Math 8202 Graduate abstract algebra- Spring 2014, Vic Reiner
Final exam - Due Wednesday May 14 by 5pm, at my office Vincent 256
(If I'm not there, put it in the envelope taped to the door, or slide it under that door.)

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used. All calculations must be shown, written out by hand.

1. (40 points total; 5 points each)
Prove or disprove the following statements.

(a) For permutations σ, τ in the symmetric group S_n, the products $\sigma\tau$ and $\tau\sigma$ have the same cycle types.

(b) The field $\mathbb{K} := \mathbb{Q}(\sqrt{375}, \sqrt{135})$ has $[\mathbb{K} : \mathbb{Q}] = 4$.

(c) The field $\mathbb{K} := \mathbb{Q}(\sqrt{375}, \sqrt{135})$ has $\mathbb{K}/\mathbb{Q}$ Galois.

(d) The splitting field $\mathbb{K}$ for $x^3 - 20x + 19$ over $\mathbb{Q}$ has $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ isomorphic to the symmetric group S_3.

(e) A tower $\mathbb{F} \subset \mathbb{L} \subset \mathbb{K}$ of fields with $\mathbb{K}/\mathbb{L}$ and $\mathbb{L}/\mathbb{F}$ both finite Galois extensions will have $\mathbb{K}/\mathbb{F}$ also Galois.

(f) There are no solutions to $a^2 + b^2 + c^2 + d^2 = -1$ with $(a, b, c, d) \in \mathbb{K}^4$ if $\mathbb{K} := \mathbb{Q}(\omega \sqrt[3]{2})$ with $\omega := e^{\frac{2\pi i}{3}}$

(g) If $A \in \mathbb{C}^{n \times n}$ has $\det(xI - A) = \prod_{i=1}^{r}(x - \lambda_i)^{m_i}$ for distinct $\lambda_1, \ldots, \lambda_r$ in $\mathbb{C}$ and $m_i > 0$, then there exists a $\mathbb{C}$-vector space direct sum decomposition $\mathbb{C}^n = \bigoplus_{i=1}^{r} V_i$ where for each $i = 1, 2, \ldots, r$,

- $\dim_{\mathbb{C}} V_i = m_i$,
- V_i is A-stable, that is, $A(V_i) \subset V_i$, and
- $A - \lambda_i I$ acts *nilpotently* on V_i; specifically, $(A - \lambda_i I)^{m_i}(V_i) = 0$.

(h) Any $A \in \mathbb{C}^{n \times n}$ can be written $A = A_1 + A_2$ with A_1 diagonalizable, $A_2^n = 0$, and $A_1 A_2 = A_2 A_1$.

2. (25 points total) Let $c(n), r(n), q(n)$, respectively, be the number of irreducible factors of $x^n - 1$ when considered as elements of $\mathbb{C}[x], \mathbb{R}[x], \mathbb{Q}[x]$, respectively.

(a) (5 points) Write down $c(n), r(n), q(n)$ as functions of n in the simplest form that you can for each.

For parts (b),(c), let A be any matrix in $\mathbb{Q}^{n\times n}(\subset \mathbb{R}^{n\times n} \subset \mathbb{C}^{n\times n})$ having $\det(xI - A) = x^n - 1$. For example, one can take A to be the permutation matrix representing an n-cycle in the symmetric group S_n.

(b) (10 points) Regarding $V = \mathbb{R}^n$ as an $\mathbb{R}[x]$-module with x acting on V as multiplication by the matrix A, how many $\mathbb{R}[x]$-submodules $W \subset V$ will there be, including both $\{0\}$ and V itself among them?

(c) (10 points) Answer the same question as in (b), replacing $\mathbb{R}$ with $\mathbb{Q}$, so $V = \mathbb{Q}^n$ is a $\mathbb{Q}[x]$-module in which x acts as multiplication by A.

3. (20 points; 10 points each part) Let $V = \mathbb{C}^n$ with a positive definite Hermitian form $(-,-)$ making it a complex inner product space. Given $V \xrightarrow{T} V$ a $\mathbb{C}$-linear operator, recall one has an adjoint operator $V \xrightarrow{T^*} V$, and that T is called *self-adjoint* if $T = T^*$, and T is called *normal* if it commutes with its adjoint, that is, $T^*T = TT^*$.

(a) Show that T is normal if and only if it can be written as $T = T_1 + iT_2$ where T_1, T_2 are both self-adjoint and commute, that is, $T_1T_2 = T_2T_1$.

(b) Assume $T^2 = T$. Show that in this situation, T is normal if and only if T is self-adjoint.

4.(15 points total) Let $\mathbb{K}/\mathbb{F}$ be a field extension with $[\mathbb{K} : \mathbb{F}] = n$, and for k in the range $0 \leq k \leq n$, let $G_{\mathbb{F}}(k, \mathbb{K})$ denote the set of all k-dimensional $\mathbb{F}$-linear subspaces inside $\mathbb{K}(\cong \mathbb{F}^n)$.

(a) (5 points) Show that every element α in $\mathbb{K}^\times$ acts $\mathbb{F}$-linearly and invertibly on $\mathbb{K}$ via

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\alpha} & \mathbb{K} \\ \kappa & \longmapsto & \alpha \cdot \kappa \end{array}$$

(b) (5 points) Show that this gives rise to an action of the group $\mathbb{K}^\times$ on the set $G_{\mathbb{F}}(k, \mathbb{K})$, where α in $\mathbb{K}^\times$ sends a subspace $V \subset \mathbb{K}$ to $\alpha(V)$. Show that this $\mathbb{K}^\times$-action on $G_{\mathbb{F}}(k, \mathbb{K})$, where α in $\mathbb{K}^\times$ actually descends to an action of the quotient group $\mathbb{K}^\times/\mathbb{F}^\times$ on $G_{\mathbb{F}}(k, \mathbb{K})$.

(c) (5 points) Show that if $\gcd(k, n) = 1$ then this quotient group action $\mathbb{K}^\times/\mathbb{F}^\times$ on $G_{\mathbb{F}}(k, \mathbb{K})$ is *free*, that is, only the identity element of $\mathbb{K}^\times/\mathbb{F}^\times$ has any fixed points.