Math 8202 Graduate abstract algebra- Spring 2011, Vic Reiner
Midterm exam 1- Due Friday February 25, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult.

1. (10 points) Dummit and Foote, §7.4, Problem 10, on page 257.

2. (10 points) Dummit and Foote, §8.3, Problem 4, on page 293.

3. (10 points) Dummit and Foote, §9.4, Problem 11, on page 312.

4. (10 points) Dummit and Foote, §9.4, Problem 13, on page 312.

5. (10 points) Dummit and Foote, §9.5, Problem 3, on page 315.

6. (10 points) Let $f(x) = x^4 + 4$.
(a) Show that $f(x)$ fails the hypotheses of Eisenstein's criterion for irreducibility in $\mathbb{Z}[x]$, no matter for which prime p in $\mathbb{Z}$ one tries to apply it.
(b) Is $f(x)$ irreducible in $\mathbb{Z}[x]$? Either factor it nontrivially, or prove that it is irreducible.

7. (20 points total) In each of the following problems, give an explicit isomorphism between the two rings, making sure that you *prove* it is an isomorphism. As notation, let $R[x, x^{-1}]$ denote the ring of Laurent polynomials in x with coefficients in R, that is

$$R[x, x^{-1}] := \{a_n x^n + a_{n+1} x^{n+1} + ... + a_N x^N : a_i \in R, n, N \in \mathbb{Z}\}$$

with obvious ring operations. If R is a subring of some other ring S, and $\alpha_1, \ldots, \alpha_m \in S$, then $R[\alpha_1, \ldots, \alpha_m]$ is the smallest subring of S containing R and all the $\alpha_1, \ldots, \alpha_m$.

(a) (5 points) $\mathbb{Z}[x, u]/(xu - 1) \cong \mathbb{Z}[t, t^{-1}]$

(b) (5 points) $\mathbb{Z}[x, xy, xy^2] \cong \mathbb{Z}[u^2, uv, v^2]$.
(Hint: show both are isomorphic to the ring $\mathbb{Z}[a, b, c]/(b^2 - ac)$.)

(c) (10 points) $\mathbb{F}_2[x]/(x^3 + x + 1) \cong \mathbb{F}_2[y]/(y^3 + y^2 + 1)$

8. (20 points total, 5 points each)
Let $\mathbb{F}$ be a field. Then a polynomial $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ is *homogeneous of degree* d if every monomial occurring in f has the same degree d. By segregating monomials according to their degree, one can express any polynomial f uniquely as $f = f_0 + f_1 + \cdots + f_d$ with f_i homogeneous of degree i. The f_i are called the *homogeneous components of* f. Prove the following simple facts:

(a) For $f(\mathbf{x})$ homogeneous of degree d and any λ in $\mathbb{F}$, one has

$$f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x}).$$

(b) If $f(\mathbf{x})$ is homogeneous of degree d, then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

(c) An ideal I in $\mathbb{F}[x_1, \ldots, x_n]$ is said to be *homogeneous* if every homogeneous component f_i of any f in I also lies in I.

Show that I is a homogeneous ideal if and only if it can be generated by a collection of homogeneous polynomials.

(d) If $n = 2$ and $\mathbb{F}$ is algebraically closed, show that every homogeneous polynomial f in $\mathbb{F}[x, y]$ in two variables can be factored as a product of linear (degree 1) polynomials, that is, $f(x, y) = \prod_{i=1}^{d}(a_i x + b_i y)$ for some a_i, b_i in $\mathbb{F}$.