Real reflection groups and their root systems

(Humphreys §§ 1.1–1.5;
after planning on Humphreys Chap. 5 + §§ 6.1–6.4)

People had motivation to write down a "reflection representation" for any Coxeter group $W \cong \langle S \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$ from features they knew occur in (finite) real ref'n groups W.

Let's understand this motivation, which comes from some root system geometry of W

As before, $V = \mathbb{R}^n$
with a chosen inner product $(\cdot,\cdot)$
$\mathbb{R}\times\mathbb{R} \to \mathbb{R}$

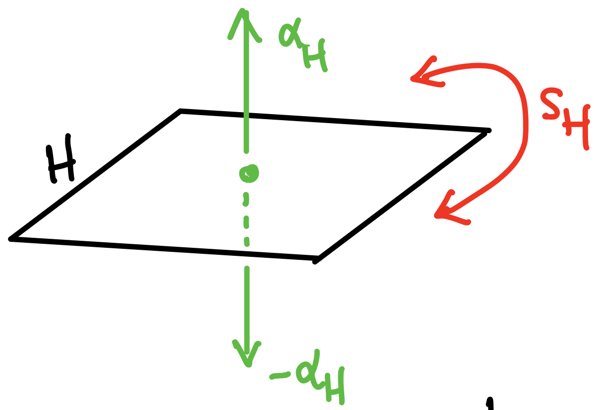
which means $(\cdot,\cdot)$ is
- $\mathbb{R}$-bilinear: $(x+y,z) = (x,z)+(y,z)$
  $(x,y+z) = (x,y)+(x,z)$
  $(cx,y) = c\cdot(x,y) = (x,cy)$
- symmetric: $(x,y) = (y,x)$
- positive definite: $(x,x) \geq 0$
  with $(x,x)=0$
  only if $x = \underline{0}$

Using Gram-Schmidt process, there's always an orthonormal basis $e_1,\ldots,e_n$ for $\mathbb{R}^n$ making

$$(x,y) = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \text{, if we want.}$$

DEF'N: For a hyperplane $H \subset V$
$(=(n-1)\text{-dim'l linear subspace})$
the reflection $s_H : V \longrightarrow V$
fixes H pointwise, and negates $\pm\alpha_H := $ unit normals to H

i.e. $\alpha_H^{\perp} = H$ and $(\alpha_H, \alpha_H) = 1$



In general, if $\alpha \in V$ is any vector, then $s_\alpha := s_H$ where $H = \alpha^\perp = \{x \in V : (\alpha, x) = 0\}$ has this formula:

$$\boxed{s_\alpha(x) = x - 2 \cdot \frac{(x,\alpha)}{(\alpha,\alpha)}\alpha}$$

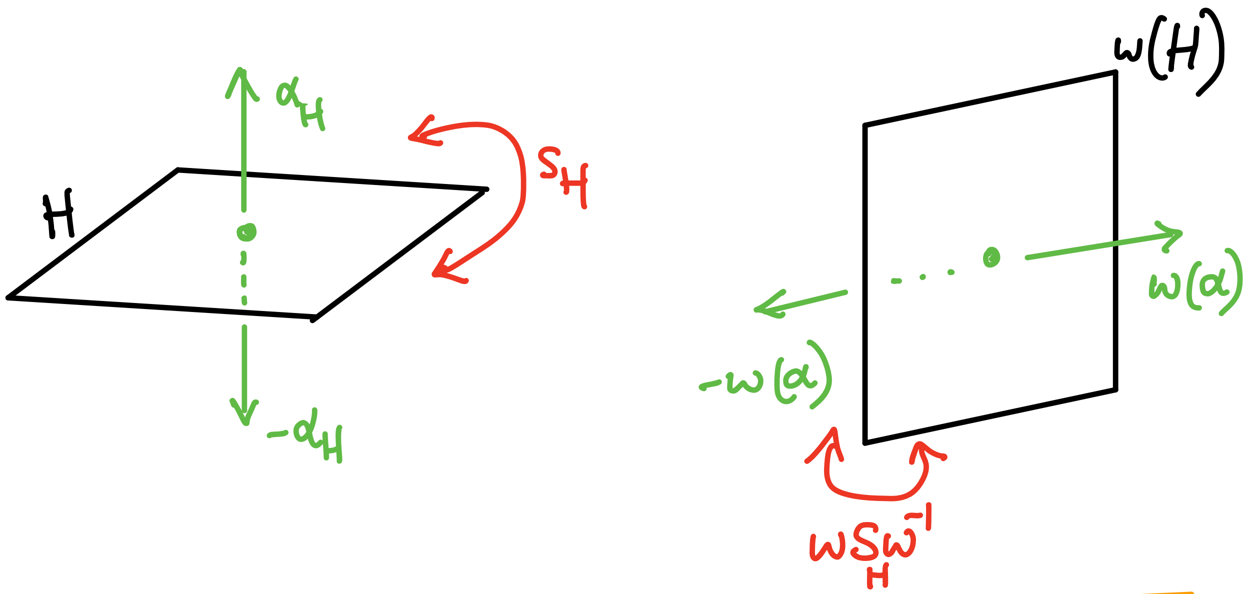
since it's $\begin{cases} \text{linear in } x \\ \text{correct for } x \in H = \alpha^\perp: \ s_\alpha(x) = x - 2\frac{0}{(\alpha,\alpha)}\alpha = x \\ \text{correct for } x = \alpha: \ s_\alpha(\alpha) = \alpha - 2\frac{(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = -\alpha \end{cases}$

An important fact: Conjugating a ref'n $s_H = s_\alpha$ by an orthogonal transformation

$w \in O(V, (\cdot,\cdot)) := \{ w \in GL(V) : (w(x), w(y)) = (x,y) \}$

gives another ref'n, namely

$$\boxed{\begin{array}{ccc} w s_H w^{-1} & = & s_{w(H)} \\ \| & & \| \\ w s_\alpha w^{-1} & & s_{w(\alpha)} \end{array}}$$



Check: For $x \in w(H)$, say $x = w(y)$, $y \in H$

$$\begin{aligned} ws_Hw^{-1}(x) &= ws_Hw^{-1}(w(y)) \\ &= ws_H(y) \\ &= w(y) \\ &= x \\ &= s_{w(H)}(x) \checkmark \end{aligned}$$

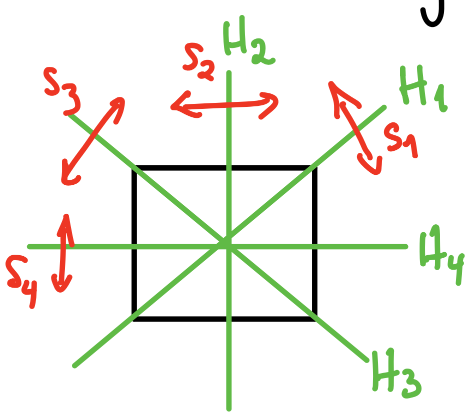
and
$$\begin{aligned} ws_Hw^{-1}(w(\alpha)) &= ws_H(\alpha) \\ &= w(-\alpha) \\ &= -w(\alpha) \\ &= s_{w(H)}(w(\alpha)) \checkmark \end{aligned}$$

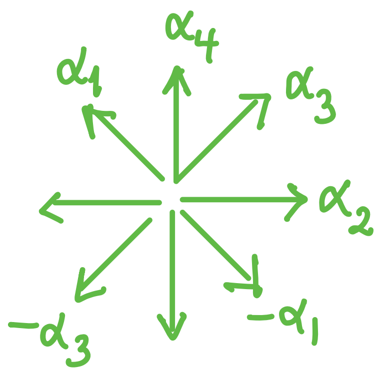
Recall

DEF'N: $W \subset GL(V)$, $V = \mathbb{R}^n$ with $(\cdot,\cdot)$ is a real ref'n group if it is finite and generated by the reflections $\{s_H\}$ contained within it.

e.g. $W = I_2(m)$ = dihedral group of order $2m$
= symmetries of regular m-gon

$m=4$

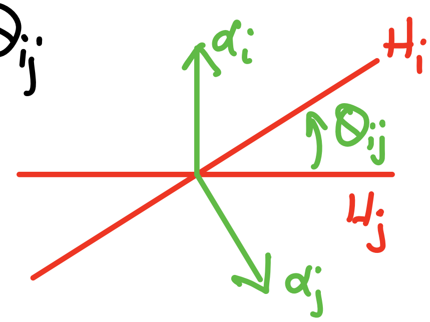




IDEA: If we had a basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ for V consisting of unit normals to ref'n hyperplanes $H_1, \ldots, H_n$, the inner products (α_i, α_j) determine $(\cdot,\cdot)$, and are computable from

$$(\alpha_i, \alpha_j) = \|\alpha_i\| \|\alpha_j\| \cos\theta_{ij} = -\cos\theta_{ij}$$

if H_i, H_j have dihedral angle θ_{ij}



So we could have recovered the V and $(\cdot,\cdot)$ just from the Coxeter diagram/matrix $(m_{ij})_{i,j=1,\dots,n}$ which predict the dihedral angles $\theta_{ij} = \frac{\pi}{m_{ij}}$.

(since $s_1 s_2$ = rotation through $\frac{2\pi}{m_{ij}}$, $(s_i s_j)^{m_{ij}} = 1$)

This gives the idea for how to make a a faithful geometric representation

$$W \longrightarrow O(V, (\cdot,\cdot)) \subset GL(V)$$

$V \times V \xrightarrow{(\cdot,\cdot)} \mathbb{R}$ will be symmetric, bilinear, but not always positive definite or even non-degenerate!

for any Coxeter system (W, S).

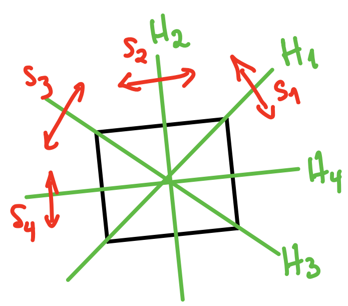
(Humphreys §5.3)

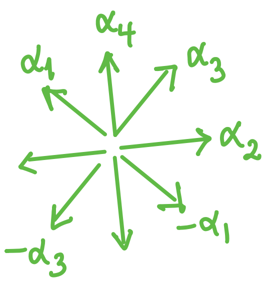
How to find these $\Pi = \{\alpha_1, \ldots, \alpha_m\}$ given the real ref'n group $W \subset O(V, (\cdot,\cdot))$? Start with...

DEF'N: The **(unit length) root system** of W

$$\Phi := \{\pm \alpha_H : \text{all ref'ns } s_H \text{ in } W\}$$

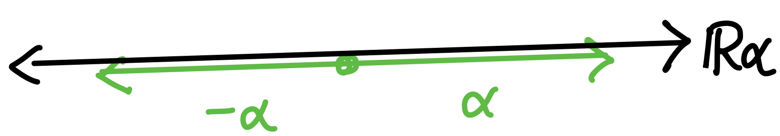
W





Φ

Note it satisfies
$$\begin{cases} s_\alpha(\Phi) = \Phi \quad \forall \alpha \in \Phi \\ \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\} \end{cases}$$



We'll decompose it into two halves using...

DEF'N: **Lexicographic order** on $\mathbb{R}^n$ sets $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} <_{lex} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = y$ if
$$\begin{matrix} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_\ell = y_\ell \\ x_{\ell+1} < y_{\ell+1} \end{matrix}$$
for some ℓ

Note $x <_{lex} y \iff y - x >_{lex} \underline{0}$

$$x <_{lex} y \implies \begin{cases} cx <_{lex} cy & \text{for } c \in \mathbb{R}_{>0} \\ cx >_{lex} cy & \text{for } c \in \mathbb{R}_{<0} \end{cases}$$

$$\implies x + z <_{lex} y + z \quad \forall z \in \mathbb{R}^n$$

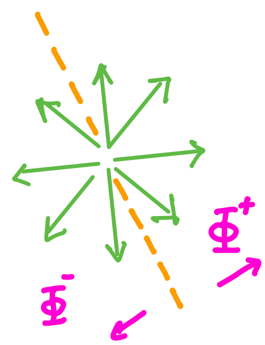
$$x, y >_{lex} \underline{0} \implies x + y >_{lex} x >_{lex} \underline{0}$$

$>_{lex}$ is a total/linear order:

either $x <_{lex} y$ or $x = y$ or $x >_{lex} y$

DEF'N: Disjointly decompose

$$\underset{\text{positive roots}}{\Phi = \Phi^+} \sqcup \underset{\text{negative roots}}{\Phi^-}$$



where $\Phi^+ := \{ \alpha \in \Phi : \alpha >_{lex} \underline{0} \}$

$\Phi^- := \{ \alpha \in \Phi : \alpha <_{lex} \underline{0} \} = -\Phi^+$

DEF'N: Define a set of simple roots $\Pi \subset \Phi^+$ to be any subset with these properties:

(a) $\Phi^+ = \mathbb{R}_{\geq 0} \Pi$

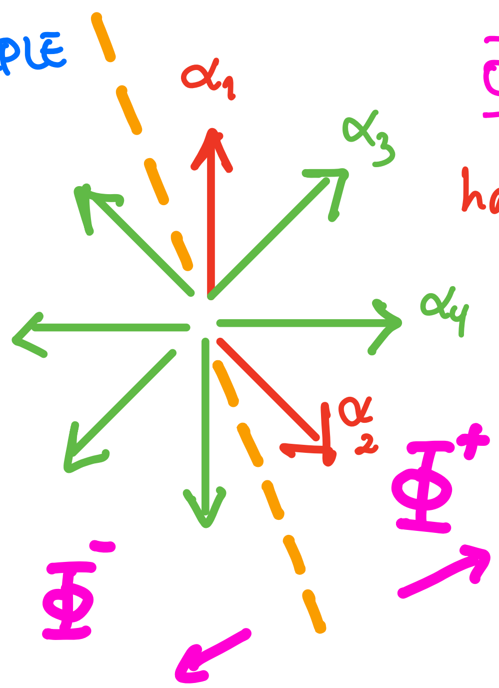
meaning every $\beta \in \Phi^+$ can be written

$$\beta = \sum_{\alpha \in \Pi} c_\alpha \cdot \alpha \quad \text{with} \quad c_\alpha \in \mathbb{R}_{\geq 0}$$

(b) Π is minimal with respect to inclusion having property (a)

(no $\alpha \in \Pi$ can be removed, retaining (a))

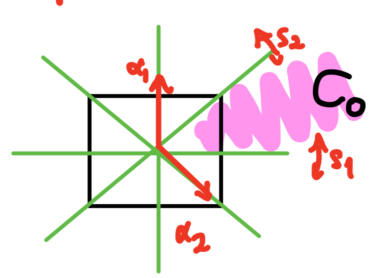
EXAMPLE



$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

has $\Pi = \{\alpha_1, \alpha_2\}$ as the only choice of simple roots

Corresponds to walls of C_0 here:



EXAMPLE $W = \mathfrak{S}_n$ = symmetric group

$= \left\{ \begin{matrix} n\times n \\ \text{permutation} \\ \text{matrices} \end{matrix} \right\} \subset GL_n(\mathbb{R}) = GL(V)$

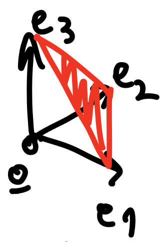
$V = \mathbb{R}^n$

= symmetries of regular $(n-1)$-simplex

$\int$ ← see Björner-Brenti EXERCISE 1.1.5 (or later PROP 1.5.4)

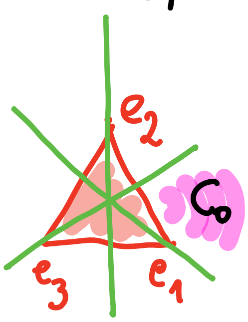
$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & \\ & 0 & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} = (i, i+1) \longleftarrow s_i$$

n=3



$$W\left(\underset{s_1}{\circ} - \underset{s_2}{\circ} - \cdots - \underset{s_{n-1}}{\circ} \right) =: W(A_{n-1})$$

$$:= \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{matrix} s_i^2 = 1 = (s_i s_j)^2 \text{ if } |i-j| \geq 2 \\ = (s_i s_{i+1})^3 \text{ for } i = 1, \ldots, n-2 \end{matrix} \right\rangle$$



Reflections: $\{\text{all transpositions } (i,j) : 1 \leq i < j \leq n\}$

Roots: $\Phi = \{ e_i - e_j,\ e_j - e_i \ : \ 1 \leq i < j \leq n \}$

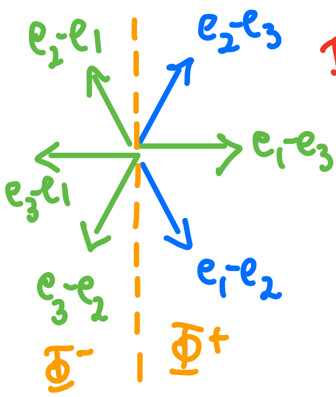
($\sqrt{2}$ length, not unit length)

$\cup$

Positive roots $\Phi^+ = \{ e_i - e_j \ : 1 \leq i < j \leq n \}$

$\cup$

Simple roots $\Pi = \{ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n \}$

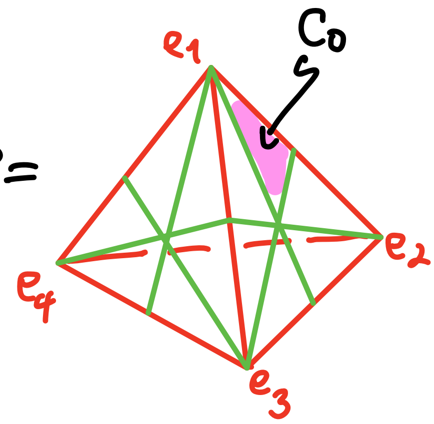


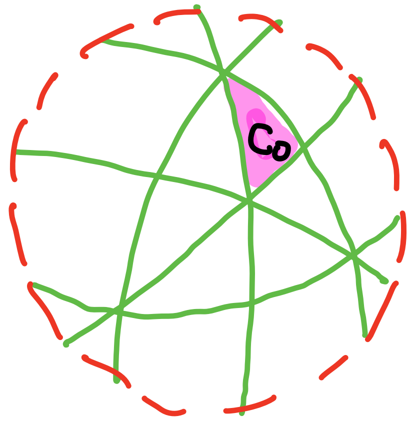
$W = \mathfrak{S}_4$

$=$ symmetries of regular tetrahedron

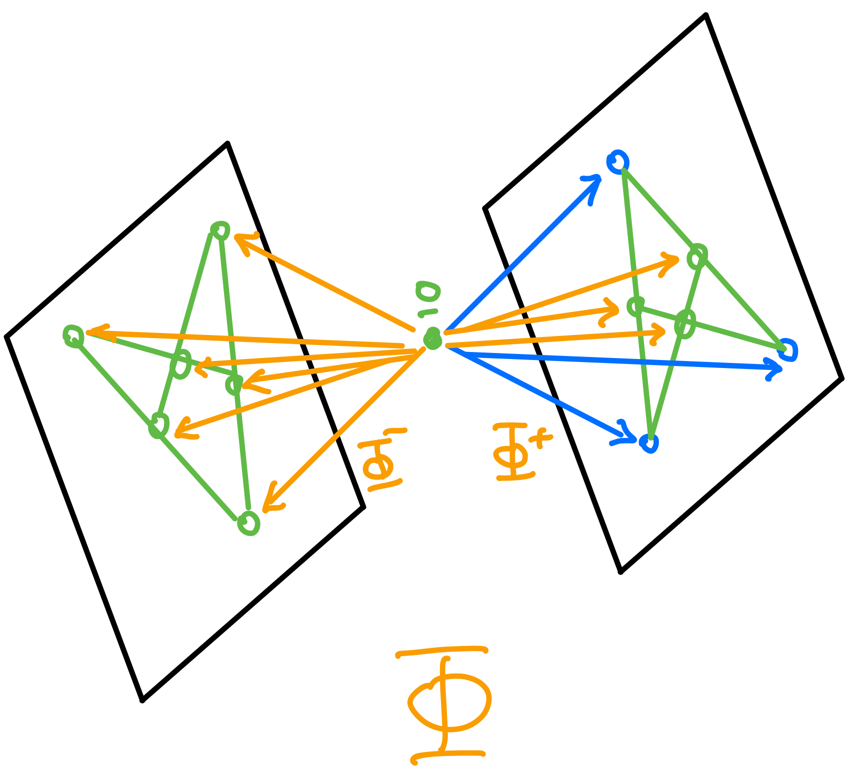
$P = \mathrm{conv}(e_1+e_2+e_3+e_4) - \frac{1}{4}(e_1+e_2+e_3+e_4)$

$P =$

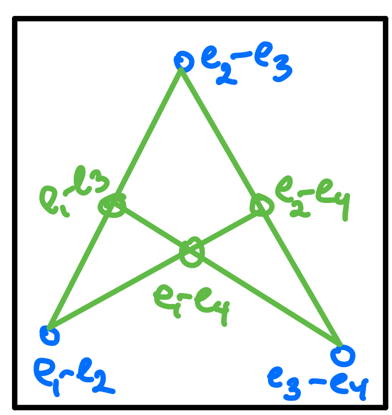




refl'n hyperplanes drawn intersecting unit sphere in $\underbrace{(e_1+e_2+e_3+e_4)^{\perp}}_{\cong \mathbb{R}^3} \subset \mathbb{R}^4$



Φ^+



$\Pi =$ simple roots $= \{e_1-e_2, e_2-e_3, e_3-e_4\}$

PROPOSITION: Simple roots $\Pi \subset \Phi^+$ satisfy:

(i) $(\alpha,\beta) \leq 0 \quad \forall \alpha \neq \beta$ in Π (pairwise non-acute)

(ii) Π is linearly independent, and hence a basis for $\mathrm{span}_{\mathbb{R}} \Phi^+ = \mathrm{span}_{\mathbb{R}} \Phi$ in V

---

proof: Let's see why (i) $\Rightarrow$ (ii) first.

Assuming (i), if we had a nontrivial dependence write it $c_1\alpha_1 + \dots + c_m\alpha_m = d_1\beta_1 + \dots + d_\ell\beta_\ell$ with $\alpha_i, \beta_j \in \Pi$ and $c_i, d_j \in \mathbb{R}_{>0}$

and note $\gamma := \sum_{i=1}^{m} c_i\alpha_i = \sum_{j=1}^{\ell} d_j\beta_j >_{lex} \underline{0}$,

but $0 \leq (\gamma,\gamma) = \left(\sum_i c_i\alpha_i, \sum_j d_j\beta_j\right) = \sum_{i,j} \underbrace{c_i d_j}_{>0} \underbrace{(\alpha_i,\beta_j)}_{\leq 0} \leq 0.$

Contradiction.

Proof of (ii): If $(\alpha,\beta) > 0$, note

$s_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - c\cdot\alpha$ with $c > 0$.

We'll reach a contradiction to

$s_\alpha(\beta) \in \Phi = \Phi^+ \sqcup \Phi^-$

in two cases for whether $s_\alpha(\beta) \in \Phi^+$ or Φ^-.

Case 1: $s_\alpha(\beta) \in \Phi^+$

Write $s_\alpha(\beta) = 1\cdot\beta - c\cdot\alpha = \sum\limits_{\gamma\in\Pi} c_\gamma\cdot\gamma = c_\beta\cdot\beta + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\beta}} c_\gamma\cdot\gamma$, $c_\gamma \geq 0$

If $c_\beta < 1$, $(1-c_\beta)\beta = c\cdot\alpha + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\beta}} c_\gamma\cdot\gamma \Rightarrow \beta\notin\Pi$ $\lightning$

If $c_\beta > 1$, $\underline{0} = c\cdot\alpha + (c_\beta - 1)\beta + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\beta}} c_\gamma\cdot\gamma >_{lex} \underline{0}$ $\lightning$

"contradiction"

Case 2: $s_\alpha(\beta) \in \Phi^-$

Write $s_\alpha(\beta) = 1\cdot\beta - c\cdot\alpha = \sum\limits_{\gamma\in\Pi} c_\gamma\cdot\gamma = c_\alpha\cdot\alpha + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\alpha}} c_\gamma\cdot\gamma$, $c_\gamma \leq 0$

If $c + c_\alpha > 0$, $\beta + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\alpha}} (-c_\gamma)\cdot\gamma = (c+c_\alpha)\cdot\alpha \Rightarrow \alpha\notin\Pi$ $\lightning$

If $c + c_\alpha \leq 0$, $\underline{0} = -\beta + (c+c_\alpha)\alpha + \sum\limits_{\substack{\gamma\in\Pi\\ \gamma\neq\alpha}} c_\gamma\cdot\gamma <_{lex} \underline{0}$ $\lightning$

$\blacksquare$

end of proof

Two important consequences

LEMMA: $\forall$ simple roots $\alpha \in \Pi$,

$$s_\alpha\left(\Phi^+ \setminus \{\alpha\}\right) = \Phi^+ \setminus \{\alpha\}$$

(but $s_\alpha(\alpha) = -\alpha \in \Phi^-$, of course)

Proof: Given $\beta \in \Phi^+ \setminus \{\alpha\}$, write

$\beta = \sum_{\gamma \in \Pi} c_\gamma \cdot \gamma$, so $c_\gamma \geq 0$ and some $c_{\gamma_0} > 0$ for $\gamma_0 \neq \alpha$.

But then $s_\alpha(\beta) = \beta - c \cdot \alpha$ has same coefficient $c_{\gamma_0} > 0$ on γ_0, and hence $s_\alpha(\beta) \in \Phi^+$, not Φ^-.

And $s_\alpha(\beta) \neq \alpha$, else

$$\beta = s_\alpha\left(s_\alpha(\beta)\right) = s_\alpha(\alpha) = -\alpha \notin \Phi^+ \qquad \blacksquare$$

COROLLARY: $W = \langle \{s_\alpha\}_{\alpha \in \Pi} \rangle$

simple reflections

---

proof: Since $W \overset{W \text{ is a real refl'n group}}{=} \langle \{s_H\} \rangle = \langle \{s_\alpha\}_{\alpha \in \Phi^+} \rangle$

and $\beta = w(\alpha) \Rightarrow s_\beta = w s_\alpha w^{-1}$,

it's enough to show every $\beta \in \Phi^+$ is in the W'-orbit of some $\alpha \in \Pi$, where $W' := \langle \{s_\alpha\}_{\alpha \in \Pi} \rangle$.

Prove this via induction on the height of $\beta = \sum_{\alpha \in \Pi} c_\alpha \cdot \alpha$

defined as $ht(\beta) := \sum_{\alpha \in \Pi} c_\alpha$.

Pick $\beta' \in \Phi^+ \cap \{W'\text{-orbit of } \beta\}$ minimizing $ht(\beta')$.

Since $0 < (\beta', \beta') = \left(\beta', \sum_{\alpha \in \Pi} c_\alpha \cdot \alpha\right) = \sum_{\alpha \in \Pi} c_\alpha (\beta', \alpha)$,

there exists $\alpha_0 \in \Pi$ with $(\beta', \alpha_0) > 0$.

Either $\beta' = \alpha_0 \in \Pi$ and we're done,

or else $\beta'' = s_{\alpha_0}(\beta') = \beta' - c\alpha_0$ with $c = 2\frac{(\beta', \alpha_0)}{(\alpha_0, \alpha_0)} > 0$

has $ht(\beta'') < ht(\beta')$ and $\beta'' \in \Phi^+$ by previous LEMMA; contradiction. ▨

UPSHOT:

For a real refln group $W \subset GL(V)$, $V = \mathbb{R}^n$ with $(\cdot,\cdot)$,

creating $\underset{\text{roots}}{\Phi} \supset \underset{\text{positive roots}}{\Phi^+} \supset \underset{\text{simple roots}}{\Pi} = \{\alpha_1, \ldots, \alpha_m\}$

and replacing V with $\mathrm{span}_{\mathbb{R}}(\Pi)$ W.L.O.G.,

we can recover both

- the bilinear form $(\cdot,\cdot)$ on V via its Gram matrix on Π:

$$\Big((\alpha_i, \alpha_j) \Big)_{i,j=1,2,\ldots,m}$$

$$\| $$

$$-\cos\left(\frac{\pi}{m_{ij}}\right) \text{ if } m_{ij} = \text{order of } s_{\alpha_i} s_{\alpha_j}$$

- the group $W \subset O(V, (\cdot,\cdot))$ via

$$W = \langle \{s_\alpha\}_{\alpha \in \Pi} \rangle$$

where $s_\alpha(x) := x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$

REMARK: Later we'll discuss for crystallographic (finite) root systems Φ^+

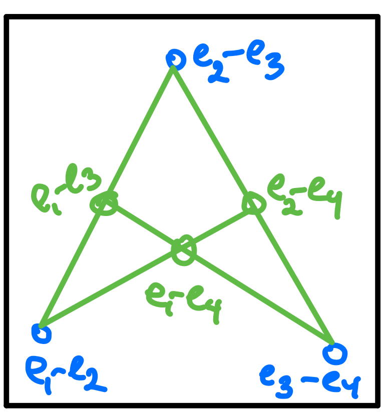
(= those where we choose roots to make $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ $\forall \alpha,\beta \in \Phi$

= those coming from Lie groups/algebras)

how $ht(\alpha)$ = rank of α in the graded/ranked root poset on Φ^+ defined by $\alpha <_{root} \beta$ if $\beta - \alpha \in \Phi^+$

EXAMPLE $W = \mathfrak{S}_4$

Φ^+



root poset $<_{root}$ on Φ^+