Math 5707 Spring 2023 Brooks's Theorem

(Bondy-Murty §8.2)

We saw how the greedy coloring algorithm shows

$$\underbrace{\chi(G)}_{\substack{\text{chromatic}\\ \#\text{ of } G}} \leq 1 + \underbrace{\Delta(G)}_{\substack{\text{max}\\ \text{vertex degree}\\ \text{in } G}}$$

THEOREM (Brooks 1941) For a connected simple graph G, unless $G = \begin{cases} C_n & n\text{-cycle with } n \text{ odd} \\ \text{or} & \\ K_n & \text{complete graph} \end{cases}$

one has $\chi(G) \leq \Delta(G)$.

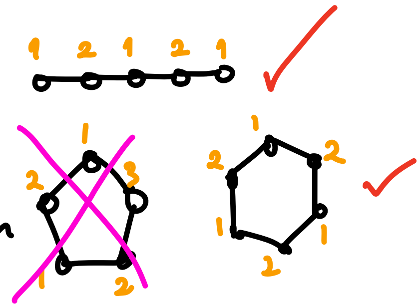
proof: Assuming $G \neq K_n$ or C_n for n odd, we'll show $\chi(G) \leq \Delta(G)$ by induction on $n = |V|$.

The base case where $n=1$ is easy: $G=\cancel{K_1}$

In the inductive step, we can also quickly deal with the cases where

$\Delta(G)=1 \Rightarrow G=\cancel{K_2}$

$\Delta(G)=2 \Rightarrow G=\begin{cases}\text{path } P_n \\ \text{or} \\ \text{cycle } C_n\end{cases}$



So without loss of generality, $\Delta(G)\geq 3$ in the inductive step.

We'll consider 3 cases:

CASE 1: G has a cut-vertex x.

CASE 2: G has no cut-vertex, but does have a pair $x,y\in V$ with no edge $\{x,y\}$ such that $G-\{x\}-\{y\}$ is disconnected

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CASE 3: $G-\{x\}-\{y\}$ is connected for all $x,y\in V$ with no edge $\{x,y\}$

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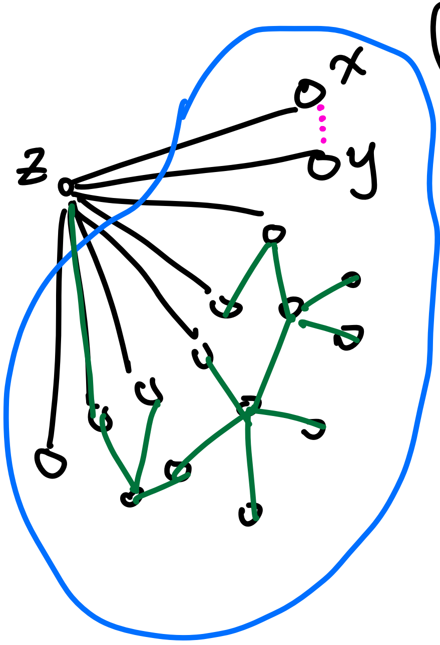
and deal with them in this order: CASE 3, CASE 1, CASE 2.

CASE 3: $G - \{x\} - \{y\}$ is connected for all $x, y \in V$ with no edge $\{x,y\}$

Pick $z \in V$ achieving $\deg_G(z) = \Delta(G)$.
Then pick any 2 neighbors x, y of z in G such that $\{x,y\} \notin E(G)$.



(We know such a pair exists, else $z \cup \{\text{its neighbors}\}$ gives a $K_{\Delta(G)+1}$ as a subgraph of G, but then it must be all of G since $\deg_G(z) = \Delta(G)$.)

Now color G greedily using the order

$$x_1, x_2, \ldots x_j \ldots \mid x_{i+1} \ldots\ldots x_{n-1}, x_n$$

$x_1 = x$, $x_2 = y$, $x_n = z$

final segments of vertices always have $G[x_{i+1}, \ldots, x_{n-1}, x_n]$ connected.

Then $f(\underset{x}{x_1}) = 1$
$f(\underset{y}{x_2}) = 1$

We also have $f(x_j) \in \{1, 2, \dots, \Delta(G)\}$

for $j = 3, 4, \dots, x_{n-1}$

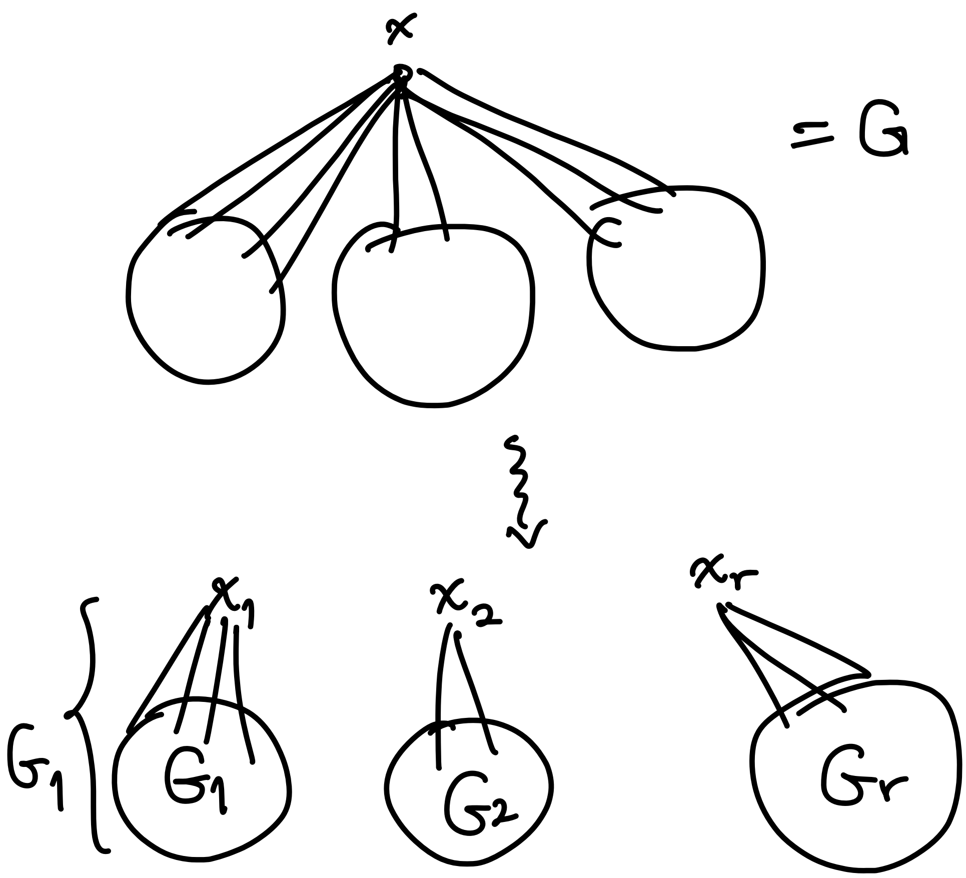
because x_j has at least one neighbor among $x_{j+1}, x_{j+2}, \dots, x_n$ (since $G[x_j, x_{j+1}, \dots, x_n]$ is connected)

hence the neighbors of x_j among $x_1, \dots, x_{j-1}$ use at most $\Delta(G) - 1$ colors.

Finally z only needs $\Delta(G)$ colors, since two of its neighbors (x & y) use the same color.

CASE 3 proved. ∎

CASE 1: G has a cut-vertex x.

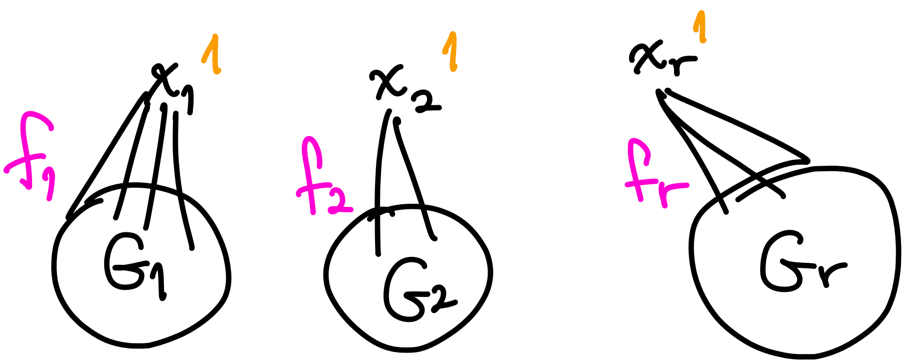


Each of the blocks G_i has a $\Delta(G)$-coloring by induction on # vertices.

(check that the blocks G_i can't be odd cycles, and even if a block G_i happens to be a complete graph K_s, one can check that $s < \Delta(G)+1$ i.e. $s \le \Delta(G)$, so needs at most s colors

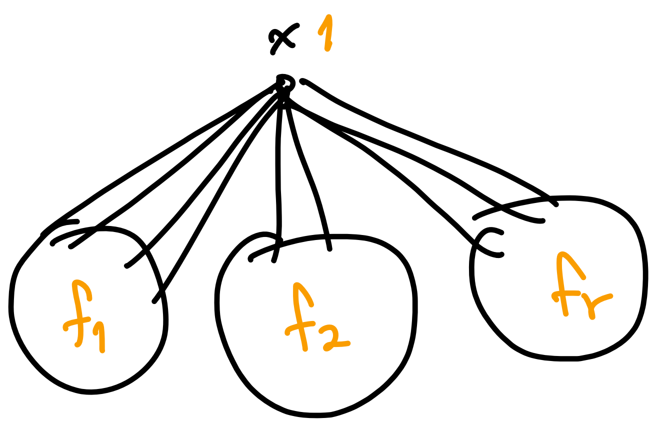
$s \le \Delta(G)$.)

Given the proper $\Delta(G)$-vertex colorings f_i for each block G_i,



one can always re-name the colors within a connected component G_i to make $f_i(x_i)=1$ for $i=1,2,\dots,r$.

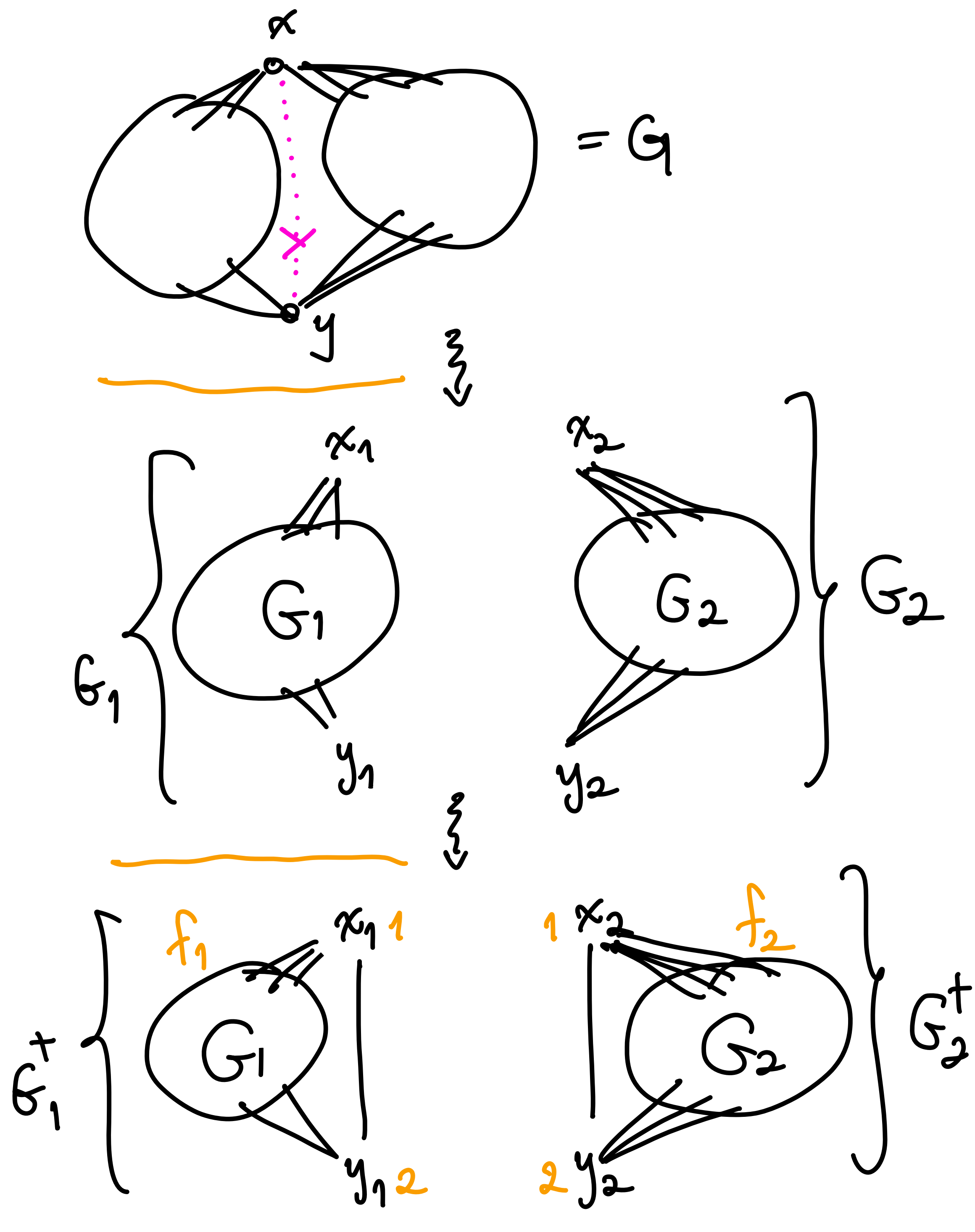
Then we can glue the colorings to get a $\Delta(G)$-coloring of G



$= G$

End of case 1 $\blacksquare$

CASE 2: G has no cut-vertex, but does have a pair $x, y \in V$ with no edge $\{x,y\}$ such that $G - \{x\} - \{y\}$ is disconnected

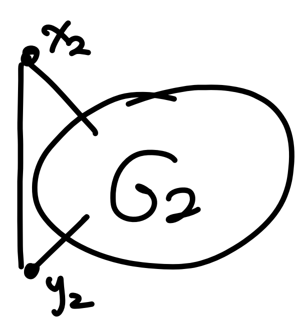


G_1^+, G_2^+ have $\Delta(G)$-colorings by induction,
and also $f_1(x_1)\neq f_1(y_1)$
and $f_2(x_2)\neq f_2(y_2)$
allowing them to be glued,
UNLESS one of G_1^+ or G_2^+ or both
is a complete graph $K_{\Delta(G)+1}$
(they can't be odd cycles, else we were
in the $\Delta(G)=2$ case for G, not $\Delta(G)\geq 3$).

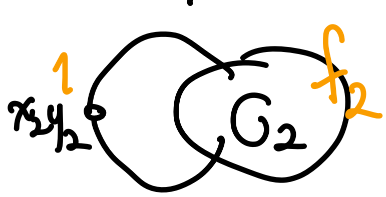
If that happens, say $G_1^+\cong K_{\Delta(G)+1}$

then $\deg_{G_2}(x_2)=1=\deg_{G_2}(y_2)$.

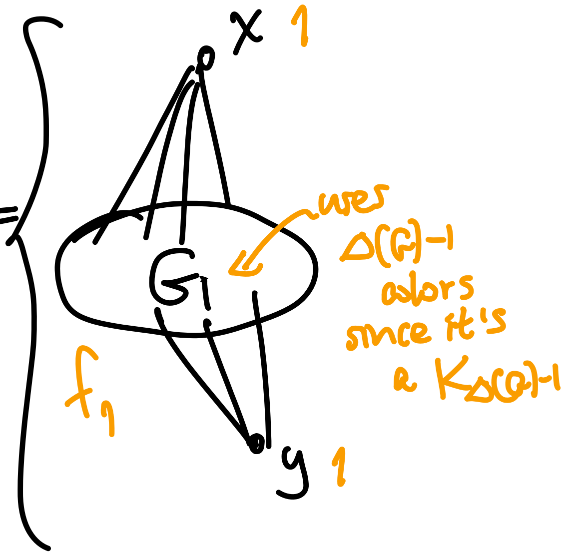
In this case, we form



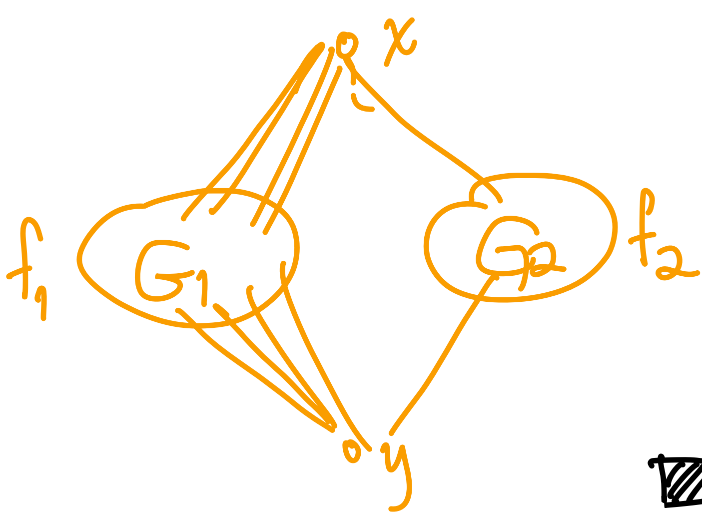
$G_2^+/\{x_2,y_2\}$

$=$ 

Then $G_1 =$



glue f_1, f_2 $\rightsquigarrow$



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