Math 8202 Graduate abstract algebra- Spring 2014, Vic Reiner
Midterm exam 2- Due Wednesday April 9, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used. All calculations must be shown, written out by hand.

1. (25 points total; 5 points each)
Recall that for a PID R, a finitely generated R-module M can be expressed uniquely in either of two forms:

- (invariant factors)
$$M = R^r \oplus \bigoplus_{i=1}^{t} R/(a_i),$$
with $r, t \geq 0$ and a_i in $R \setminus R^\times$ such that a_i divides a_{i+1} for $1 \leq i \leq t-1$, or
- (elementary divisors)
$$M = R^r \oplus \bigoplus_{j=1}^{s} R/(p_j^{\alpha_j}),$$
with $r, s \geq 0$ and $\alpha_j > 0$, and the p_j are irreducibles in R.

For parts (a), (b), consider the cyclic $\mathbb{Z}[i]$-module $\mathbb{Z}[i]/I$ where I is the principal ideal
$$I = (\ 1300 \cdot (1+i)^3 \cdot (3-2i)^4\ ) \subset \mathbb{Z}[i].$$

(a) Express this cyclic $\mathbb{Z}[i]$-module in its invariant factor form.
(b) Express this cyclic $\mathbb{Z}[i]$-module in its elementary divisor form.

For parts (c),(d),(e), regard this matrix T
$$T = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}$$
as defining a $\mathbb{Z}$-module map $\mathbb{Z}^3 \xrightarrow{T} \mathbb{Z}^3$.
(c) Express the $\mathbb{Z}$-module $\mathrm{coker}(T)$ in invariant factor form.
(d) Express the $\mathbb{Z}$-module $\ker(T)$ in invariant factor form.
(e) Express the $\mathbb{Z}$-module $\mathrm{im}(T)$ in invariant factor form.

2. (25 points total; 5 points each)
Prove or disprove the following statements.

(a) Every $\mathbb{Z}/4\mathbb{Z}$-submodule of the free module $(\mathbb{Z}/4\mathbb{Z})^{100}$ is free.

(b) Every $\mathbb{Z}/5\mathbb{Z}$-submodule of the free module $(\mathbb{Z}/5\mathbb{Z})^{100}$ is free.

(c) Every $\mathbb{Z}[i]$-submodule of the free module $(\mathbb{Z}[i])^{100}$ is free.

(d) Matrices T in $\mathbb{F}_5^{n\times n}$ satisfying $T^5 = T$ are diagonalizable over $\mathbb{F}_5$.

(e) Matrices T in $\mathbb{F}_2^{n\times n}$ satisfying $T^5 = T$ are diagonalizable over $\mathbb{F}_2$.

3. (10 points) How many different similarity classes of matrices in $\mathbb{Q}^{14\times 14}$ have (simultaneously)

- their *minimal polynomial* equal to $x^5(x+1)^3(x+2)^2$, and
- their *characteristic polynomial* equal to $x^8(x+1)^4(x+2)^2$?

4. (a) (5 points) Regard $V = \mathbb{R}^2$ as an $\mathbb{R}[x]$-module in which x acts on V via the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

How many $\mathbb{R}[x]$-submodules $W \subseteq V$ are there, counting both $W = \{0\}$ and $W = V$ among them?

(b) (10 points) Regard $V = \mathbb{C}^n$ as a $\mathbb{C}[x]$-module in which x acts on V via a matrix T in $\mathbb{C}^{n\times n}$. Assuming that $\det(xI - T) = x^n - 1$, how many $\mathbb{C}[x]$-submodules $W \subseteq V$ are there? (Your answer should be a function of n.)

5. (10 points) Given R a PID and M a finitely generated R-module, prove that there exist $\beta_0, \beta_1 \geq 0$ and an exact sequence of R-modules of the form

$$0 \to R^{\beta_1} \to R^{\beta_0} \to M \to 0.$$

6. (a) (5 points) Consider a field extension $\mathbb{F} \subset \mathbb{K}$ of degree $[\mathbb{K} : \mathbb{F}] = n$. Show that irreducible polynomials $f(x)$ in $\mathbb{F}[x]$ having degree $\deg(f)$ *relatively prime to* n always remain irreducible in the larger ring $\mathbb{K}[x]$.

(b) (10 points) Show that if a prime $p \neq 5$ has neither $p-1$ nor p^2-1 divisible by 5, then the polynomial

$$x^4 + x^3 + x^2 + x + 1 \left(= \frac{x^5 - 1}{x - 1} \right)$$

will be irreducible in $\mathbb{F}_p[x]$.