Math 8202 Graduate abstract algebra
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Extra practice problems related to finitely generated modules over a PID

1. Let $G = \mathbb{Z}^3/(\mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3)$ where v_1, v_2, v_3 are given by

$$v_1 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 4 \\ -8 \end{bmatrix}, v_3 = \begin{bmatrix} -6 \\ -8 \\ 14 \end{bmatrix}.$$

Express G explictly as a direct sum of cyclic groups.

2. Let L be the following matrix in $\mathbb{Z}^{n\times n}$:

$$L = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

(a) Compute the Smith normal form for L.
(b) Express $\mathbb{Z}^n/\mathrm{im}L$ explicitly as a direct sum of cyclic groups.

3. Let R be a Euclidean domain and $A = (a_{ij}) \in R^{m\times n}$ a matrix with Smith normal form

$$S = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where D is a diagonal matrix with diagonal entries $d_1|d_2|\cdots|d_r$, and we set $d_{r+1} = \cdots = d_m = 0$. We'll prove a different way of computing the d_i without using row or column operations. For $k = 0, 1, \ldots, m$, let g_k be the GCD of all determinants of square $k \times k$ submatrices of A, with the convention that $g_0 := 1$.

(a) Prove that $g_0 | g_1 | \cdots | g_m$.
(b) Prove that $g_1 = d_1$.
(c) Prove that $g_k = d_1 d_2 \cdots d_k$ for all k, and hence that $d_k = \frac{g_k}{g_{k-1}}$. (Note: doing this part of course takes care of parts (a), (b). A hint: prove this assertion for matrices in Smith normal form first. Then show that the operations required to bring a matrix into Smith normal form don't affect the values g_k.)
Remark: The same results hold when R is a PID, but then require an alternate proof. In fact, an elegant proof can be given using the fact that g_k is the same as g_1 for the k^{th} exterior power of A, and utilizing functoriality of exterior powers.

4. For the matrix A shown below, find the rational canonical form in $\mathbb{K}^{4\times 4}$, and the Jordan canonical form in $\mathbb{C}^{4\times 4}$. You do not need to find the change-of-basis matrices that put it into these forms.

$$A = \begin{bmatrix} 1 & 4 & -4 & 0 \\ -1 & -3 & 2 & 2 \\ 0 & -2 & 1 & 2 \\ -1 & -4 & 2 & 3 \end{bmatrix}.$$

5. Let $T : V \to V$ be an $\mathbb{F}$-linear operator on a finite dimensional vector space V over an algebraically closed field $\mathbb{F}$. (a) Show that T can expressed as a sum

$$T = T_{ss} + T_{nil}.$$

where T_{ss} is diagonalizable, T_{nil} is nilpotent, and T_{ss}, T_{nil} commute.
(b) Show that such an expression is unique.
(c) (Harder) Show that one can choose T_{ss} (and hence also T_{nil}) to be a polynomial in T, i.e. $T_{ss} = p(T)$ for some $p(x) \in \mathbb{F}[x]$.