# Math 5251 Hamming distance & decoding (§4.4)

If we send codewords $\mathbb{C} \subset \{0,1\}^*$ of length $\ell$
$$\mathbb{C} = \{x_1, ---, x_m\}$$

through a BSC with error probability $p < \frac{1}{2}$

safe to assume; Why?

and receive the word $y$, which word $x_i$ should we decode it as ?

You would think we should pick an $x_i$ that minimizes this...

DEF'N : Given two words in $\Sigma^*$ of same length $\ell$

$x = x^{(1)} x^{(2)} -- x^{(\ell)}$
$y = y^{(1)} y^{(2)} --, x^{(\ell)}$
their Hamming distance is

$$d(x,y) := \# \{ p = 1,2,--,\ell : x^{(p)} \neq y^{(p)} \}$$

EXAMPLES
$d(101,000) = 2$,
$d(102,000) = 2$
$d(101,010) = 3$
$d(101,101) = 0$

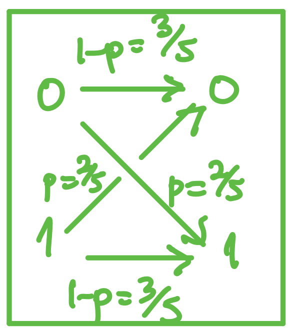
If we don't know much about the source word probabilities, this is a good rule to follow and called maximum likelihood estimation, in that it picks $x_i$ maximizing

$$P\left(\underset{\text{received}}{y} \,\middle|\, \underset{\text{sent}}{x_i}\right) = p^{d(x_i,y)}(1-p)^{\ell-d(x_i,y)}$$

---

EXAMPLE If we send words in $\{0,1\}^*$ of length 2 via repetition code of length $\ell=4$ through a BSC of error prob $p=2/5$

$$C = \begin{array}{ll} x_1 & 0000 \\ x_2 & 0101 \\ x_3 & 1010 \\ x_4 & 1111 \end{array}$$




receiving $y=0111$

then how should we decode?

$$P(y=0111 \text{ rec'd} \mid x_1=0000 \text{ sent}) = \left(\tfrac{2}{5}\right)^3\left(\tfrac{3}{5}\right)^1 = \frac{24}{625}$$

$$\boxed{x_2=0101 \qquad = \left(\tfrac{2}{5}\right)^1\left(\tfrac{3}{5}\right)^3 = \frac{54}{625}}$$

$$x_3=1010 \qquad = \left(\tfrac{2}{5}\right)^3\left(\tfrac{3}{5}\right)^1 = \frac{24}{625}$$

$$\boxed{x_4=1111 \qquad = \left(\tfrac{2}{5}\right)^1\left(\tfrac{3}{5}\right)^3 = \frac{54}{625}}$$

decode $y$ as one of these two

What would be an alternative? If we had more info about source probabilities $p_1, \dots, p_m$, then one could use ideal observer/minimum error rule, maximizing $P\left(\underset{\text{sent}}{x_i} \,\middle|\, \underset{\text{received}}{y}\right)$ via a Bayesian calculation:

$$P\left(\underset{\text{sent}}{x_i} \,\middle|\, \underset{\text{received}}{y}\right) = \frac{P\left(\underset{\text{sent}}{x_i} \cap \underset{\text{received}}{y}\right)}{P\left(\underset{\text{received}}{y}\right)}$$

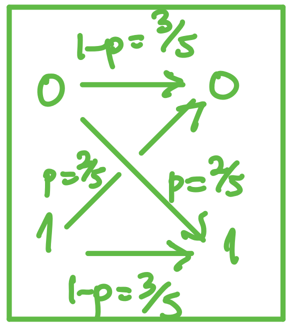
$$= \frac{P\left(\underset{\text{received}}{y} \,\middle|\, \underset{\text{sent}}{x_i}\right) \cdot P\left(\underset{\text{sent}}{x_i}\right)}{P\left(\underset{\text{received}}{y}\right)}$$

$$= \frac{p^{d(y,x_i)}(1-p)^{\ell - d(y,x_i)} \cdot p_i}{P\left(\underset{\text{received}}{y}\right)}$$

Pick $x_i$ maximizing this numerator

**EXAMPLE** What if in previous example we had these source probabilities?:

| | $p_i$ | |
|---|---|---|
| "OK" | $1/2$ | $x_1 = 0000$ |
| various distress signals? | $1/6$ | $x_2 = 0101$ |
| | $1/6$ | $x_3 = 1010$ |
| | $1/6$ | $x_4 = 1111$ |




receiving $y = 0111$

Then the ideal observer maximizes the numerator in

$$P\left(\begin{matrix} x_i \\ \text{sent}\end{matrix} \,\middle|\, \begin{matrix} 0111 \\ \text{received}\end{matrix}\right) = \frac{\left(\frac{2}{5}\right)^{d(x_i,0110)} \cdot \left(\frac{3}{5}\right)^{4-d(x_i,0110)} \cdot p_i}{P\left(\begin{matrix} 0111 \\ \text{received}\end{matrix}\right)}$$

This numerator is

$$\left(\tfrac{2}{5}\right)^3\left(\tfrac{3}{5}\right)^1 \cdot \tfrac{1}{2} = \boxed{\tfrac{12}{625} \quad \text{for } x_1 = 0000}$$

$$\left(\tfrac{2}{5}\right)^1\left(\tfrac{3}{5}\right)^3 \cdot \tfrac{1}{6} = \tfrac{9}{625} \quad \text{for } x_2 = 0101$$

$$\left(\tfrac{2}{5}\right)^3\left(\tfrac{3}{5}\right)^1 \cdot \tfrac{1}{6} = \tfrac{4}{625} \quad \text{for } x_3 = 1010$$

$$\left(\tfrac{2}{5}\right)^1\left(\tfrac{3}{5}\right)^3 \cdot \tfrac{1}{6} = \tfrac{9}{625} \quad \text{for } x_4 = 1111$$

The ideal observer decoding

# Channel capacity & Shannon's Noisy Coding Theorem (§4.4, 4.5)

As with avglength($f$) for u.d. codes, is there a limit on the rate $\dfrac{\log_2(|\mathcal{C}|)}{\ell}$ of a choice of length $\ell$ binary codewords $\mathcal{C} \subset \{0,1\}^*$ being sent through a noisy channel if we would like the probability of undetected error to be made arbitrarily small?

EXAMPLE

Note that if we began with codewords being all $2^\ell$ words $w_i$ of length $\ell$ in $\{0,1\}^*$, and encoding them with the r-fold repetition code

$$\mathcal{C} = \{w_1 w_2 \cdots w_r\} \subset \{0,1\}^* \text{ with words of length } r\ell$$

then the max error probability $\longrightarrow 0$ as $r \to \infty$

but also

$$\text{rate}(\mathcal{C}) = \frac{\log_2(|\mathcal{C}|)}{r\ell} = \frac{\log_2(2^\ell)}{r\ell} = \frac{1}{r} \longrightarrow 0 \text{ as } r \to \infty.$$

**Q**: Can we do better with the rate, still having error probability $\to 0$?

Yes, and how much better again relates to quantifying information/entropy of the source $X = \{x_1, \ldots, x_m\}$, probs $p_1, \ldots, p_m$

and the new source $Y = \{y_1, \ldots, y_n\}$, that has probabilities calculable from the channel probabilities matrix $p_{ij} := P(\underset{\text{rec'd}}{y_j} \mid \underset{\text{sent}}{x_i})$

$$\begin{array}{cc} p_1 & x_1 \\ \vdots & \vdots \\ p_m & x_m \end{array} \rightsquigarrow \boxed{\begin{array}{c} \text{channel} \\ @ \\ (p_{ij}) \end{array}} \rightsquigarrow \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}$$

$$P(\underset{\text{rec'd}}{y_j}) = \sum_{i=1}^{m} p_{ij} x_i$$

For each event $\{y_j \text{ received}\}$, one can calculate $P\left(\begin{smallmatrix} x_i \\ \text{sent} \end{smallmatrix} \middle| \begin{smallmatrix} y_j \\ \text{received} \end{smallmatrix}\right)$ for $j = 1, 2, \ldots, n$ and use it to define ...

---

DEF'N: The conditional entropies

$$H\left(X \middle| \begin{smallmatrix} y_j \\ \text{received} \end{smallmatrix}\right) := \sum_{i=1}^{m} P\left(\begin{smallmatrix} x_i \\ \text{sent} \end{smallmatrix} \middle| \begin{smallmatrix} y_j \\ \text{rec'd} \end{smallmatrix}\right) \log_2\left( \frac{1}{P\left(\begin{smallmatrix} x_i \\ \text{sent} \end{smallmatrix} \middle| \begin{smallmatrix} y_j \\ \text{rec'd} \end{smallmatrix}\right)} \right)$$

and then the entropy of X given Y,

$$H(X|Y) = \sum_{j=1}^{n} P\left(\begin{smallmatrix} y_j \\ \text{rec'd} \end{smallmatrix}\right) H\left(X \middle| \begin{smallmatrix} y_j \\ \text{rec'd} \end{smallmatrix}\right)$$

and finally the information about X given by Y

$$I(X|Y) = H(X) - H(X|Y).$$

---

That is, we expect that despite the noise, knowing Y should decrease our surprise about X, by $I(X|Y)$ bits.

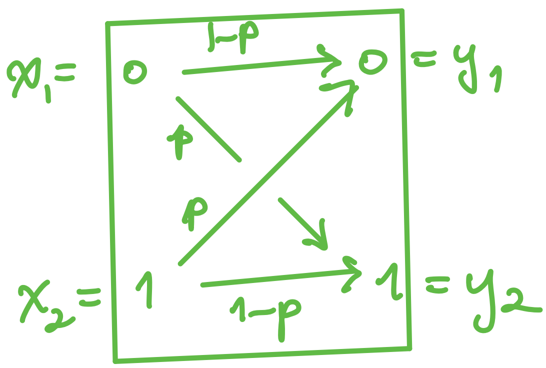
Finally, we can define...

DEF'N: The **channel capacity** of $C$

$$\text{capacity}(C) := \max \left\{ I(X|Y) : \begin{array}{l} \text{source probabilities} \\ p_1, \ldots, p_m \text{ for} \\ X = \{x_1, \ldots, x_m\} \end{array} \right\}$$

---

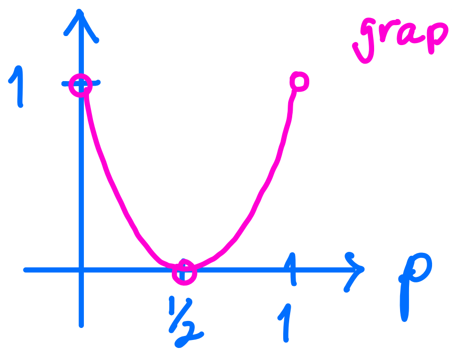
EXAMPLE Garrett calculates with some easy calculus that the BSC with error prob $p$




has

$$\text{capacity}(C) = 1 + p\log_2(p) + (1-p)\log_2(1-p)$$

(and max of $I(X|Y)$ is achieved for $P(x_1) = P(x_2) = \frac{1}{2}$, regardless of the BSC error probability $p$)

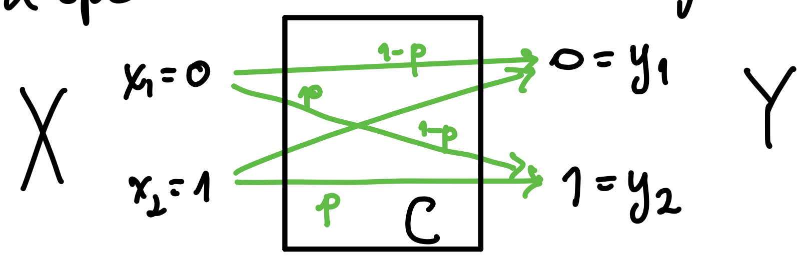



graph of $y = \text{capacity}(\text{BSC with error prob } p)$
$= 1 + p\log_2(p) + (1-p)\log_2(1-p)$

---

**EXAMPLE** Note BSC with error prob $p = 1/2$ has capacity $1 + \frac{1}{2}\log_2\left(\frac{1}{2}\right) + \frac{1}{2}\log_2\left(\frac{1}{2}\right) = 1 - \frac{1}{2} - \frac{1}{2} = 0$

It's a special case of this family of useless channels:




where one can check that for any choice of source probabilities $X = \{x_1, x_2\}$, one has
probs: $p_1$ $p_2$

$X, Y$ independent $\Rightarrow H\left(X \middle| \begin{smallmatrix} y_j \\ \text{occurs} \end{smallmatrix}\right) = H(X) \quad \forall y_j \in Y$

$\Rightarrow H(X|Y) = H(X)$

i.e. $I(X|Y) = 0$

$\Rightarrow \text{capacity}(C) = 0.$

No way to detect errors, even with long repetition codes and very low rates!

## Shannon's Noisy Coding Thm (§4.5)

Let $C$ be a memoryless channel, and pick any $R$ in the range $0 < R < \text{capacity}(\mathbb{C})$.

Then one can find a sequence of codes

$$\mathcal{C}_n \subset \{0,1\}^* \text{ for } n = 1, 2, 3, \ldots$$

with
- $\mathcal{C}_n$ consists of words of length $n$
- $\text{rate}(\mathcal{C}_n) \to R$ as $n \to \infty$
- using max likelihood (= min Hamming distance) decoding, the max probability of a word in $\mathcal{C}_n$ being decoded wrong $\to 0$ as $n \to \infty$.

Roman §3.4.4 states it, but both he and Garrett prove it only for the BSC with error probability $p$.

---

An interesting feature of the proof is, using fairly easy probabilistic estimates, one can pick $\mathcal{C}_n$ with high probability to be

$2^{\lfloor R \cdot n \rfloor}$ randomly chosen (!) words of length $n$

$$\left(\text{so } \operatorname{rate}(\mathcal{C}_n) = \frac{\lfloor R \cdot n \rfloor}{n} \longrightarrow R \text{ as } n \to \infty\right)$$

---

DRAWBACK: Efficiently doing minimum-distance decoding with $\mathcal{C}$ chosen randomly is hard.