Math 8202 Graduate abstract algebra
Spring 2002, Vic Reiner
Midterm exam 1- Due Friday March 8, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points) Dummit and Foote, Section 14.6, Problem 29 on pp. 600–601. Except please ignore where they say "common root", and replace it with "common factor" (they probably meant "common root in some extension field").

2. (a) (5 points) Let R be a commutative ring with 1, and recall that the *nilradical* $\mathcal{N}(R)$ is an ideal in R consisting of all the nilpotent elements. Prove that $\mathcal{N}(R)$ is the intersection of all prime ideals in R.

(Hint: One inclusion is easy. For the other, given a non-nilpotent element x in R, exhibit a prime ideal not containing it as follows. Show (with careful explanation) that the collection of all ideals in R that contain no power of x, when ordered by inclusion, is a partially ordered set to which Zorn's lemma applies, and that a maximal element is a prime ideal in R not containing x.)

(b) (10 points)Dummit and Foote, Section 7.4, Problem 40 on p. 261.

3. (5 points each) In each of the following problems, give an explicit isomorphism between the two rings, making sure that you *prove* it is an isomorphism.
Notation: $R[x, x^{-1}]$ is the ring of Laurent polynomials in x with coefficients in R, that is

$$R[x, x^{-1}] := \{a_n x^n + a_{n+1}x^{n+1} + ... + a_N x^N : a_i \in R, n, N \in \mathbb{Z}\}$$

with obvious ring operations. If R is a subring of some other ring S, and $\alpha_1, \ldots, \alpha_m \in S$, then $R[\alpha_1, \ldots, \alpha_m]$ is the smallest subring of S containing R and all the $\alpha_1, \ldots, \alpha_m$.

(a) $\mathbb{Z}[x, u]/(xu - 1) \cong \mathbb{Z}[t, t^{-1}]$
(b) $\mathbb{Z}[u]/(2u - 1) \cong \mathbb{Z}[\frac{1}{2}](\subset \mathbb{Q})$
(c) $\mathbb{Z}[x, xy, xy^2] \cong \mathbb{Z}[u^2, uv, v^2]$. (Hint: show both are isomorphic to the ring $\mathbb{Z}[a, b, c]/(b^2 - ac)$.)

4. (5 points each) Say a polynomial $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ is *homogeneous of degree d* if every monomial occurring in f has the same degree d. By segregating monomials according to their degree, one can express any polynomial f uniquely as

$$f = f_0 + f_1 + \cdots + f_d$$

with f_i homogeneous of degree f_i. Here the f_i's are called the *homogeneous components of f*.

Prove the following simple facts:

(a) If $f(\mathbf{x})$ is homogeneous of degree d, then

$$f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x}).$$

(b) If $f(\mathbf{x})$ is homogeneous of degree d, then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

(c) An ideal I in $\mathbb{F}[x_1, \ldots, x_n]$ is said to be *homogeneous* if every homogeneous component f_i of any f in I also lies in I.

Show that I is a homogeneous ideal if and only if it can be generated by a collection of homogeneous polynomials.

(d) If $n = 2$ and $\mathbb{F}$ is algebraically closed, show that every homogeneous polynomial f in $\mathbb{F}[x_1, x_2]$ can be written as a product of linear (degree 1) polynomials.

5. (15 points) Prove that for two ideals $I, J \subset \mathbb{F}[x_1, \ldots, x_n]$ one has

$$I \cap J = (tI + (1-t)J) \cap \mathbb{F}[x_1, \ldots, x_n]$$

where

$$(tI + (1-t)J) = \{tf(\mathbf{x}) + (1-t)g(\mathbf{x}) : f \in I, g \in J\}$$

is an ideal in $\mathbb{F}[t, x_1, \ldots, x_n]$.

6. (10 points each) The goal of this problem is to prove ...

Theorem 1. *(Fermat's Last Theorem for $\mathbb{F}[x_1, \ldots, x_n]$) Let $\mathbb{F}$ be an algebraically closed field of characteristic zero (e.g. the complex numbers). Assume f, g, h, in $\mathbb{F}[x_1, \ldots, x_n]$ with no common irreducible factors satisfy*

$$f^n + g^n = h^n$$

where n is an integer greater than 2. Then f, g, h are all constants in $\mathbb{F}$.

We will reduce the Theorem to the following lemma:

Lemma 2. *Fix $n \geq 3$. Assume g, h have no common irreducible factors in $\mathbb{F}[x_1, \ldots, x_n]$, and there exist at least 3 distinct $\mathbb{F}$-linear combinations $a_i g + b_i h$ which are perfect n^{th} powers in $\mathbb{F}[x_1, \ldots, x_n]$ (i.e. $a_i g + b_i h = r_i^n$ for some polynomials r_i and the ratios $[a_i : b_i]$ are all different for $i = 1, 2, 3$). Then g, h are constants in $\mathbb{F}$.*

(a) Show that the theorem follows from the Lemma.
(Hint: First assume there is a counterexample f, g, h to the theorem, and show why one may assume that g, h have no common irreducible factors. Then write

$$f^n = h^n - g^n = \prod_{i=1}^{n} (h - \zeta^i g)$$

where ζ is a (primitive) n^{th} root of 1 in $\mathbb{F}$. Show using unique factorization in $\mathbb{F}[x_1, \ldots, x_n]$ that g, h satisfy the hypotheses of the lemma).

(b) Prove the lemma by Fermat's "method of infinite descent": Assume that one has such a pair g, h satisfying the hypotheses of the lemma, with $\max(\deg(g), \deg(h))$ *as small as possible* but bigger than 0. Show using unique factorization that one can produce a new pair of polynomials $\tilde{g}, \tilde{h}$ which satisfy the hypotheses of the lemma, but with

$$\max(\deg(g), \deg(h)) > \max(\deg(\tilde{g}), \deg(\tilde{h})) > 0.$$