 0_{n\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}, \quad 1 = I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

EXERCISE:

$\mathrm{Mat}_{n\times n}(\mathbb{k})$ contains ...

- exactly 2^n left ideals given by zeroing out columns

  e.g. $n=3$: $\mathrm{Mat}_{3\times 3}(\mathbb{k}) \supset \left\{ \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix} \right\}$

  $\supset \left\{ \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix} \right\}$, etc.

- exactly 2^n right ideals given by zeroing out rows

  e.g. $\mathrm{Mat}_{3\times 3}(\mathbb{k}) \supset \left\{ \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \right\}$

- only 2 two-sided ideals: $I = \mathrm{Mat}_{n\times n}(\mathbb{k})$

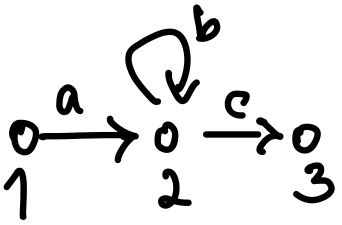
  and $I = \{0\} = \left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \right\}$

EXERCISE: Check that the space

$$T_n := \left\{ \text{upper triangular matrices } \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix} \right\}$$

is a subalgebra of $\mathrm{Mat}_{n\times n}(\mathbb{k})$

③ A **quiver** $Q=(\underset{\text{nodes}}{Q_0},\underset{\text{arcs}}{Q_1})$ is a finite directed graph

e.g. $Q=$ 

$Q_0=\{1,2,3\}$
$Q_1=\{a,b,c\}$

and has a **path algebra** $\Bbbk Q$ over a field $\Bbbk$:

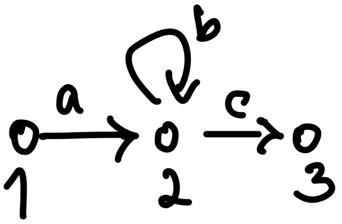
$\Bbbk Q :=$ $\Bbbk$-vector space with basis all **paths**

$a_1 a_2 \cdots a_n$ having $\text{head}(a_i)=\text{tail}(a_{i+1})\ \forall i$,

including the **lazy paths** e_v at each node $v \in Q_0$.

Multiplication of the path basis elements is by **concatenation** if possible, zero otherwise, and then extended $\Bbbk$-linearly to all of $\Bbbk P$.

$1=e_1+e_2+\ldots+e_n$ = sum of all lazy paths.

e.g. $Q=$ 

has $\Bbbk Q$ with $\Bbbk$-basis

$$\{ e_1,\ a,\ ab,\ ab^2,\ ab^3,\ \dots$$
$$ac,\ abc,\ ab^2c,\ ab^3c,\ \dots$$

$$e_2,\ b,\ b^2,\ b^3,\ \dots$$
$$bc,\ b^2c,\ b^3c,\ \dots$$
$$c,$$

$$e_3 \}$$

and some examples of multiplications are

$$\left.\begin{array}{l} e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3 \\ 0=e_1e_2=e_2e_1=e_1e_3=e_3e_1=e_2e_3=e_3e_2 \end{array}\right\} \Rightarrow$$

$1=e_1+e_2+e_3$ has

$$1^2=(e_1+e_2+e_3)^2$$
$$=e_1^2+e_2^2+e_3^2+0+0+0+0+0+0$$
$$=1$$

$$e_1a=a,\quad e_1b=0=e_1c$$
$$ae_2=a,\quad be_2=b,\quad ce_2=0$$

$$a\cdot c = ac$$
$$c\cdot a=0$$

$$ab^5\cdot b^3c = ab^8c$$

$$1\cdot c=(e_1+e_2+e_3)c=e_2c=c$$
$$c\cdot 1 = c(e_1+e_2+e_3)=ce_3=c$$

EXERCISE :

(a) In $\mathrm{Mat}_{n\times n}(\mathbb{k})$, show the matrix units

$$\left\{ E_{ij} := \begin{array}{cc} & \text{j^{th} column} \\ & \downarrow \\ \text{i^{th} row} \rightarrow & \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \end{array} \right\}_{\substack{i=1,2,\dots,n \\ j=1,2,\dots,n}}$$

form a $\mathbb{k}$-basis.

(b) They satisfy $E_{ij}E_{k\ell} = \begin{cases} 0 & \text{if } j\neq k \\ E_{i\ell} & \text{if } j=k \end{cases}$

(c) The upper triangular subalgebra $T_n = \left\{ \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix} \right\}$

has $\{E_{ij}\}_{1\le i\le j\le n}$ as a $\mathbb{k}$-basis.

(d) The path algebra $\mathbb{k}Q$ for this quiver Q

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, "a_1"'] & \overset{2}{\circ} \arrow[r, "a_2"'] & \overset{3}{\circ} \arrow[r, "a_3"'] & \cdots \arrow[r, "a_{n-2}"'] & \overset{n-1}{\circ} \arrow[r, "a_{n-1}"'] & \overset{n}{\circ} \end{tikzcd}$$

has an algebra isomorphism $\mathbb{k}Q \cong T_n$

sending

$$\begin{aligned} \mathbb{k}Q &\xrightarrow{f} T_n \\ e_i &\longmapsto E_{ii} \quad \text{for } i=1,2,\dots,n \\ a_i &\longmapsto E_{i,i+1} \quad \text{for } i=1,2,\dots,n-1. \end{aligned}$$

④ The free associative algebra $\Bbbk\langle x_1,\dots,x_n\rangle$ on n letters $x_1,\dots,x_n$ over a field $\Bbbk$ has a $\Bbbk$-basis given by all words

$$\{\ x_{i_1}x_{i_2}\cdots x_{i_\ell}\ \}$$

in the alphabet $x_1,\dots,x_n$, with multiplication of two words by concatenation

$$x_{i_1}x_{i_2}\cdots x_{i_\ell}\cdot x_{j_1}x_{j_2}\cdots x_{j_m} = x_{i_1}x_{i_2}\cdots x_{i_\ell}x_{j_1}x_{j_2}\cdots x_{j_m}$$

EXAMPLE

$\Bbbk\langle x,y\rangle$ has $\Bbbk$-basis

$$\left\{\begin{array}{llll} 1, & x, & x^2, & x^3, \quad \dots \\ & y, & xy, & x^2y, \\ & & yx, & xyx, \\ & & y^2, & yx^2, \\ & & & xy^2, \\ & & & yxy, \\ & & & y^3, \end{array}\right\}$$

Why free associative algebra?

It has a universal property:

Given any ring R with a ring homom $\Bbbk \hookrightarrow R$

and a set map $\{x_1,\dots,x_n\} \xrightarrow{f} R$

$$\begin{array}{ccc} x_1 & \longmapsto & r_1 = f(x_1) \\ \vdots & & \vdots \\ x_n & \longmapsto & r_n = f(x_n) \end{array}$$

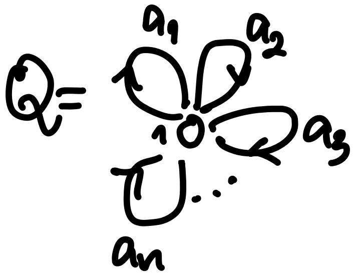
there exists a unique map $\Bbbk\langle x_1,\dots,x_n\rangle \xrightarrow{\bar{f}} R$ making this diagram commute:

$$\begin{tikzcd}
 & x_i \arrow[rr, mapsto] & & r_i & \\
x_i \arrow[d, mapsto] & \{x_1,\dots,x_n\} \arrow[rr, "f"] \arrow[d, hook] & & R & \sum c_{i_1 i_2\cdots i_l}\, r_{i_1} r_{i_2}\cdots r_{i_l} \\
x_i & \Bbbk\langle x_1,\dots,x_n\rangle \arrow[urr, dashed, "\exists!\, \bar{f}"'] & & & \\
 & & \sum c_{i_1 i_2\cdots i_l}\, x_{i_1} x_{i_2}\cdots x_{i_l} \arrow[uurr, mapsto] & &
\end{tikzcd}$$

EXERCISE: Prove there is a ring isomorphism

$$\Bbbk\langle x_1,\dots,x_n\rangle \longrightarrow \Bbbk Q \quad \text{for the quiver}$$

path algebra



sending

$$\begin{array}{ccc} x_1 & \longmapsto & a_1 \\ \vdots & & \vdots \\ x_n & \longmapsto & a_n \end{array}$$

REMARK: Later we'll see $\Bbbk\langle x_1,\dots,x_n\rangle$ is isomorphic to the tensor algebra

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V) \quad \text{where } T^n(V) = n\text{-fold tensor product } V\otimes V\otimes\dots\otimes V$$

and V is a $\Bbbk$-vector space with basis $x_1,\dots,x_n$

⑤ For any group G and field $\Bbbk$, the **group algebra** (ring) $\Bbbk G$ has a $\Bbbk$-basis $\{T_g\}_{g\in G}$ that multiply via $T_g \cdot T_h := T_{gh}$, which one extends $\Bbbk$-bilinearly to all of $\Bbbk G$:

$$\Big(\sum_{g\in G} a_g T_g\Big)\cdot\Big(\sum_{h\in G} b_h T_h\Big) = \sum_{(g,h)\in G\times G} a_g b_h T_{gh}$$

$$= \sum_{u\in G}\Big(\sum_{\substack{(g,h)\in G\times G\\ gh=u}} a_g b_h\Big) T_u$$

A presentation for G via generators and relations leads to a presentation for $\Bbbk G$ as a quotient of a free associative algebra

EXAMPLE: $G = \{1, g, g^2, \ldots, g^{m-1}\} \cong \mathbb{Z}/m\mathbb{Z}$ cyclic of order m

$= \langle g \mid g^{m-1} = 1\rangle$

has $\Bbbk G = \Big\{\sum_{i=0}^{m-1} c_i T_{g^i}\Big\} \cong \Bbbk\langle x\rangle / \underbrace{(x^m - 1)}_{\text{2-sided ideal generated by } x^m-1 \text{ in } \Bbbk\langle x\rangle}$

via $T_g \longmapsfrom x$

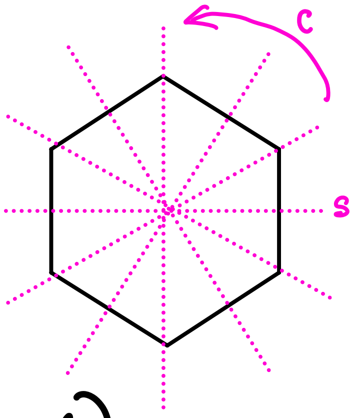
$T_{g^i} \longmapsfrom x^i$

A noncommutative

EXAMPLE: D_m = dihedral group of order $2m$
= linear symmetries of regular m-gon

$$D_m \cong \langle s, c \mid s^2 = 1,\ c^m = 1,\ scs = c^{-1} \rangle$$

$$= \langle s, c \mid s^2 = 1,\ c^m = 1,\ scsc = 1 \rangle$$



has $$\Bbbk D_m \cong \Bbbk\langle s, c\rangle \Big/ \underbrace{(s^2-1,\ c^m-1,\ scsc-1)}_{\text{2-sided ideal gen'd by } \{s^2-1,\ c^m-1,\ scsc-1\} \text{ in } \Bbbk\langle s,c\rangle}$$

REMARK We'll hopefully discuss later...

- $\left\{ \begin{array}{c} \text{representations} \\ G \xrightarrow{\rho} GL_n(\Bbbk) \\ \text{of the group } G \end{array} \right\} \longleftrightarrow \{ \text{(left-) } \Bbbk G\text{-modules} \}$

- $\left\{ \begin{array}{c} \text{representations} \\ \text{of the quiver} \\ Q = (Q_0, Q_1) \end{array} \right\} \longleftrightarrow \{ \text{(left-) } \Bbbk Q\text{-modules} \}$