Math 4707
The Matching Theorem

Dennis White

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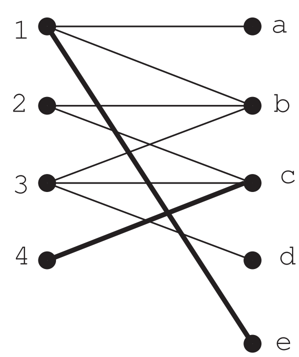
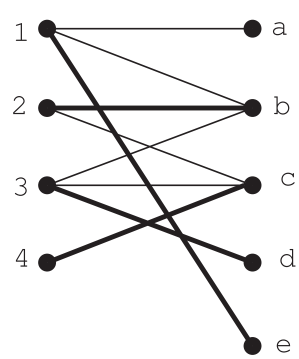
A *matching* in a bipartite graph is a subset of the edges with no common vertices. Alternatively, it is a 1-regular subgraph. Here is an example of a matching having two edges:



A complete match from X to Y in a bipartite graph with vertex bipartition (X, Y) is a matching containing every vertex in X. Here is an example of a complete match:



Suppose G is a bipartite graph with bipartition (X, Y). Suppose $A \subseteq X$. Let $N_G(A)$ denote the *neighbors* of A in G, that is, all the vertices in Y which are adjacent to at least one vertex in A. For example, in the bipartite graph above, if $A = \{2, 4\}$, then $N_G(A) = \{b, c\}$.

The Matching Theorem gives a simple condition which tells exactly when there exists a complete match in a bipartite graph.

Theorem 1. *Suppose G is a bipartite graph with vertex bipartition (X, Y). There is a complete match from X to Y if and only if for every $A \subseteq X$, $|A| \leq |N_G(A)|$.*

The condition that for every $A \subseteq X$, $|A| \leq |N_G(A)|$ is called the *matching condition.*

Proof. Suppose there is a complete match from X to Y and suppose $A \subseteq X$. Then the complete match identifies each vertex in A with a unique vertex in $N_G(A)$, so that the number of vertices in $N_G(A)$ is at least as great as the number of vertices in A.

The converse is more difficult. We must show that if the matching condition is satisfied, then there is a complete match. The proof is by induction on the number of vertices in X. Suppose $|X| = 1$, that is, $X = \{v\}$. If the matching condition is satisfied, then v is adjacent to at least once vertex in Y, and the edge to such a vertex gives the complete match.

Now suppose that every bipartite graph H with bipartition (U, V) and with $1 \leq |U| < n$ which satisfies the matching condition has a complete match from U to V. Let G be a bipartite graph with bipartition (X, Y), with $|X| = n$, which satisfies the matching condition. We must show that there is a complete match from X to Y.

We consider two cases. The first case is that for every $\emptyset \subset A \subset X$, $|A| < |N_G(A)|$. That is, for non-empty proper subset A, not only is the matching condition satisfied, but it is satisfied *strictly.*

The second case is that there is some A in X, which is neither $\emptyset$ nor X, such that $|A| = |N_G(A)|$. Note that exactly one of these two cases must occur.

In the first case, pick any vertex $v \in X$. The matching condition implies that v is adjacent to at least one vertex $w \in Y$. Set the edge (v, w) aside and remove v from X and w from Y to form a new bipartite graph H with bipartition $(X - v, Y - w)$. Let $\emptyset \subset A \subseteq X - v$. Since A is a proper subset of X, we know that $|A| < |N_G(A)|$. If $N_G(A)$ did not include w, then $N_H(A) = N_G(A)$ and so $|A| < |N_H(A)|$. If $N_G(A)$ did include w, then $N_H(A) = N_G(A) - \{w\}$, so $|A| < |N_G(A)|$ implies $|A| \leq |N_H(A)|$.

Therefore the matching condition is satisfied in H. Since $X - v$ has one fewer vertex than X, the inductive hypothesis implies that there is a complete match from $X - v$ to $Y - w$. That match together with the edge (v, w) gives a complete match from X to Y.

Now suppose there is $A \subset X$, $A \neq \emptyset$, such that $|A| = |N_G(A)|$. We construct the match from X to Y by first matching A to $N_G(A)$, then matching $X - A$ to $Y - N_G(A)$.

Let H be the bipartite graph induced by the bipartition $(A, N_G(A))$. Pick $B \subseteq A$. Since $N_G(B) \subseteq N_G(A)$, it follows that $N_H(B) = N_G(B)$, and so the matching condition is satisfied in H because it is satisfied in G. Furthermore, $1 \leq |A| \leq n-1$, so that by induction there is a complete match M from A to $N_G(A)$.

Now let K be the bipartite graph induced by the bipartition $(X - A, Y - N_G(A))$. Let $B \subseteq X - A$. We must have $N_G(A \cup B) = N_G(A) \cup N_K(B)$ since there are no edges between vertices in A and vertices in $Y - N_G(A)$.

But A was chosen so that $|N_G(A)| = |A|$, and the matching condition gives $|A \cup B| \leq |N_G(A \cup B)|$. Therefore

$$\begin{aligned}
|A| + |B| &= |A \cup B| \\
&\leq |N_G(A \cup B)| \\
&= |N_G(A) \cup N_K(B)| \\
&= |N_G(A)| + |N_K(B)| \\
&= |A| + |N_K(B)|
\end{aligned}$$

so that $|B| \leq |N_K(B)|$. That is, the matching condition is satisfied in K. But $1 \leq |X - A| \leq n-1$, so by induction, there is a complete match M' from $X - A$ to $Y - N_G(A)$. Putting M and M' together gives a complete match from X to Y. $\square$

Notice that the first case used weak induction, but the second used strong induction. Also, note that the requirement that A not be either X or $\emptyset$ played a key role in the second case: it guaranteed that induction could be applied both to A and to $X - A$.

An important consequence of the Matching Theorem is the König-Egerváry Theorem. A *vertex cover* in G is a subset of vertices Q such that every edge in G is incident upon at least one vertex in Q. Let $\alpha'(G)$ be the maximum size matching and let $\beta(G)$ be the minimum size vertex cover in G.

Theorem 2. *In a bipartite graph G, the maximum size matching equals the minimum size vertex cover, i. e., $\alpha'(G) = \beta(G)$.*

Proof. Each edge in any match will require at least one vertex in the vertex cover. Therefore $\alpha'(G) \leq \beta(G)$ for any graph (not necessarily bipartite) G. To show equality, we need only produce a single match and a single vertex cover of the same size.

Let Q be the minimum vertex cover of bipartite $G = (X, Y)$. Let H be the bipartite graph $(Q \cap X, Y - Q \cap Y)$ and let K be the bipartite graph $(Q \cap Y, X - Q \cap X)$. Note that there are no edges between $Y - Q \cap Y$ and $X - Q \cap X$. We will construct a complete match from $Q \cap X$ to $Y - Q \cap Y$ in H and from $Q \cap Y$ to $X - Q \cap X$ in K. Putting these two matches together will give a match in G with $|Q|$ edges, thus proving the result.

By symmetry, we need only show one of the complete matches exists. We use the Matching Theorem. Pick $A \subseteq Q \cap X$. Suppose $|A| > |N_H(A)|$. Then $N_H(A)$ can be used instead of A in Q to get a smaller vertex cover, since $N_H(A)$ covers all edges incident to A that are not covered by $Q \cap Y$. Since Q was smallest, this is impossible, and so $|A| \leq |N_H(A)|$. The existence of the complete match then follows from the Matching Theorem. $\square$