Math 5707 Spring 2023

Matching theory Snippet 1: P. Hall's Matching Theorem ("Marriage")

REVIEW of matching theory so far...

$G=(V,E)$ simple graph

$\nu(G) := \max\{|M| : M \subset E \text{ a matching}\}$

$\leq \tau(G) := \min\{|W| : W \subset V \text{ a vertex cover}\}$

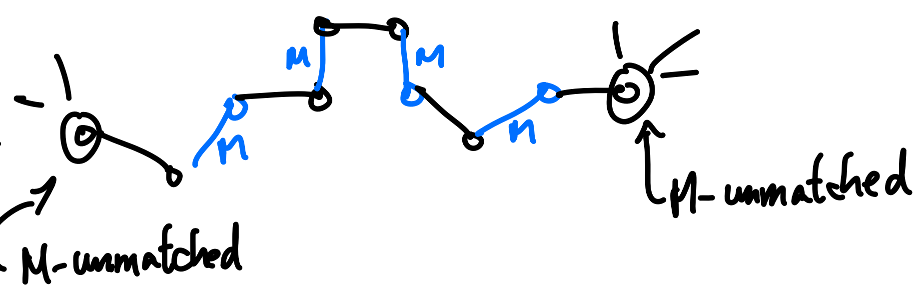
part of Gallai's Thm.

PROPOSITION

A matching $M \subset E$ is max-sized

$\Longleftrightarrow$

G contains no M-augmenting paths P

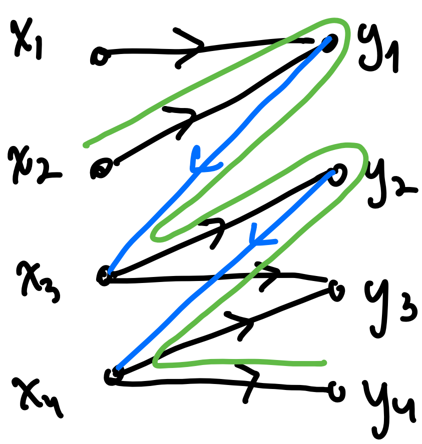
$P=$ 

PROPOSITION: In a bipartite graph $G = (V, E)$, $V = X \sqcup
Y$

$$\left\{ \begin{array}{c} M\text{-augmenting} \\ \text{paths } P \end{array} \right\} = \left\{ \begin{array}{l} \text{directed paths } P \\ \text{from } X\text{-unmatched } x \in X \\ \text{to } Y\text{-unmatched } y \in Y \\ \text{in this digraph } D: \end{array} \right.$$

$x \rightarrow y$ non-M edges

$x \leftarrow y$ M edges

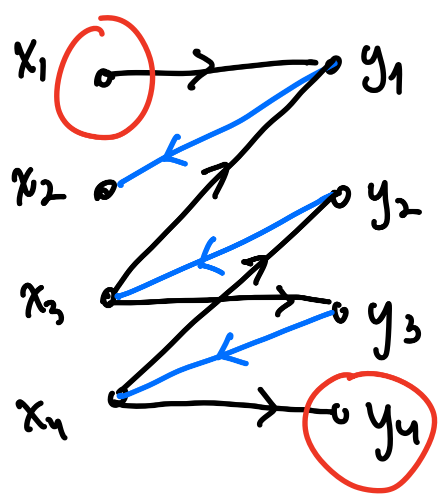


G

M

P

augment along P



no directed paths from X-unmatched x's to Y-unmatched y's

so max-sized

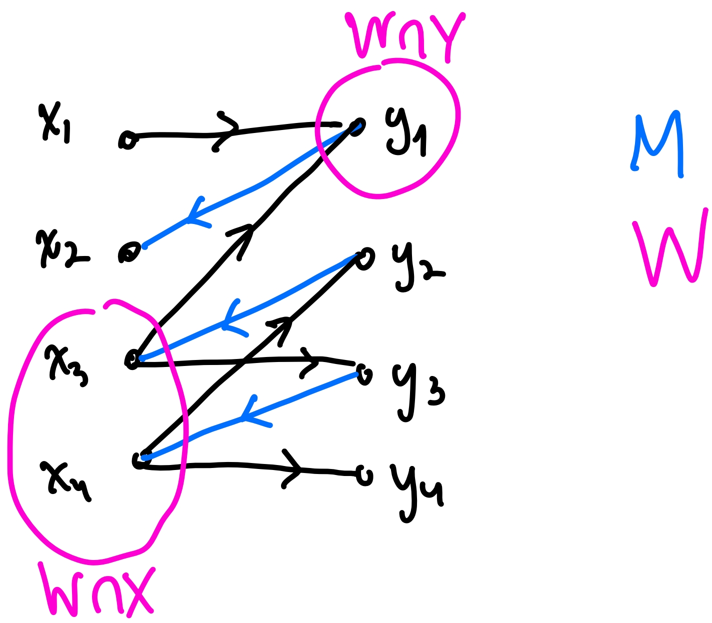
This gives the so-called Hungarian algorithm to find $\nu(G)$ and max-sized matchings M

COROLLARY (König-Egerváry 1931) For G bipartite,

$$\underbrace{\nu(G)}_{\substack{\text{max-size of} \\ \text{a matching } M}} = \underbrace{\tau(G)}_{\substack{\text{min-size of} \\ \text{a vertex cover } W}}$$

In fact, at the end of the Hungarian algorithm one finds a vertex cover W with $|W| = |M|$ same size as the max-sized matching M, by letting

$$W := \left\{ \begin{array}{c} x \in X \text{ not reachable in } D \\ \text{from the } M\text{-unmatched} \\ X\text{-vertices} \end{array} \right\} \sqcup \left\{ \begin{array}{c} y \in Y \text{ reachable in } D \\ \text{from the } M\text{-unmatched} \\ X\text{-vertices} \end{array} \right\}$$



Another corollary...

COROLLARY (P. Hall's Matching Thm.): 1935 "Marriage"

A bipartite graph $G=(X \sqcup Y, E)$ has a matching M that matches all of X

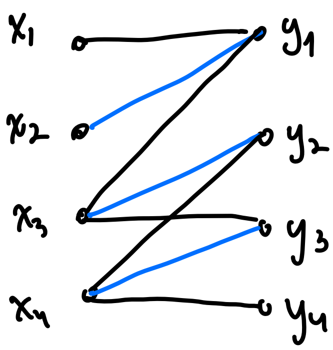
$\iff$ $\forall$ subsets $X' \subseteq X$ one has

$N(X') := \{y \in Y : \exists \text{ some } x \in X' \text{ with } \{x,y\} \in E\}$

neighbors of X'

of size $|N(X')| \geq |X'|$.

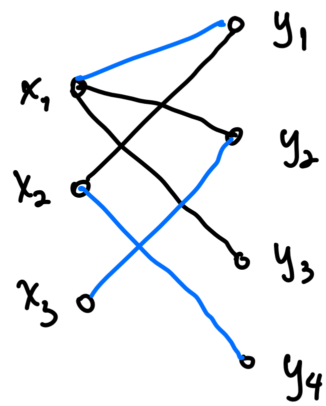
EXAMPLE



M of max size does not match all of X;

$X' = \{x_1, x_2\}$ has

$N(X') = \{y_1\}$ too small



M matches all of X

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proof: ($\Longrightarrow$) is pretty easy to see, since if we had a matching M that matched all of X, then for every subset $X' \subset X$, the matching M gives an injective map

$$\begin{aligned} X' &\hookrightarrow N(X') \\ x &\longmapsto \text{its match } y \end{aligned}$$

so $|X'| \leq |N(X')|$.

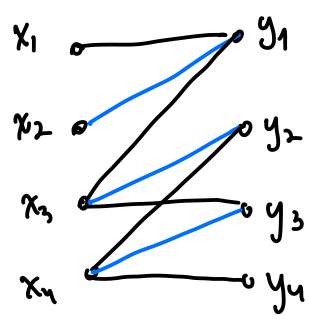
For ($\Longleftarrow$), assume there is no matching M that matches all of X. So

$\nu(G) < |X|$

$\| \leftarrow$ by König-Egerváry Thm,

$\tau(G)$

so $\exists$ a vertex cover W of size $|W| < |X|$.



M of max size does not match all of X

$X' = \{x_1, x_2\}$ has $N(X') = \{y_1\}$ too small

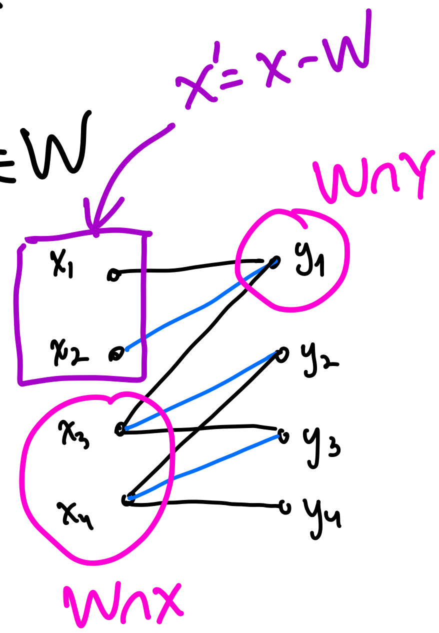
We use W to exhibit a subset $X' \subseteq X$ with too few neighbors, i.e. $|N(X')| < |X|$ as follows: Let $X' = X - W$

$$= \{x \in X : x \notin W\}$$

Note that every $y \in N(X')$ must be in W because W is a vertex cover:

$$X' \ni x - y \Rightarrow y \in W$$

so $x \notin W$



$$W \supseteq (X \cap W) \sqcup N(X')$$

$$|X| > |W| \geq \underbrace{|X \cap W|}_{= |X| - |X - W|} + |N(X')|$$

$$\Rightarrow \cancel{|X|} > \cancel{|X|} - \underbrace{|X - W|}_{= |X'|} + |N(X')|$$

$$\Rightarrow |X'| > |N(X')| \qquad \blacksquare$$

NEXT: Applying Hall's Thm. to find matchings of entire left sides X!

APPLICATION 1: regular bipartite graphs

THEOREM (König 1931):
Every d-regular bipartite multigraph $G=(X \sqcup Y, E)$ with $d \geq 1$

(a) has $|X|=|Y|$.

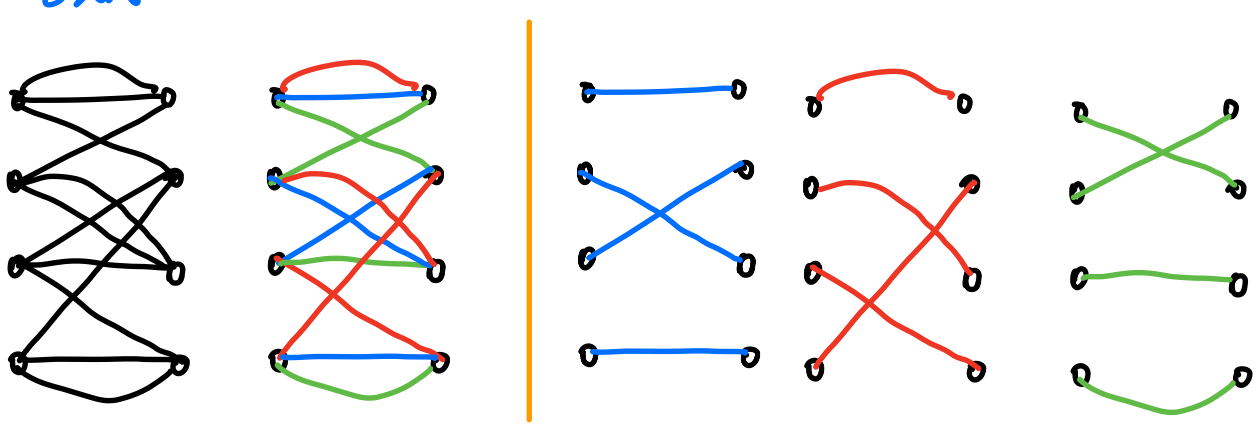
(b) contains a perfect matching $M \subset E$ (or a 1-factor)

a matching of all the vertices, i.e. $\deg_M(v)=1 \ \forall v \in V$

so $\nu(G)=|X|=\frac{|V|}{2}$.

(c) and in fact, one can express E as a disjoint union $E = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_d$ of d perfect matchings inside G.

EXAMPLE $d=3$



G = $M_1 \sqcup M_2 \sqcup M_3$

3-regular, bipartite

THEOREM (König 1931):

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(c) and in fact, one can express E as a disjoint union $E=M_1\sqcup M_2\sqcup \dots \sqcup M_d$ of d perfect matchings inside G.

proof:

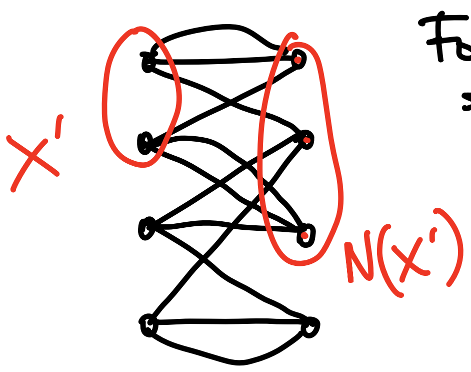
(a): We've seen this comes from counting $|E|$ in two ways:

$$|X|\cdot d = \sum_{x\in X}\underbrace{\deg_G(x)}_{=d} = |E| = \sum_{y\in Y}\underbrace{\deg_G(y)}_{=d} = |Y|\cdot d$$

$$|X|\cdot\cancel{d} = |Y|\cdot\cancel{d}$$

$$|X|=|Y|$$

(b):



For (b), we want to use Hall's Thm, so need to check $\forall X'\subset X$

$$|N(X')|\geq |X'|$$

Let's count all the X' to $N(X')$ edges in G two ways:

$$\#\left\{\begin{array}{l}\text{edges } (x',y) \\ \text{with } x'\in X', \\ y\in N(X')\end{array}\right\}$$

$$= \sum_{x'\in X'} \underbrace{\deg_G(x')}_{=d}$$

$$= d\cdot|X'|$$

$$\| \quad \sum_{y\in N(X')} \underbrace{\#\left\{\begin{array}{l}\text{edges } (x',y) \\ \text{with } x'\in X'\end{array}\right\}}_{\le \underbrace{\deg_G(y)}_{=d}}$$

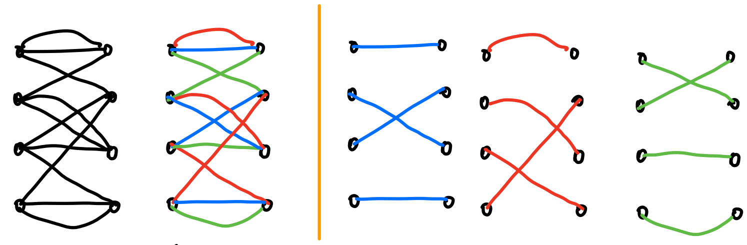
$$\le d\cdot|N(X')|$$

$$\cancel{d}|X'| \le \cancel{d}|N(X')|$$

$$|X'| \le |N(X')|$$

Hence $\exists$ a perfect matching $M\subset E$.

<u>(c)</u>:



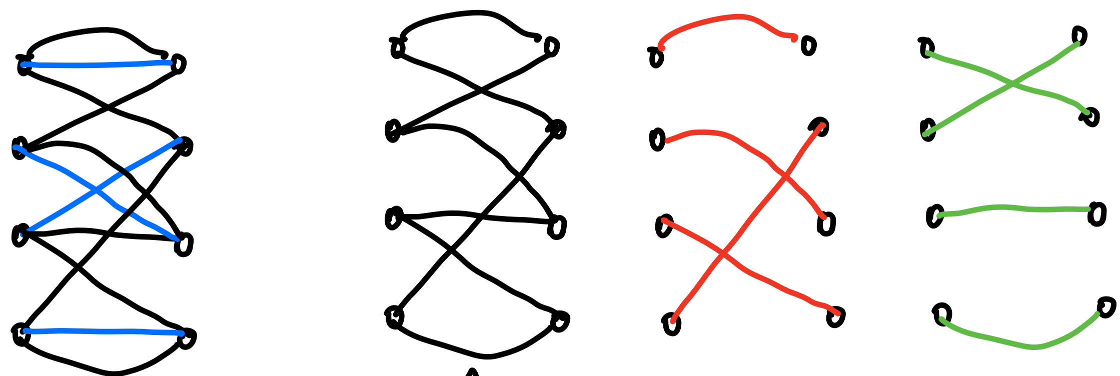
$G = M_1 \sqcup M_2 \sqcup M_3$

For (c), we use induction on d.

<u>BASE CASE $d=1$:</u> Then the perfect matching M found in part (b) must have $M=E$.

INDUCTIVE STEP $d \geq 2$:

Use the perfect matching M from part (b), and create $\hat{G} := G$ with the edges of M removed.



G, M $\hat{G} \overset{\text{by induction}}{=} M_1 \sqcup M_{\substack{2 \\ \shortparallel \\ d-1}}$

d-regular   (d-1)-regular

$$\Rightarrow \quad G = M_1 \sqcup M_2 \sqcup \dots \sqcup M_{d-1} \sqcup \underset{\substack{\shortparallel \\ M}}{M_d}$$

$\blacksquare$

APPLICATION 2: doubly-stochastic matrices

DEFINITION: In probability theory, a **stochastic matrix** is a $n \times n$ square matrix $A = (a_{ij})$ with entries $a_{ij} \in \mathbb{R}_{\geq 0}$ whose **rows all sum to 1**, that is, $\sum_{j=1}^{n} a_{ij} = 1 \quad \forall\, i = 1, 2, \ldots, n$.

It's called **doubly stochastic** if the columns also all sum to 1, that is

$$\begin{cases} \sum_{j=1}^{n} a_{ij} = 1 & \forall\, i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} a_{ij} = 1 & \forall\, j = 1, 2, \ldots, n \end{cases}$$

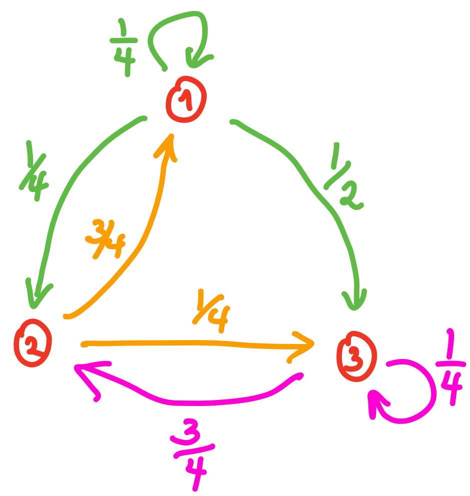
Stochastic matrices arise in theory of **Markov chains**, where there are n possible states, and

$$a_{ij} = \text{Prob}\,(\text{starting in state } i \text{ one transitions to state } j)$$

EXAMPLE

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 3/4 & 0 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

states, transition probabilities:



DEFINITION: A special case of doubly-stochastic matrices are permutation matrices P that have exactly one 1 in each row and column, and all other entries 0.

EXAMPLES:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Identity matrix

THEOREM (Birkhoff-von Neumann) 1946: ← Bondy Murty EXER 5.2.8

Every doubly stochastic matrix A can be written as a weighted average of permutation matrices

i.e. $A = c_1 P_1 + c_2 P_2 + \ldots + c_r P_r$ with $c_1, \ldots, c_r \in \mathbb{R}_{\geq 0}$

$c_1 + c_2 + \ldots + c_r = 1$

EXAMPLE:

$$A = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 3/4 & 0 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= c_1 \quad P_1 \quad + \quad c_2 \quad P_2 \quad + \quad c_3 \quad P_3$$

THEOREM (Birkhoff-von Neumann). 1946

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EXAMPLE:

$$A = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 3/4 & 0 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

proof: Let's prove the more general-sounding statement that if $\underset{(a_{ij})}{\overset{}{A}}$ an $n \times n$ matrix

with $a_{ij} \in \mathbb{R}_{\geq 0}$ and all row sums $= d > 0$
all col sums $= d$

then $A = c_1 P_1 + c_2 P_2 + \ldots + c_r P_r$ where $P_i =$ permutation matrices

and $c_1, c_2, \ldots, c_r \in \mathbb{R}_{\geq 0}$
with $c_1 + c_2 + \ldots + c_r = d$.

We'll prove it by induction on $\#\{$nonzero entries a_{ij} in $A\}$

BASE CASE: no nonzero entries, so $A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

EXAMPLE:

$$A=\begin{bmatrix}1/4&1/4&1/2\\3/4&0&1/4\\0&3/4&1/4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

INDUCTIVE STEP:

Build from A a bipartite graph

$$G=(V,E)$$

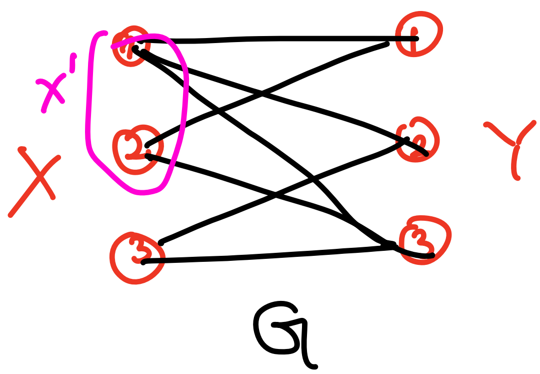
$$V = X \sqcup Y, \qquad E=\{(i,j) : a_{ij} > 0\} \quad (a_{ij}\neq 0)$$

X = row indices $\{1,2,\dots,n\}$

Y = col indices $\{1,2,\dots,n\}$

e.g. for A above

$$A=\begin{array}{c} \\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{array}{ccc}1&2&3\end{array}\\ \begin{bmatrix}1/4&1/4&1/2\\3/4&0&1/4\\0&3/4&1/4\end{bmatrix}\end{array}$$



Let's check G satisfies

Hall's condition: $\forall X' \subset X,\ |N(X')| \geq |X'|$

by counting in 2 ways the sum of all entries of A lying in rows indexed by X':

$$\sum_{\substack{i\in X'\\ j=1,2,\dots,n}} a_{ij} = \sum_{i\in X'} \underbrace{\sum_{j=1}^{n} a_{ij}}_{=d} = d\cdot|X'|$$

$$\sum_{\substack{i\in X'\\ j=1,2,\dots,n}} a_{ij} = \sum_{j=1}^{n} \sum_{i\in X'} a_{ij} = \sum_{j\in N(X')} \underbrace{\sum_{i\in X'} a_{ij}}_{\le \underbrace{\sum_{i=1}^{n} a_{ij}}_{=d}} \le d\cdot|N(X')|$$

$a_{ij}=0$ unless $j\in N(X')$ since $i\in X'$

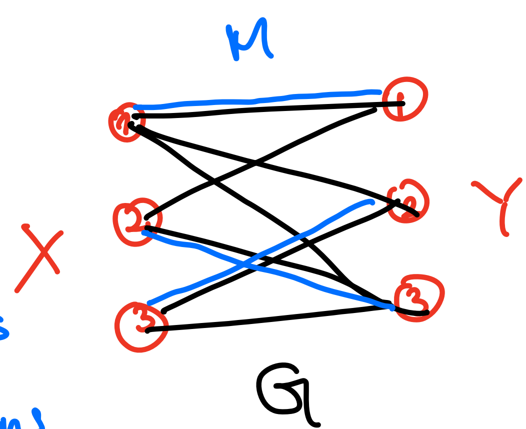
$$\Rightarrow \cancel{d}\cdot|X'| \le \cancel{d}\cdot|N(X')|$$

$$\Rightarrow |X'| = |N(X')|$$

Hence by Hall's Thm, $\exists$ a perfect matching $M\subset E$

$$A=\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \frac14 & \frac14 & \frac12 \\ 2 & \frac34 & 0 & \frac14 \\ 3 & 0 & \frac34 & \frac14 \end{array}$$

M ⟵ circles the positions of a permutation matrix P_1



Let $c_1 =$ min of the entries $\{a_{ij} : (i,j) \in M\}$

Then $\hat{A} := A - c_1 P_1$ has entries in $\mathbb{R}_{\geq 0}$,

fewer nonzero entries,

and its rows and columns sum to $d - c_1$.

$$A = \begin{array}{c|ccc} & ① & ② & ③ \\ \hline ① & & & \\ ② & & & \\ ③ & & & \end{array} \left[\begin{array}{ccc} 1/4 & 1/4 & 1/2 \\ 3/4 & 0 & 1/4 \\ 0 & 3/4 & 1/4 \end{array}\right] \rightsquigarrow \hat{A} = \begin{bmatrix} 0 & 1/4 & 1/2 \\ 3/4 & 0 & 0 \\ 0 & 1/2 & 1/4 \end{bmatrix}$$

rows/cols all sum to $3/4$
$= 1 - 1/4$
$= d - c_1$

so by induction,

$\hat{A} = c_2 P_2 + \ldots + c_r P_r$ with $c_2, \ldots, c_r \geq 0$
$c_2 + \ldots + c_r = d - c_1$

$A = c_1 P_1 + c_2 P_2 + \ldots + c_r P_r$

$\blacksquare$