Möbius function and topology of Bruhat intervals

Recall from enumerative combinatorics of posets ...

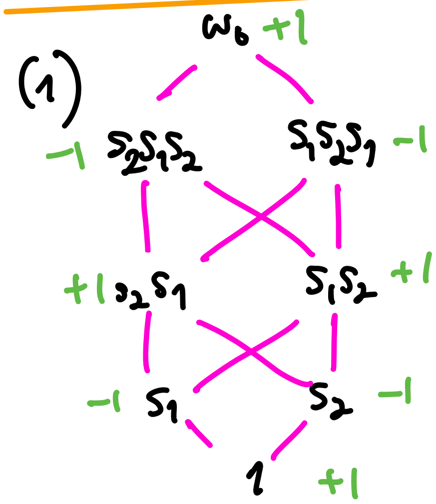
DEF'N: For a poset P with finite intervals $[x,y]$, the Möbius function $\mu(x,y) := \begin{cases} 1 & \text{if } x=y \\ -\sum_{z:\, x\le z<y} \mu(x,z) & \text{for } x<_P y \end{cases}$

which shows up in the Möbius inversion formula:

$f,g : P \to R$ (a ring or abelian group) satisfy

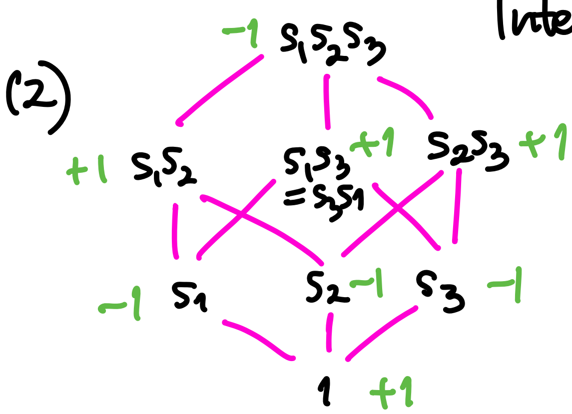
$$g(y) = \sum_{x:\, x\le_P y} f(x) \iff f(y) = \sum_{x:\, x\le_P y} \mu(x,y)\, g(x)$$

EXAMPLES



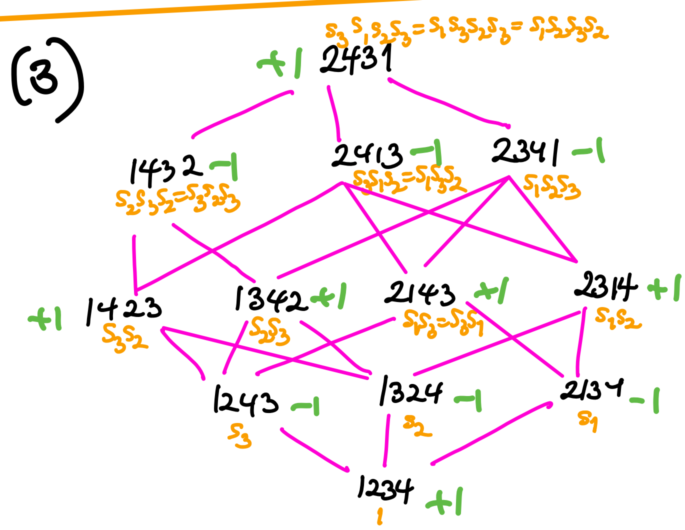
Bruhat on $W(\underset{s_1}{\circ}\overset{4}{\text{—}}\underset{s_2}{\circ}) = I_2(4) = B_2/C_2$

values of $\mu(1,w)$ labeled



Interval $[1, s_1s_2s_3]$

in $\mathfrak{S}_4 = W(\underset{s_1}{\circ}\text{—}\underset{s_2}{\circ}\text{—}\underset{s_3}{\circ})$



$= [1234, 2431]$

$= [1, s_3s_1s_2s_3]$

in $\mathfrak{S}_4$

One might be tempted to guess ...

THEOREM (Verma 1971) For any Cox. system (W,S), any $u \le w$ in Bruhat order have $\mu(u,w) = (-1)^{\ell(w)-\ell(u)}$.

[That is, Bruhat order is an Eulerian poset]

ranked $r: P \to \mathbb{Z}$

and $\mu(x,y) = (-1)^{r(y)-r(x)}$ $\forall x,y$

We'll approach this topologically (see B-B App. A2), starting with...

THEOREM (P. Hall 1936) In any poset P,

$$\mu(x,y) = \tilde{\chi}(\Delta \underbrace{(x,y)})$$

(reduced) Euler characteristic $:= -f_{-1} + f_0 - f_1 + f_2 - f_3 + \ldots$

where $f_i = \#$ i-dimensional simplices/faces

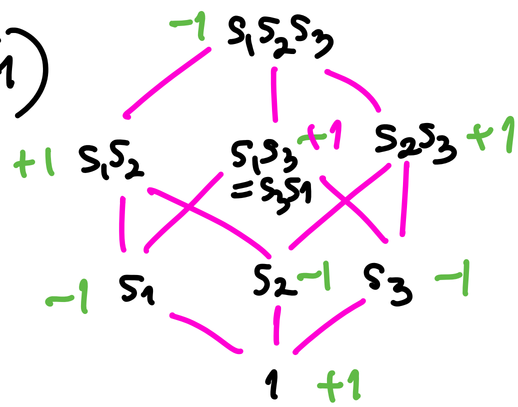
($f_{-1} = 1$ counts the empty simplex $\varnothing$ of dimension -1)

order complex := simplicial complex whose simplices are the totally ordered subsets

open interval $(x,y) := \{z \in P : x < z < y\}$

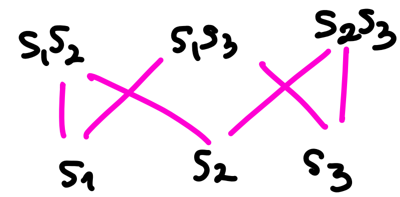
EXAMPLES

(1)

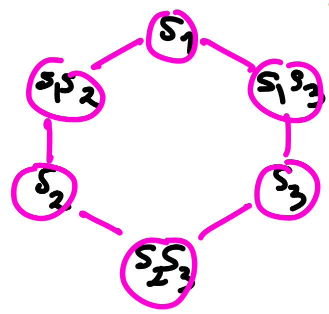


$$\mu(1, s_1s_2s_3) = \tilde{\chi}\left(\Delta(1, s_1s_2s_3)\right)$$

$(1, s_1s_2s_3) =$



$\Delta(1, s_1s_2s_3) =$



homeomorphic

$\cong S^1$

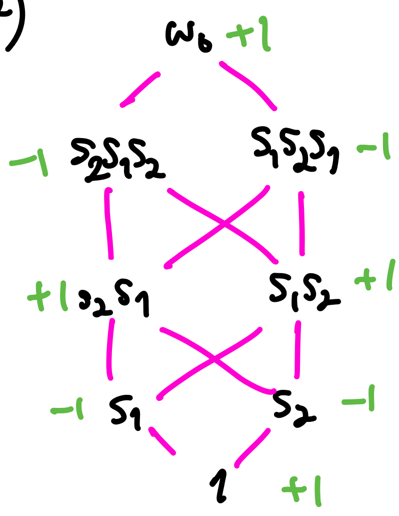
1-sphere = circle

1 empty face, 6 vertices, 6 edges

$$\tilde{\chi}\left(\Delta(1, s_1s_2s_3)\right) = -f_{-1} + f_0 - f_1$$

$$= -1 + 6 + 6 = -1 = \mu(1, s_1s_2s_3)$$

(2)



$$\mu(1, s_1s_2) = \tilde{\chi}\left(\Delta(1, s_1s_2)\right)$$

$= \tilde{\chi}\Big($

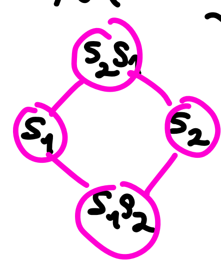
$\Big)$

0-sphere S^0

$$= -f_1 + f_0 = -1 + 2 = +1$$

$$\mu(1, s_2s_1s_2) = \tilde{\chi}\left(\Delta(1, s_2s_1s_2)\right)$$

$= \tilde{\chi}\Big($



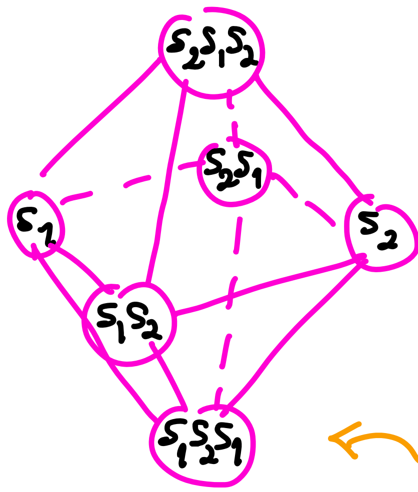
$\Big) = -1 + 4 - 4$

$= -1$

$$\mu(1,w_0) = \tilde{\chi}(\Delta(1,w_0))$$

$$= \tilde{\chi}\left(\Delta\left(\begin{tikzcd}[column sep=small, row sep=small] s_2s_1s_2 \arrow[d, dash] \arrow[dr, dash] & s_1s_2s_1 \arrow[d, dash] \arrow[dl, dash] \\ s_2s_1 \arrow[d, dash] \arrow[dr, dash] & s_1s_2 \arrow[d, dash] \arrow[dl, dash] \\ s_1 & s_2 \end{tikzcd}\right)\right)$$

$$= \tilde{\chi}\Bigg($$



$$\Bigg)$$

homeomorphic to $\mathbb{S}^2$ = 2-sphere

$$= -f_{-1} + f_0 - f_1 + f_2$$

$$= -1 + 6 - 12 + 8 = -1$$

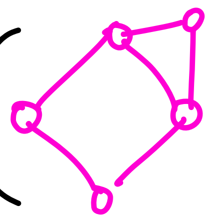
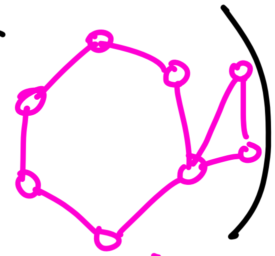
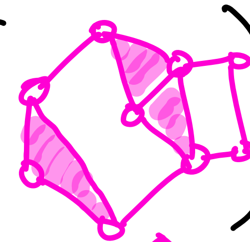
Recall reduced Euler characteristic is computable from homology groups,

$$\tilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i = -\tilde{\beta}_{-1} + \tilde{\beta}_0 - \tilde{\beta}_1 + \tilde{\beta}_2 - \cdots$$

where $\tilde{\beta}_i = \operatorname{rank} \tilde{H}_i(\Delta, \mathbb{Z})$ (or could take $\tilde{\beta}_i = \dim_k \tilde{H}_i(\Delta, k)$ for a field k)

This makes $\tilde{\chi}(\Delta)$ a homeomorphism invariant, and even a homotopy type invariant.

EXAMPLE

$\tilde{\chi}\Bigg($  $\Bigg) = \tilde{\chi}\Bigg($  $\Bigg) = \tilde{\chi}\Bigg($  $\Bigg)$

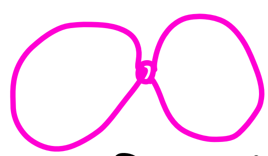
Δ_1 Δ_2 Δ_3

$= -1+5-6 \quad = -1+8-9 \quad = -1+9-13+3 = -2$

$= -\tilde{\beta}_{-1} + \tilde{\beta}_0 - \tilde{\beta}_1 + \tilde{\beta}_2 - \tilde{\beta}_3 + \cdots$

$= -0 + 0 - 2 + 0 - 0 + \cdots$

because $\Delta_1, \Delta_2, \Delta_3$ are all homotopy equivalent to a (1-point) wedge $\mathbb{S}^1 \vee \mathbb{S}^1$



which has $\tilde{H}_i(\mathbb{S}^1 \vee \mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} 0 & \text{if } i = -1, 0, 2, 3, 4, \ldots \\ \mathbb{Z}^2 & \text{if } i = 1 \end{cases}$

So what we will actually try to show is this:

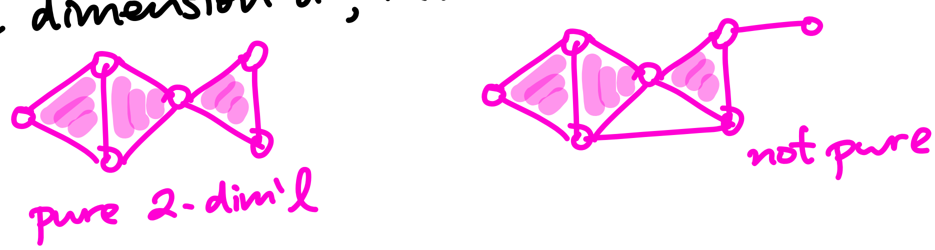
THEOREM (Björner–Wachs 1982) For any Cox. sys. (W,S) and $u \le w$,

$$\Delta(u,w) \overset{\text{homeomorphic}}{\cong} \mathbb{S}^{\ell(w)-\ell(u)-2}$$

$$\left[\overset{\text{P. Hall}}{\Rightarrow} \mu(u,w) = \tilde{\chi}\left(\mathbb{S}^{\ell(w)-\ell(u)-2}\right) = (-1)^{\ell(w)-\ell(u)} \right]$$

The approach is via these useful concepts.

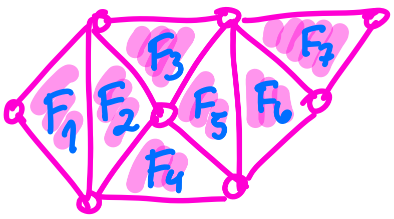
DEF'N: Say that a simplicial complex Δ is pure of dimension d if all of its facets (:= inclusion-maximal faces) have dimension d, i.e. $d+1$ vertices.

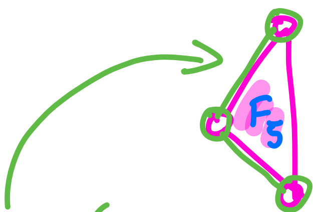


DEF'N: Say a pure d-dim'l simplicial complex Δ is shellable if one can order its facets $F_1, F_2, \ldots, F_t$ with $\forall i \ge 2$, $F_i \cap \left(\begin{smallmatrix}\text{subcomplex} \\ \text{gen'd by } F_1,\ldots,F_{i-1}\end{smallmatrix}\right)$ forming a pure $(d-1)$-dim'l subcomplex.

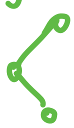
EXAMPLES

(1) shellable:



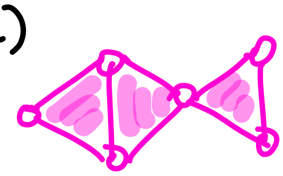


$F_5 \cap$ (subcomplex gen'd by F_1, F_2, F_3, F_4)

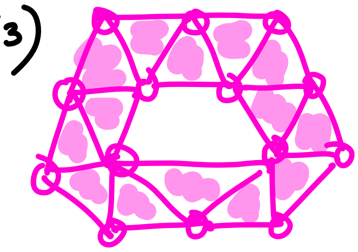
= 

pure, but not shellable

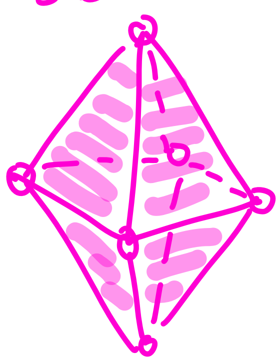
(2)



(3)

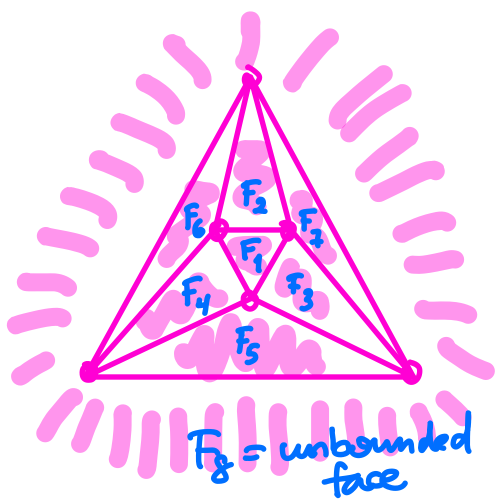


(4) shellable:



boundary of octahedron

flatten as a planar graph ⇝

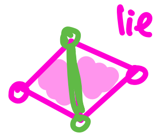


A useful PL-topology fact will apply to Bruhat intervals.

↑ piecewise-linear

THEOREM: (Fact A.2.4.3 in B–B) If a pure d-dim'l complex Δ is both shellable and thin, then $\Delta \cong \mathbb{S}^d$.

thin: every $(d-1)$-simplex lies in exactly two facets



triangulations of d-manifolds are thin

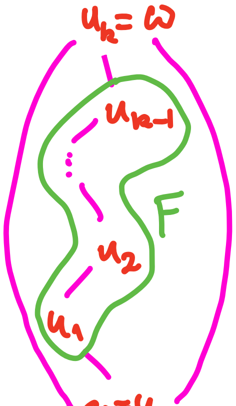
So we'll try to show any $u \leq w$ in Bruhat has $\Delta(u,w)$

- pure d-dim'l where $d = \ell(w) - \ell(u) - 2$
- shellable
- thin.

The purity is immediate from the fact that Bruhat order on W is ranked by $\ell(w)$,

so all maximal chains $u = u_0 <\cdot\, u_1 <\cdot \cdots <\cdot\, u_{k-1} <\cdot\, u_k = w$ have $k = \ell(w) - \ell(u)$

$\Updownarrow$



all facets $F = \{u_1, u_2, \ldots, u_{k-1}\}$ in $\Delta(u,w)$ have $\#F = k-1$, $\dim F = k-2$ where $k = \ell(w) - \ell(u)$

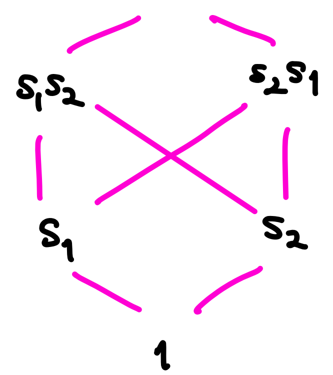
The shellability and thin-ness both will come from a certain way to label edges in max chains.

Fix a reduced expression $w = s_1 s_2 \cdots s_q$ with its positions labeled $1\ 2 \cdots q$.

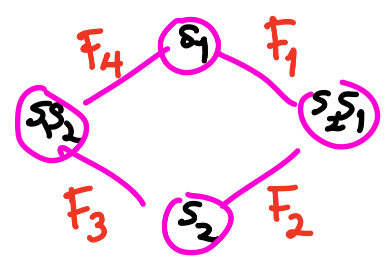
Then in any max chain $w = w_0 > w_1 > \cdots > w_{k-1} > w_k = u$ as you work down from the top, each step w_j to w_{j+1} knocks out a unique s_k (by Strong Exchange applied to $w_{j+1} = w_j t$) whose original position k labels the edge $w_j > w_{j+1}$

The proposed shelling order is via lex order on label sequences read top-to-bottom

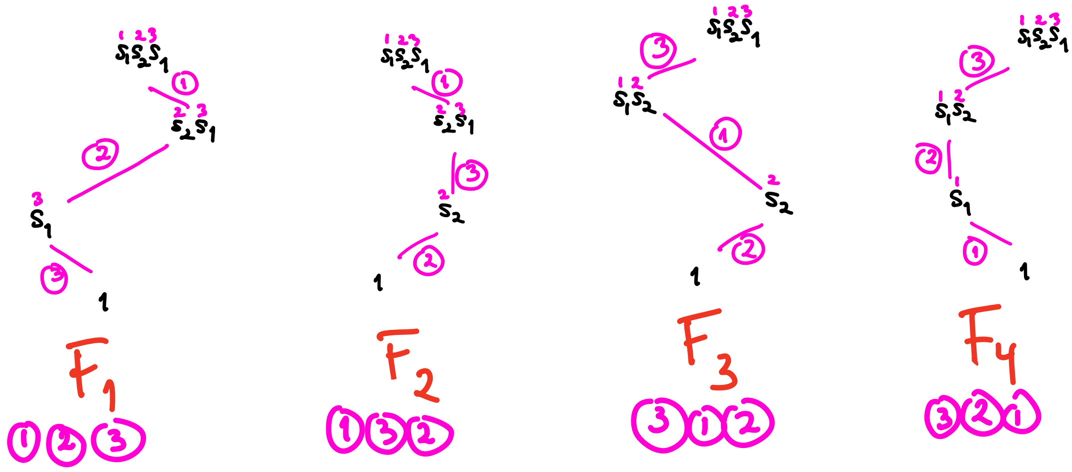
EXAMPLE In $\mathfrak{S}_3$, $w = w_0 = \overset{1}{s_1}\overset{2}{s_2}\overset{3}{s_1}$ ← fixed



$\Delta(1, w)$



Chain labels:



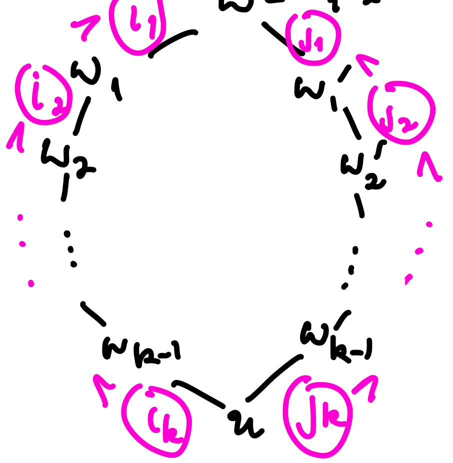
LEMMA Every Bruhat interval $[u,w]$ has a unique chain with increasing labels $i_1 < i_2 < \cdots < i_k$.

proof: We constructed one when we showed Bruhat was ranked: if $w = s_1 s_2 \cdots s_q$ and $u = s_1 s_2 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots s_q$, we exhibited one with labels $(i_1, i_2, \ldots, i_k)$ and i_k leftmost.

To show one can't have two of them, induct on $\ell(w) - \ell(u)$, with **BASE CASE** $\ell(w) = \ell(u)+1$ easy.

In the inductive step, given two of them

$w = s_1 s_2 \cdots s_q$



one knows

$$s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots s_q = u = s_1 \cdots \hat{s}_{j_1} \cdots \hat{s}_{j_k} \cdots s_q$$

and if $i_k < j_k$ WLOG, then one gets a contradiction that

$$w_{k-1}' = ut = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_k} \cdots \hat{s}_{j_k} \cdots s_q$$

$t = s_q s_{q-1} \cdots s_{j_k} \cdots s_{q-1} s_q$ has length $< \ell(w)$.

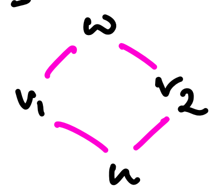
Therefore $i_k = j_k$, so $w_{k-1} = w_{k-1}'$. Then there is at most one increasing labeled chain in $[w_{k-1}, w]$ by induction, implying $(i_1, \ldots, i_{k-1}) = (j_1, \ldots, j_{k-1})$

∎

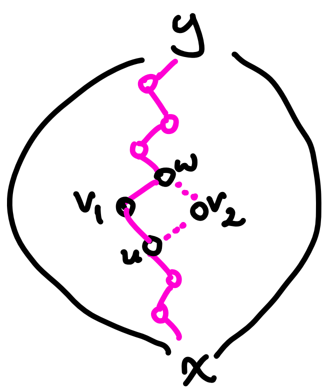
One can reverse left-to-right choices and also prove:
(pick $i_1 < \dots < i_k$ with i_1 rightmost)

LEMMA Every Bruhat interval $[u,w]$ has a unique chain with decreasing labels $i_1 > i_2 > \dots > i_k$. ∎

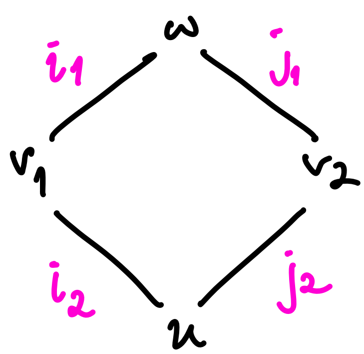
COROLLARY Bruhat intervals $[u,w]$ with $\ell(w)-\ell(u)=2$ all look like this:



Equivalently, all Bruhat intervals $[x,y]$ have $\Delta(x,y)$ thin:



Proof: In a length 2 interval, every maximal chain is either increasing or decreasing



$i_1 < i_2$ or $i_1 > i_2$

∎

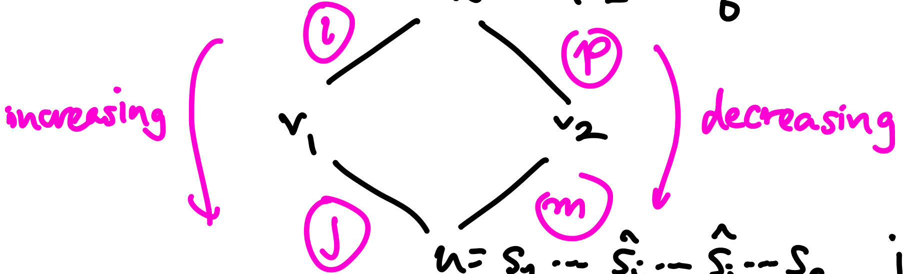
LEMMA: The lex smallest labeled max chain in $[u,w]$ is the unique increasing one.

proof: Induct on $\ell(w)-\ell(u)$.

BASE CASE $\ell(w)-\ell(u)=2$.

There by construction they are

$$w = s_1 s_2 \cdots s_q$$



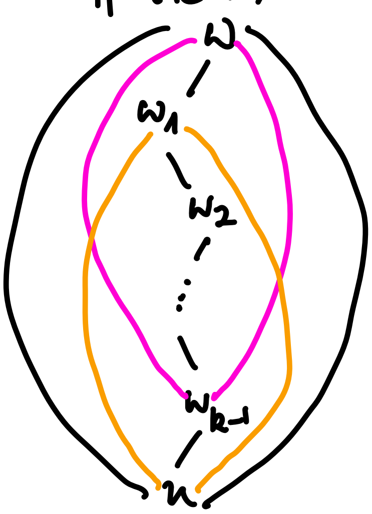
$$u = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_q \quad j \text{ minimal}$$

$$= s_1 \cdots \hat{s}_m \cdots \hat{s}_p \cdots s_q \quad m \text{ maximal}$$

$$\Rightarrow m \geq i \Rightarrow p > i \Rightarrow (i,j) <_{lex} (p,m)$$

INDUCTIVE STEP $\ell(w)-\ell(u) \geq 3$.

If the lex smallest chain is this one



then its segment in $[w_{k-1},u]$ and its segment in $[u,w_1]$ are also lex smallest, so increasing by induction, but then they overlap enough to show the big one is increasing

∎

Finally ...

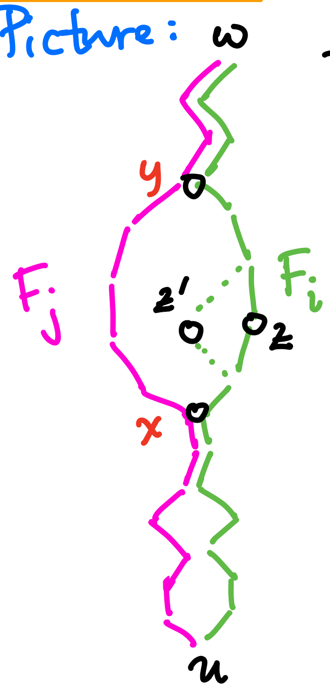
THEOREM Lex order on the labels of facets $F_1, F_2, \ldots$ of $\Delta(u,w)$ gives a shelling.

[Hence $\Delta(u,w)$ is pure d-dim'l, thin, shellable so homeomorphic to $\mathbb{S}^d$ where $d = \ell(w) - \ell(u) - 2$.]

Proof: A typical face of $F_i \cap \left(\begin{smallmatrix}\text{subcomplex} \\ \text{gen'd by } F_1, F_2, \ldots, F_{i-1}\end{smallmatrix}\right)$ is a face $F_i \cap F_j$ with $F_j <_{lex} F_i$. We must exhibit some face $F_k <_{lex} F_i$ with

$$F_i \cap F_j \subseteq F_i \cap F_k \text{ and } \dim(F_i \cap F_k) = d-1.$$

Picture:



Find y nearest w where F_i, F_j first deviate, and let x be the first place where they coincide again.

Inside the interval $[x,y]$, since $F_j <_{lex} F_i$, it cannot be that F_i is lex earliest, so it cannot be increasing in $[x,y]$, so it has some adjacent decreasing labels surrounding some z. Let F_k replace z with z' in the same length 2 interval, and note F_k has $F_k <_{lex} F_i$,

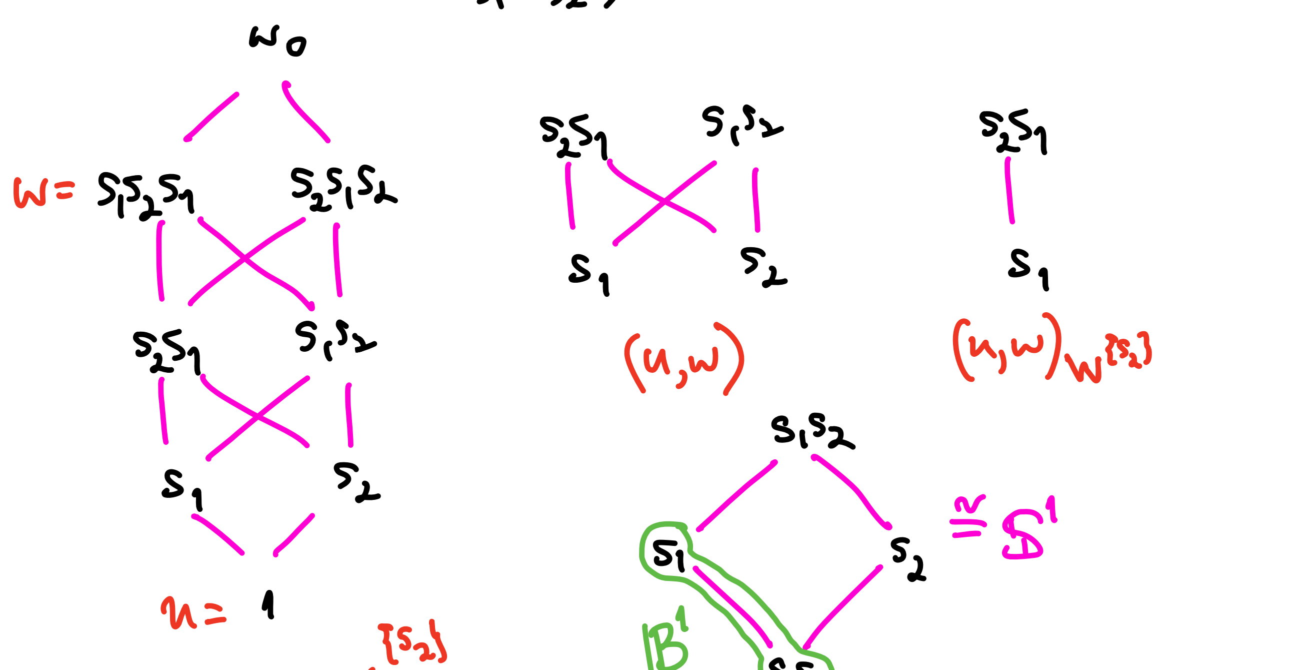
$$F_i \cap F_j \subseteq F_i \cap F_k \text{ and } \dim(F_i \cap F_k) = d-1. \blacksquare$$

REMARKS

(1) B-B prove more strongly (THM 2.7.5) that for any $J \subseteq S$ and $u \leq w$ in W^J, the interval $(u,w)_{W^J}$ is also pure d-dim'l and shellable where $d = \ell(w) - \ell(u) - 2$. Furthermore, it is not thin but subthin, i.e. $(d-1)$-faces lie in ≤ 2 facets. This implies

$$\Delta(u,w)_{W^J} \underset{\text{homeomorphic}}{\cong} \begin{cases} \mathbb{S}^d & \text{if } [u,w]_{W^J} = [u,w]_W \\ \underset{d\text{-ball}}{\mathbb{B}^d} & \text{if } [u,w]_{W^J} \subsetneq [u,w]_W \end{cases}$$

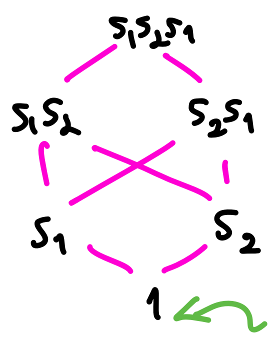
EXAMPLE $W(\overset{4}{\underset{s_1\quad s_2}{\circ\!-\!\!-\!\circ}})$



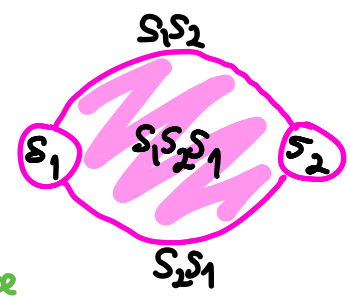
$u, w \in W^{\{s_2\}}$

(2) The fact that (open) Bruhat intervals $\Delta(u,w)$ are all spherical implies $[u,w]$ is also the face poset of a regular CW-ball - see B–B THM 2.7.12 & App. A2.5

EXAMPLES

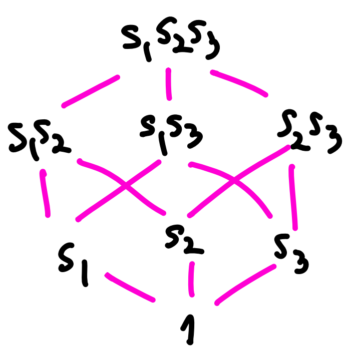


= face poset of

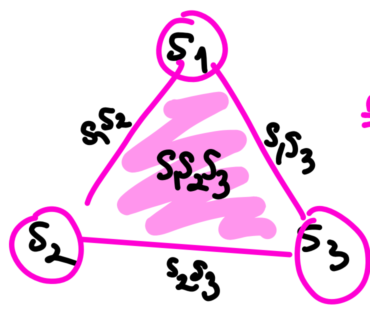


$\cong \mathbb{B}^2$

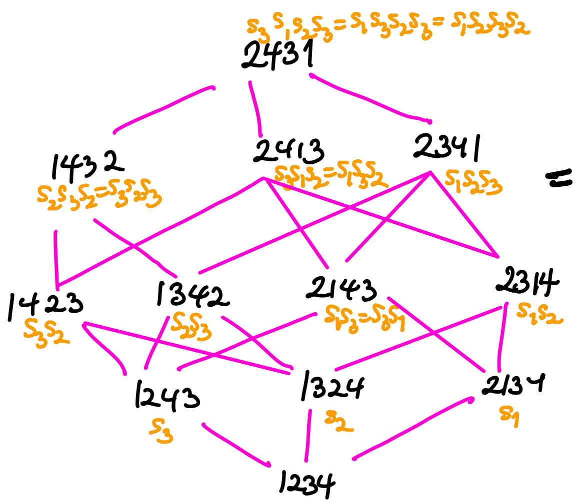
corresponds to empty face $\varnothing$



= face poset of



$\cong \mathbb{B}^2$



= face poset of

$s_3s_1s_2s_3$ = interior cell of $\mathbb{B}^3$

$\cong \mathbb{B}^3$

These regular CW-balls have been interpreted geometrically (for Weyl groups W) in terms of total positivity using ideas of Lusztig,

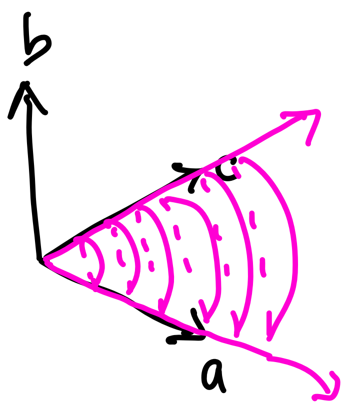
via Fomin-Shapiro-Shapiro,

Hersh,

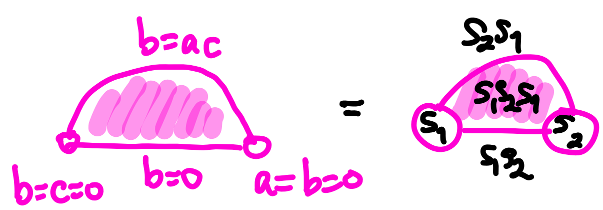
Galashin-Karp, Lam.

EXAMPLE Totally nonnegative part of the unipotents $\mathcal{U} = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

is defined by inequalities
$$a \geq 0$$
$$b \geq 0$$
$$c \geq 0$$
$$ac - b = \begin{vmatrix} a & b \\ 1 & c \end{vmatrix} \geq 0 \quad \text{i.e. } b \leq ac$$



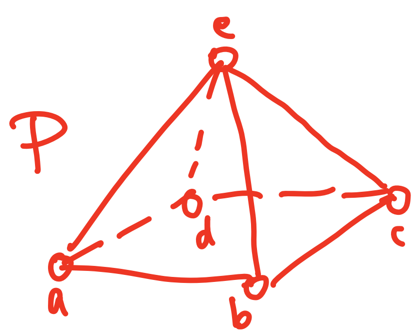
cross-section slice $\rightsquigarrow$



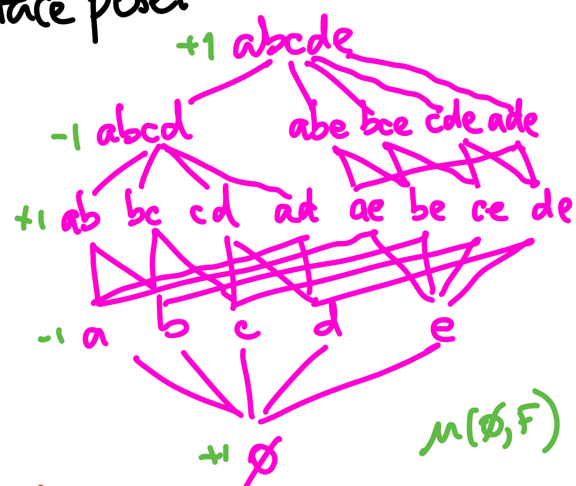
(3) The analogy between these $\left\{\begin{matrix}\text{Bruhat interval} \\ \text{balls } \mathbb{B}^d\end{matrix}\right\}$ and $\left\{\begin{matrix}\text{d-dim'l} \\ \text{convex polytopes} \\ P\end{matrix}\right\}$ is very strong.

Face posets of polytopes are also Eulerian:

$$\mu(F,G) = (-1)^{\dim G - \dim F}$$



$\rightsquigarrow$ face poset



There is a toric variety one can associate to P when it has vertices in $\mathbb{Q}^n$, analogous to the Schubert varieties and their strata associated to $[u,w]$ when W is crystallographic.