Math 8202 Graduate abstract algebra- Fall 2025, Vic Reiner
Midterm exam 2- in class on Wednesday November 12.

Instructions: This is a 50 minute exam with 3 problems worth a total of 20 points: 8 true/false problems worth 1 point each, and 3 other problems worth 4 points each.

It is closed book, with no electronic devices, phones, watches, calculators allowed. You are only allowed handrwitten notes on one 3x5 index card, front and back. Put all answers in the blue exam books, not on this sheet or scrap paper. Answers without justifiction or proof will not receive full credit. Results proven in lecture or homework are fine to quote.

1. True or False? If true, give a proof, and if false, provide a counterexample or disproof.
Each part is worth 1 points.

(a) If R is a Euclidean domain, then the polynomial ring $R[x_1, x_2, x_3]$ is a Noetherian ring.

(b) If R a Euclidean domain, then the polynomial ring $R[x]$ is a Euclidean domain.

(c) For a field $\mathbf{k}$ and ring R, all ring maps $\mathbf{k} \xrightarrow{f} R$ are either injective or the *zero map*, i.e., $f(\alpha) = 0\ \forall \alpha \in \mathbf{k}$.

(d) One has a ring isomorphism $\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

(e) For $a, b \neq 0$ in R a PID, $\bar{a} := a + (b)$ is a unit in $R/(b)$ if and only if $\bar{b} := b + (a)$ is a unit in $R/(a)$.

(f) The polynomial $f(x) = 5x^3 + 10$ is an irreducible in the ring $\mathbb{Q}[x]$.

(g) There are infinitely many solutions to $6x + 21y = 50$ with x, y in $\mathbb{Z}$.

(h) For any field $\mathbf{k}$, the $\mathbf{k}$-vector space $V = \mathbf{k}^2$ has $\dim_{\mathbf{k}} \wedge^4(S^2(V)) = 0$.

2. (4 points) Consider the power series ring $\mathbb{Q}[[x]]$ as a PID, and these two elements of $\mathbb{Q}[[x]]$:

$$a(x) = x^3 - x^4,$$

$$b(x) = 12x^5 - 100x^{42} + \sum_{i=1}^{\infty} \left(\left(\frac{3}{7}\right)^i - 8i \right) \cdot x^{70000+i}$$

Compute their greatest common divisor in $\mathbb{Q}[[x]]$

$$g(x) = \gcd(\ a(x),\ b(x)\ ),$$

which is unique up to associates, and explicitly express $g(x) = r(x)a(x) + s(x)b(x)$ with $r(x), s(x)$ in $\mathbb{Q}[[x]]$.

3. (4 points) Let A be a matrix in $\mathbb{R}^{4\times 4}$, giving an $\mathbb{R}$-linear map $\mathbb{R}^4 \to \mathbb{R}^4$. Its second exterior power $\wedge^2(A)$ gives an $\mathbb{R}$-linear map $\wedge^2(\mathbb{R}^4) \to \wedge^2(\mathbb{R}^4)$, and is represented by a matrix in $\mathbb{R}^{\binom{4}{2}\times\binom{4}{2}} = \mathbb{R}^{6\times 6}$. Assuming that

$$\det(xI_4 - A) = (x^2 - 2)(x + 4)(x - 5),$$

prove that both matrices A and $\wedge^2(A) \in \mathbb{R}^{6\times 6}$ are diagonalizable, and calculate $\det(xI_6 - \wedge^2(A))$.

4. (4 points) Let R be an integral domain, and $R \hookrightarrow \mathrm{Frac}(R)$ the inclusion map into its fraction field. Assume S is an intermediate subring between the two, that is, one has inclusions of rings

$$R \subset S \subset \mathrm{Frac}(R).$$

Prove that S is a domain, and that there is a ring isomorphism $\mathrm{Frac}(S) \cong \mathrm{Frac}(R)$.