Math 8202 Graduate abstract algebra
Spring 2002, Vic Reiner
Final exam - Due Friday May 10, in class

Instructions: The usual: open book, open library, open notes, take-home exam, to be discussed with no humans but me.

1. (10 points) Dummit and Foote, Section 9.5, Problem 7, on page 316.

2. Let V be a 4-dimensional $\mathbb{F}_2$-vector space, and say that $g \in GL(V)$ is a *derangement* (of $V - \{0\}$) if $g(v) \neq v$ for every $v \neq 0$ in V.

(a) (5 points) Show that this property is an invariant of the conjugacy class of g, that is, if g, g' are conjugate in $GL(V)$ then g is a derangement if and only if g' is.

(b) (10 points) How many conjugacy classes of derangements are there in $GL(V)$? Write down the list of invariant factors for the rational canonical form representing each conjugacy class.

3. Let R be a P.I.D. and $A \in R^{n\times n}$ representing a map $R^n \stackrel{A}{\to} R^n$.
(a) (7 points) Show that

$$\operatorname{coker}(A) = 0 \Leftrightarrow \det(A) \in R^\times$$
$$\operatorname{coker}(A) \text{ is a torsion } R\text{-module} \Leftrightarrow \det(A) \in R - \{0\}.$$

(b) (7 points) Assume $\det(A) \in R - \{0\}$. Let J be the principal ideal $(\det(A))$ in R, and $I := \mathrm{Ann_R(coker(A))}$. Show that there exists a positive integer N for which

$$I^N \subset J \subset I.$$

(c) (7 points) Assume $R = \mathbb{F}[x]$ for some field $\mathbb{F}$, and that $\det(A) \neq 0$. Show that $\dim_{\mathbb{F}} \operatorname{coker}(A)$ is the degree (in x) of the polynomial $\det(A)$.

4. (15 points) Let R be a P.I.D., and let $A \in R^{m\times n}, B \in R^{l\times m}$ represent maps of free R-modules as shown, with $BA = 0$ and $m \geq l, n$:

$$R^n \xrightarrow{A} R^m \xrightarrow{B} R^l.$$

Let A, B be equivalent by left and right multiplication by invertible matrices over R to diagonal matrices

$$A \sim \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad B \sim \begin{bmatrix} E & 0 \end{bmatrix}$$

where $D \in R^{n\times n}$ is diagonal with diagonal entries $(d_1, \ldots, d_n)$, and $E \in R^{l\times l}$ is diagonal with diagonal entries $(e_1, \ldots, e_l)$.

Express the quotient R-module $H := \ker B/\mathrm{im} A$ as a direct sum of cyclic R-modules in terms of the data given by the d_i, e_j's.

5. Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{l\times m}$ represent $\mathbb{R}$-linear maps

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \xrightarrow{B} \mathbb{R}^l,$$

and assume that $BA = 0$. Define a linear operator $L : \mathbb{R}^m \to \mathbb{R}^m$ by

$$L = AA^T + B^T B.$$

(a) (8 points) Show that the two operators AA^T and $B^T B$ on $\mathbb{R}^m$ are positive semidefinite (and hence so is their sum L). Recall that a symmetric matrix M is *positive semidefinite* (or *non-negative definite*) if $x^T M x \geq 0$ for all vectors $x \in \mathbb{R}^n$.

Also show that their product in either order is 0, so that they are commuting and mutually annihilating.

(b) (8 points) Show that

$$\mathrm{im} AA^T = \mathrm{im} A, \quad \mathrm{im} B^T B = \mathrm{im} B^T$$
$$\ker AA^T = \ker A^T, \quad \ker B^T B = \ker B$$

and that (with respect to the standard inner product on $\mathbb{R}^m$) one has an orthogonal direct sum decomposition

$$\mathbb{R}^m = \mathrm{im} B^T \oplus \ker L \oplus \mathrm{im} A.$$

(c) (8 points) Define $H := \ker B/\mathrm{im} A$, and prove that as $\mathbb{R}$-vector spaces, there is an isomorphism $H \cong \ker L$.

6. (15 points) Dummit and Foote, Section 14.5, Problem 12, on page 584. However, in this problem, I will allow you to assume the *Normal basis theorem*: if $\mathbb{K}$ is any finite Galois extension of $\mathbb{F}$, then there exists

an $\alpha \in \mathbb{K}$ such that the set of Galois conjugates $\{\sigma(\alpha) : \sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{F})\}$ form an $\mathbb{F}$-basis for $\mathbb{K}$.