Roots, length, exchange & deletion conditions

Now that we have the (injective) faithful rep'n

$$W \overset{\sigma}{\hookrightarrow} GL(V) \text{ on } V=\mathbb{R}^n$$
$$s_i \longmapsto \sigma_i$$

$W = \langle S \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$

basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ with $B(\cdot,\cdot)$

and root system $\Phi = \Phi^+ \sqcup \Phi^-$

$\Phi = \{w(\alpha_i) : w \in W, i=1,\ldots,n\}$

many things follow.

PROPOSITION: $\ell(w) = \# N(w)$ where

$N(w) := \{\beta \in \Phi^+ : w(\beta) \in \Phi^-\} = \Phi^+ \cap w^{-1}(\Phi^-)$

(In particular, RHS is finite.)

proof: Induct on $\ell(w)$. In base case $\ell(w)=1$, so $w=s_i$, it says $s_i(\Phi^+ \setminus \{\alpha_i\}) = \Phi^+ \setminus \{\alpha_i\}$ since $s_i(\alpha_i) = -\alpha_i \in \Phi^-$. Prove this like before:

Write $\beta \in \Phi^+ \setminus \{\alpha_i\}$ uniquely as $\beta = \sum_{j=1}^{n} c_j \alpha_j$, $c_j \geq 0$ and at least one $c_{j_0} > 0$ with $j_0 \neq i$ (else $\beta = \alpha_i$).

Hence $s_i(\beta)$ has same coeff $c_{j_0} > 0$ on α_{j_0}, so $s_i(\beta) \in \Phi^+$.

In the inductive step, let's show that

$$w(\alpha_i)\in\Phi^+ \overset{(*)}{\Longrightarrow} N(ws_i)=s_iN(w)\sqcup\{\alpha_i\}.$$

This will also show

$$w(\alpha_i)\in\Phi^- \Longrightarrow N(ws_i)=s_iN(w)\setminus\{\alpha_i\}$$

by applying $(*)$ to ws_i instead of w.

But $(*)$ follows since if $w(\alpha_i)\in\Phi^+$, then

since $ws_i(\alpha_i) = -w(\alpha_i)\in\Phi^-$

$$\begin{aligned} N(ws_i) &= \{\beta\in\Phi^+ : ws_i(\beta)\in\Phi^-\} \\ &= \{\beta\in\Phi^+\setminus\{\alpha_i\} : ws_i(\beta)\in\Phi^-\} \sqcup \{\alpha_i\} \\ &= \{s_i(\gamma)\in\Phi^+\setminus\{\alpha_i\} : w(\gamma)\in\Phi^-\} \sqcup \{\alpha_i\} \\ &= s_iN(w)\sqcup\{\alpha_i\} \end{aligned}$$

$\gamma=s_i(\beta)$
$\beta=s_i(\gamma)$
using base case

Then $\ell(w)=\#N(w)$ follows by induction on $\ell(w)$:

If $\ell(ws_i)=\ell(w)+1$ then $w(\alpha_i)\in\Phi^+$, so $\#N(ws_i)=\#(s_iN(w)\sqcup\{\alpha_i\})$
$=\#N(w)+1$

(Also if $\ell(ws_i)=\ell(w)-1$ then $w(\alpha_i)\in\Phi^-$, so $\#N(ws_i)=\#(s_iN(w)\setminus\{\alpha_i\})$
$=\#N(w)-1$,
but only needed one of these)

$\blacksquare$

EXAMPLE: For $\mathfrak{S}_n = W(\overset{3}{o - o} \overset{3}{-} \cdots \overset{3}{-} o)$,
we saw we could choose $\Phi = \Phi^+ \sqcup \Phi^-$

$$\Phi^+ = \{e_i - e_j : 1 \le i < j \le n\} \qquad \Phi^- = \{e_j - e_i : 1 \le i < j \le n\}$$

and hence $w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$ has

$$w(e_i - e_j) = e_{w_i} - e_{w_j} \in \begin{cases} \Phi^+ & \text{if } w_i < w_j \quad \text{i.e. } (i,j) \text{ noninversion positions} \\ \Phi^- & \text{if } w_i > w_j \quad \text{i.e. } (i,j) \text{ inversion positions} \end{cases}$$

so $\ell(w) = \# N(w)$
$= \#\{(i,j) : 1 \le i < j \le n,\ w_i > w_j\} = \mathrm{inv}(w)$ inversion number

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DEF'N: For any Coxeter system (W,S), define the set of reflections

$$T := \{ws_iw^{-1} : w \in W,\ s_i \in S\} = \bigcup_{w \in W} wSw^{-1}$$

and note that if $w(\alpha_i) = \beta \in \Phi$, then acting on V using geom. rep'n σ, $\quad ws_iw^{-1} = ws_{\alpha_i}w^{-1} = s_\beta$

where $s_\beta(x) = x - 2\dfrac{B(x,\beta)}{B(\beta,\beta)}\beta$
$= x - 2B(x,\beta)\cdot\beta$

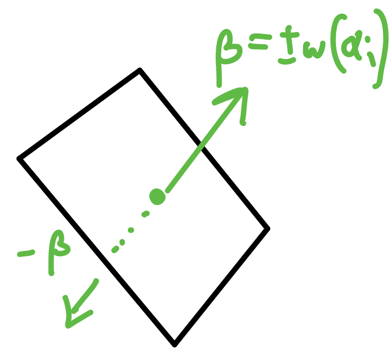
since one can check both of ws_iw^{-1} and s_β will
- negate $\beta = w(\alpha_i)$
- pointwise fix $\beta^\perp$

So this gives a bijection

$$\underset{\text{reflections}}{T} \longleftrightarrow \underset{\text{positive roots}}{\Phi^+}$$

$$s_\beta \longleftarrow\!\shortmid \beta$$

$$ws_iw^{-1} \longmapsto \mathbb{R}w(\alpha_i) \cap \Phi^+$$



Another important property of length:

PROPOSITION: $\forall \beta \in \Phi^+$

$$\ell(ws_\beta) > \ell(w) \iff w(\beta) \in \Phi^+$$

$$\ell(ws_\beta) < \ell(w) \iff w(\beta) \in \Phi^-$$

and hence $\ell(w) = \# N(w) = \# \overbrace{\{t \in T : \ell(wt) < \ell(w)\}}^{\text{Björner–Brenti call this set } T_R(w) \text{ "right associated refns to } w\text{"}}$

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proof: Induct on $\ell(w)$, with base case $\ell(w)=0$ easy.

Enough to show $\ell(ws_\beta) > \ell(w) \Rightarrow w(\beta) \in \Phi^+$, by usual dichotomy.

If $\ell(ws_\beta) > \ell(w)$, pick s_i with $\ell(s_iw) < \ell(w)$.

Then $\ell(s_iws_\beta) \geq \ell(ws_\beta) - 1 > \ell(w) - 1 = \ell(s_iw)$

$\underset{\text{induction applied to } s_iw}{\Longrightarrow}$ $s_iw(\beta) \in \Phi^+$. If $w(\beta) \in \Phi^-$, this implies $w(\beta) = -\alpha_i$

$$\Rightarrow ws_\beta w^{-1} = s_i$$

$$\Rightarrow ws_\beta = s_iw$$

contradicting $\ell(ws_\beta) > \ell(w) > \ell(s_iw)$ ∎

PROPOSITION (The strong exchange condition): Let $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ and $t \in T$.
(not necessarily reduced)
:

(i) $\ell(wt) < \ell(w)$ (i.e. $t \in T_R(w)$) $\Longrightarrow wt = s_{i_1}s_{i_2}\cdots \widehat{s_{i_j}} \cdots s_{i_\ell}$ (omit s_{i_j})

$\overset{(*)}{\Longleftrightarrow} t = s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_j}\cdots s_{i_{\ell-1}}s_{i_\ell}$

for some $j = 1, 2, \ldots, \ell$

(ii) If $\ell = \ell(w)$, so $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is reduced, the index j is unique, and hence

$$T_R(w) = \{t \in T : \ell(wt) < \ell(w)\}$$
$$= \{\underbrace{s_{i_\ell},\ s_{i_\ell}s_{i_{\ell-1}}s_{i_\ell},\ s_{i_\ell}s_{i_{\ell-1}}s_{i_{\ell-2}}s_{i_{\ell-1}}s_{i_\ell}, \ldots}_{\ell \text{ of these}}\}$$

Proof:
(i): First note when $t = s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_j}\cdots s_{i_{\ell-1}}s_{i_\ell}$ that

$$wt = s_{i_1}s_{i_2}\cdots s_{i_j}\cdots s_{i_\ell} \cdot s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_j}\cdots s_{i_{\ell-1}}s_{i_\ell}$$
$$= s_{i_1}s_{i_2}\cdots s_{i_{j-1}}\widehat{s_{i_j}}s_{i_{j+1}}\cdots s_{i_{\ell-1}}s_{i_\ell} \quad \text{as claimed.}$$

So the $(\overset{(*)}{\Longleftrightarrow})$ holds.

For 1st right implication ($\Rightarrow$): assume $\ell(wt) < \ell(w)$ and write $t = ws_iw^{-1} = s_\beta$ for some $\beta = w(\alpha_i) \in \Phi^+$.

We know $w(\beta) \in \Phi^-$, so find rightmost (largest) j with

$$\beta \in \Phi^+$$

$$s_{i_\ell}(\beta) \in \Phi^+$$

$$s_{i_{\ell-1}} s_{i_\ell}(\beta) \in \Phi^+$$

$$\vdots$$

$$s_{i_{j+1}} \cdots s_{i_{\ell-1}} s_{i_\ell}(\beta) \in \Phi^+$$

$$s_{i_j} s_{i_{j+1}} \cdots s_{i_{\ell-1}} s_{i_\ell}(\beta) \in \Phi^-$$

Since $s_{i_j}(\Phi^+ \setminus \{\alpha_{i_j}\}) = \Phi^+ \setminus \{\alpha_{i_j}\}$, $s_{i_{j+1}} \cdots s_{i_{\ell-1}} s_{i_\ell}(\beta) = \alpha_{i_j}$

$$\Rightarrow \beta = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{j+1}}(\alpha_{i_j})$$

and $t = s_\beta = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{j+1}} s_{i_j} s_{i_{j+1}} \cdots s_{i_{\ell-1}} s_{i_\ell}$.

(ii): When $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is reduced, we can't have

$t = \quad s_{i_\ell} \cdots s_{i_j} \cdots s_{i_\ell}$

and $t = s_{i_\ell} \cdots\cdots s_{i_{j'}} \cdots s_{i_\ell}$ with $j' < j$

else $w = wtt = s_{i_1} s_{i_2} - s_{i_{j'}} - s_{i_j} - s_{i_\ell} \cdot t \cdot t$

$= s_{i_1} s_{i_2} - \hat{s}_{i_{j'}} - s_{i_j} - s_{i_\ell} \cdot t$

$= s_{i_1} s_{i_2} - \hat{s}_{i_{j'}} - \hat{s}_{i_j} - s_{i_\ell}$, too short! ∎

The special case of strong exchange condition where $t \in S$ has its own name:

COROLLARY: (Weak) exchange condition:

If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $\ell(w s_i) < \ell(w)$ for $s_i \in S$

then $w s_i = s_{i_1} s_{i_2} \cdots \hat{s}_{i_j} \cdots s_{i_\ell}$ for some j.

$\Rightarrow$ **COROLLARY** (Deletion condition):

If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $\ell(w) < \ell$ then

$\exists j' < j$ with $w = s_{i_1} s_{i_2} \cdots \hat{s}_{i_{j'}} \cdots \hat{s}_{i_j} \cdots s_{i_\ell}$

proof: Find left most (smallest) j with

$$\ell(s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j}) < \ell(s_{i_1} s_{i_2} \cdots s_{i_{j-1}})$$

and apply weak exchange to find

$$s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j} = s_{i_1} \cdots \hat{s}_{i_{j'}} \cdots s_{i_{j-1}} \qquad \blacksquare$$

REMARK: Björner & Brenti prove (but we'll skip)...

THEOREM 1.5.1 For a group W generated by involutions S, T.F.A.E. ← "the following are equivalent"

(i) (W,S) is a Coxeter system, i.e. $W = \langle S \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$

(ii) (W,S) satisfies (weak) exchange condition

(iii) (W,S) satisfies deletion condition

REMARK: $w \longleftrightarrow w^{-1}$ gives left-handed versions of all the previous definitions/results

"left-associated ref'ns to w"

e.g. $T_L(w) := \{t \in T : \ell(tw) < \ell(w)\}$

$= \{s_\beta : \beta \in \Phi^+, \ w^{-1}(\beta) \in \Phi^-\}$

$= \{\underbrace{s_{i_1}, \ s_{i_1}s_{i_2}s_{i_1}, \ s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1}, \ldots}_{\ell = \ell(w) \text{ of these}}\}$

if $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ reduced

EXAMPLE:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \in \mathfrak{S}_4 = W(\underset{s_1}{\circ} \overset{3}{-} \underset{s_2}{\circ} \overset{3}{-} \underset{s_3}{\circ})$$

$s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$

$T_R(w) = \{(ij) : i<j, \ w_i > w_j\}$ = inversion pair positions

$= \{(12), (13), (14), (34)\}$

$T_L(w) = \{(w_i w_j) : i<j, \ w_i > w_j\}$ = inversion pair values

$= \{(14), (34), (24), (23)\}$

Words $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ come from bubble-sorting w to 1:

$w = \begin{pmatrix} 1&2&3&4 \\ 4&1&3&2 \end{pmatrix}$

$4132 \xrightarrow{s_3} 4123 \xrightarrow{s_1} 1423 \xrightarrow{s_2} 1243 \xrightarrow{s_3} 1234$

$\Rightarrow w s_3 s_1 s_2 s_3 = 1$

$\Rightarrow w = s_3 s_2 s_1 s_3$

reduced, since

$\ell(w) = \mathrm{inv}(w) = 4$

$w = \begin{pmatrix} 1&2&3&4 \\ 4&1&3&2 \end{pmatrix}$

$4132 \xrightarrow{s_2} 4312 \xrightarrow{s_1} 3412 \xrightarrow{s_2} 3142 \xrightarrow{s_3} 3124 \xrightarrow{s_1} 1324 \xrightarrow{s_2} 1234$

$\Rightarrow w = s_2 s_1 s_3 s_2 s_1 s_2$

not reduced

$= s_2 s_3 s_1 s_2 s_1 s_2$

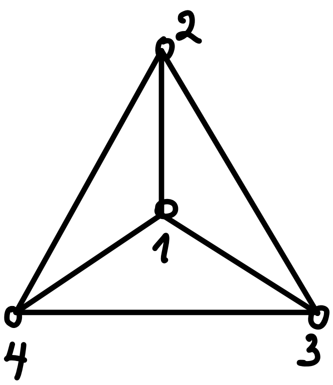
$= s_2 s_3 s_1 s_1 s_2 s_1$

$= s_2 s_3 s_2 s_1$

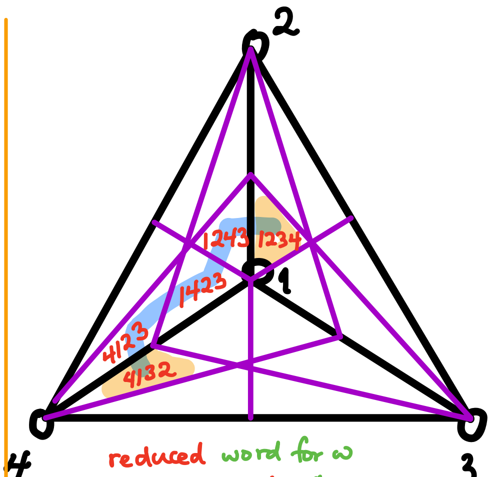
$$\Rightarrow T_R(w) = \{ \underset{(34)}{s_3},\ \underset{(12)}{s_3 s_1 s_3},\ \underset{(14)}{s_3 s_1 s_2 s_1 s_3},\ \underset{(13)}{s_3 s_1 s_2 s_3 s_2 s_1 s_3} \}$$

$$T_L(w) = \{ \underset{(34)}{s_3},\ \underset{(24)}{s_3 s_2 s_3},\ \underset{(14)}{s_3 s_2 s_1 s_2 s_3},\ \underset{(23)}{s_3 s_2 s_1 s_3 s_1 s_2 s_3} \}$$

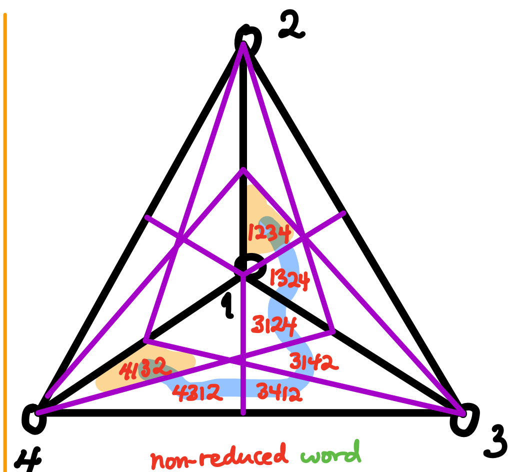
$\longleftrightarrow$ hyperplanes $\alpha^\perp = \{x_i = x_j\}$ separating 1 from w



regular tetrahedron $= \mathrm{conv}(e_1, e_2, e_3, e_4)$ in $\mathbb{R}^4$



reduced word for w $\Leftrightarrow$ minimal gallery from 1 to w



non-reduced word $\Leftrightarrow$ non-minimal gallery from 1 to w

REMARKS

1. (on Tits' solution to the word problem for (W,S))

We just saw that in $\mathfrak{S}_4 = W\big(\underset{s_1}{\circ} \overset{3}{—} \underset{s_2}{\circ} \overset{3}{—} \underset{s_3}{\circ}\big)$,

$$\begin{aligned}
s_3 s_2 s_1 s_3 &= s_2 \underbrace{s_1 s_3}\, s_2 s_1 s_2 \\
&= s_2 s_3 s_1 \underbrace{s_2 s_1 s_2} \\
&= s_2 s_3 \underbrace{s_1 s_1}\, s_2 s_1 \\
&= \underbrace{s_2 s_3 s_2}\, s_1 \\
&= s_3 s_2 \underbrace{s_3 s_1} \\
&= s_3 s_2 s_1 s_3
\end{aligned}$$

braid moves $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$

"nil-move" $s_i^2 \to 1$

This illustrates a result we'll prove later:

THEOREM (Björner–Brenti THM 3.3.1)

In a Coxeter system (W,S),

(i) every word $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ can be brought to a reduced word by a sequence of braid moves and nil-moves (in particular, never need to make it longer)

(ii) any two reduced words for w can be connected by a sequence of braid moves.

2. Björner & Brenti manage to prove lots of things early on avoiding the geom. rep'n $W \xrightarrow{\sigma} GL(V)$ until Chapter 4 !

How? They define in §1.3 a cooked-up version of the (faithful!) permutation rep'n of W acting on

$$\Phi = \Phi^+ \sqcup \Phi^-$$

$$\updownarrow \qquad \updownarrow$$

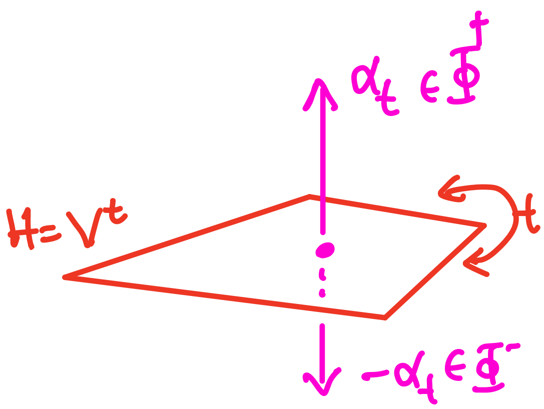
$$T \times \{\pm 1\} \overset{\text{DEF'N}}{=:} R = \{(t,+1) : t \in T\} \sqcup \{(t,-1) : t \in T\}$$

$$= \{(t, \pm 1) : t \in T\}$$

where $T \underset{\text{DEF'N}}{:=} \{wsw^{-1} : s \in S, w \in W\}$

They show a homomorphism $W \xrightarrow{\pi} \mathfrak{S}_R$ can be defined sending $s_i \longmapsto \pi_i$

where $\pi_i(t, \pm 1) = \begin{cases} (s_i t s_i, \pm 1) & \text{if } s_i \neq t \\ (s_i t s_i, \mp 1) & \text{if } s_i = t \end{cases}$



(mimicking
$s_i(\Phi^+ \setminus \{\alpha_i\}) = \Phi^+ \setminus \{\alpha_i\}$
$s_i(\alpha_i) = -\alpha_i \in \Phi^+$)