Math 8202 Graduate abstract algebra
Spring 2002, Vic Reiner
Midterm exam - Due Friday April 19, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points) Dummit and Foote, Section 12.1, Problem 8 on p. 449.

2. (15 points) Dummit and Foote, Section 12.2, Problem 13 on p. 469.

3. (10 points each)
(a) Dummit and Foote, Section 12.3, Problem 27 on p. 481
(b) Dummit and Foote, Section 12.3, Problem 28 on p. 481

4. (15 points) Dummit and Foote, Section 13.2, Problem 17 on p. 510.

5. (15 points) Let $\mathbb{F}$ be a field and $f(x) \in \mathbb{F}[x]$ irreducible of degree n. Let $\mathbb{K}$ be a field extension of $\mathbb{F}$ with $[\mathbb{K} : \mathbb{F}] = m < \infty$ satisfying $\gcd(m, n) = 1$. Show that $f(x)$ remains irreducible in $\mathbb{K}[x]$.

6. Let $\mathbb{F}$ be a field and $\mathbb{K}$ a field extension of $\mathbb{F}$ with $[\mathbb{K} : \mathbb{F}] = n < \infty$. Let k be an integer in the range $0 \leq k \leq n$, and consider the set

$$\mathbb{G}_{\mathbb{F}}(k, \mathbb{K}) := \{k\text{-dimensional } \mathbb{F}\text{-linear subspaces of } \mathbb{K}\}.$$

(a) (5 points) Show that the group of units $\mathbb{K}^\times$ acts on the set $\mathbb{G}_{\mathbb{F}}(k, \mathbb{K})$ in the following way: if V is an $\mathbb{F}$-linear subspace of $\mathbb{K}$ and $\alpha \in \mathbb{K}^\times$ then explain why

$$\alpha V := \{\alpha v : v \in V\}$$

is another $\mathbb{F}$-linear subspace of the same dimension. Also explain exactly how this induces an action of the quotient group $\mathbb{K}^\times/\mathbb{F}^\times$ on $\mathbb{G}_{\mathbb{F}}(k, \mathbb{K})$.

(b) (15 points) Show that if $\gcd(k, n) = 1$, the action of $\mathbb{K}^\times/\mathbb{F}^\times$ on $\mathbb{G}_{\mathbb{F}}(k, \mathbb{K})$ is *free*, i.e. for every non-identity element $\bar{\alpha}$ of the group $\mathbb{K}^\times/\mathbb{F}^\times$ there are no k-dimensional $\mathbb{F}$-subspaces with $\bar{\alpha}V = V$.