1)

9/9/2015 >

Math 8668 Fall 2015 Vic Reiner

Combinatorial Theory - Intro Grad Combinatorics 1st semester

SYLLABUS issues:
1. Office hours
2. Makeups for Nov. 16, 18, 20 (M W F)
3. Grading - show-up, ask questions, do some HW!

Text: Stanley Enum. Comb. Vol 1. (where HW comes from)

I'll borrow heavily from Ardila's (beautiful!) handbook chapter (on syllabus, linked), just Part I.

We'll count combinatorial objects (e.g. subsets, multisets, partitions of numbers & sets, compositions, graphs, trees, ...)

but also pay attention to natural structures they carry, most often partially ordered set structures (poset)

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§1.1 What is a good answer for a counting question?

Some are better than others, but different answers can have different advantages...

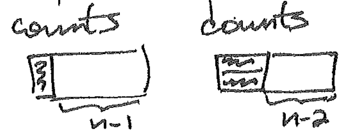
EXAMPLE (Ardila's §1.1)

Let a_n = # of tilings of a $2 \times n$ rectangle by dominoes. What is a_n?

n	a_n	rectangle	tilings	expected fraction of tiles that are vertical	expected number of vertical tiles
0	1			$\frac{0}{0}$?	0
1	1			1	$\frac{1}{1}=1$
2	2			$\frac{1}{2}$	$\frac{2+0}{2}=1$
3	3			$\frac{5}{9}$	$\frac{3+1+1}{3}=\frac{5}{3}$
4	5		(a_3) (a_2)	$\frac{1}{2}$	$\frac{4+2+2+2+0}{5}=2$

(2)

① Recurrence: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, and $a_0 = a_1 = 1$



(compare with Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0$, $F_1 = 1$

n	a_n	F_n
0	1	0
1	1	1
2	2	1
3	3	2
4	5	3
5	8	5
6	13	8

and realize $a_n = F_{n+1}$)

It will take a while to compute a_{1000} this way, and we don't have too much sense of its order of magnitude either.

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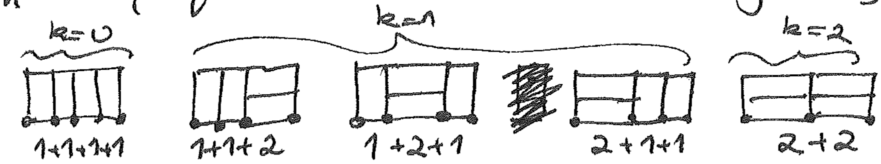
② First explicit formula:

Note $a_n = \#\{\text{sequences of 1's \& 2's totalling to } n\}$



so $a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \#\{\text{sequences of } k \text{ 2's and } n-2k \text{ 1's}\}$

e.g. $1+2+2+1+2+1+1+1+2 = n = 13$, $k=4$, $n-2k=5$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{(n-2k)+k}{k}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$

e.g. $a_4 = \binom{4}{0} + \binom{4-1}{1} + \binom{4-2}{2} = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1 + 3 + 1$

Explicit, but maybe not so helpful.

③ Second explicit formula:

We'll derive soon that

$$a_n = \frac{1}{\sqrt{5}}\left(\Big(\underbrace{\tfrac{1+\sqrt{5}}{2}}_{\text{call this } \varphi}\Big)^{n+1} - \Big(\underbrace{\tfrac{1-\sqrt{5}}{2}}_{\text{call this } \psi}\Big)^{n+1}\right)$$

which is very explicit, but still not so good for computing a_{1000} on the nose.

(Why is it even an integer?!)

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④ Asymptotic formula: Since $\begin{cases} \varphi := \frac{1+\sqrt{5}}{2} \approx 1.618 \text{ (the golden ratio)} \\ \text{and } \psi := \frac{1-\sqrt{5}}{2} \approx -0.618 \in (-1,0) \end{cases}$

one has from above that

$$a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

(and in fact, a_n is the nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$)

This tells us a lot about its growth, e.g. its number of base 10 digits is

$$\log_{10}(a_n) = (n+1)\underbrace{\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)}_{\approx\, 0.20899} + \log_{10}\left(\frac{1}{\sqrt{5}}\right)$$

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⑤ (Ordinary) generating function for $(a_0, a_1, a_2, \ldots)$

$$A(x) \overset{\text{DEF'N}}{:=} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$ as an element of $\underbrace{\mathbb{C}[[x]]}$ the ring of formal power series in x with $\mathbb{C}$ coefficients

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

$$= \sum_{n \geq 0} a_n x^n$$

9/11/15 ≥ Perhaps not clear yet why we would even consider $A(x)$, but let's find a simple formula for it now (the ~~slow~~ way; fast way later) and derive everything else from it!

(4)

The recurrence $\boxed{a_n = a_{n-1} + a_{n-2} \text{ for } n\geq 2}$ and $a_0 = a_1 = 1$

[mult by x^n + sum on n] [$\sum_n a_n x^n$] gives

$$\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2}' a_{n-1} x^n + \sum_{n\geq 2}' a_{n-2} x^n$$

[express in terms of A(x)]

$$= x^1 \sum_{n\geq 2} a_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} a_{n-2} x^{n-2}$$

($m=n-1$, $m=n-2$)

$$A(x) - a_1 x^1 - a_0 x^0 = x^1 \sum_{m\geq 1}' a_m x^m + x^2 \sum_{m\geq 0}' a_m x^m$$

$$= x^1 (A(x) - a_0 x^0) + x^2 A(x)$$

[solve for A(x)]

$$A(x) - x - 1 = x(A(x) - 1) + x^2 A(x)$$

$$A(x)(1 - x - x^2) = x + 1 - x = 1$$

GENERATING FUNCTION $\boxed{A(x) = \frac{1}{1-x-x^2}}$ ← we'll learn to write this down immediately (!) later

What good is this? Plenty! It depends on how we try to extract or estimate coefficients.

(a) $A(x) = \frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \dots$

i.e. $\sum_{n\geq 0} a_n x^n = \sum_{d\geq 0} (x+x^2)^d = \sum_{d\geq 0} \sum_{k=0}^{d} \binom{d}{k} \underbrace{(x^2)^k x^{d-k}}_{x^{d+k}}$

$$= \sum_{n\geq 0}' x^n \left(\sum_{k=0}^{\lfloor n/2 \rfloor}{}' \binom{n-k}{k} \right)$$

← $n = d+k$, $d = n-k$

$\Rightarrow a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$ from before

(b) $A(x) = \frac{1}{1-x-x^2} = \frac{\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)}{1 - \frac{1+\sqrt5}{2}x} + \frac{-\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)}{1 - \frac{1-\sqrt5}{2}x}$

↑ partial fraction computation (skipped!)

[i.e. $A(x) = \frac{1}{ax^2+bx+c} = \frac{1}{a(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2} = \frac{-A/r_1}{1-\frac{x}{r_1}} + \frac{-B/r_2}{1-\frac{x}{r_2}}$]

$$= \frac{1}{\sqrt5} \sum_{n\geq 0} \left(\frac{1+\sqrt5}{2}\right)^{n+1} x^n - \frac{1}{\sqrt5} \sum_{n\geq 0} \left(\frac{1-\sqrt5}{2}\right)^{n+1} x^n$$

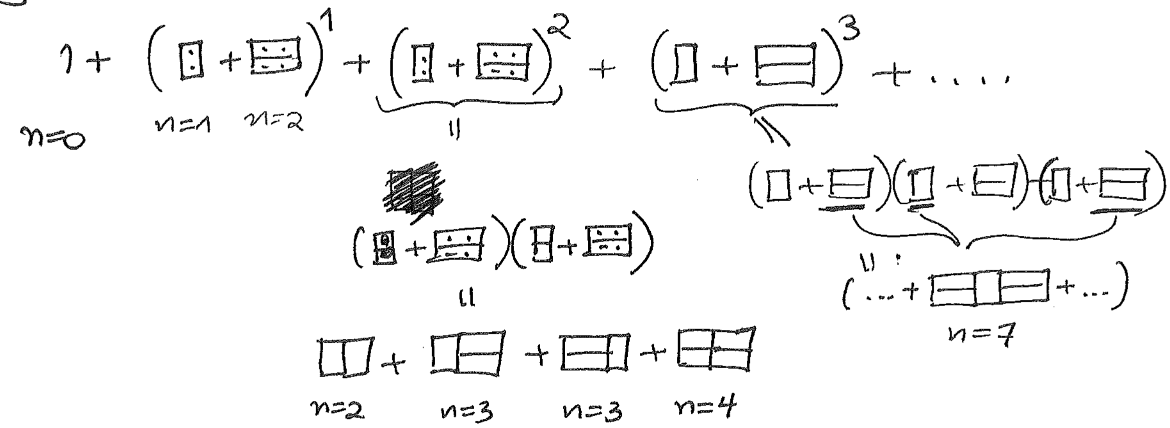
$$= \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^{n+1} - \left(\frac{1-\sqrt5}{2}\right)^{n+1} \right)$$ from before.

(4½)

The fast way (see Ardila p.20 #18) is via Polya's "picture-writing":

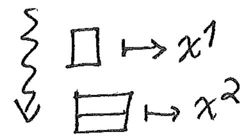
$$\frac{1}{1-(A+B)} = 1 + (A+B)^1 + (A+B)^2 + (A+B)^3 + \cdots$$

$\uparrow$ $\mathbb{C}[[A,B]]$



$A = x^1$

$B = x^2$



$$\mathbb{C}[[x]] \ni A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + (x+x^2)^1 + (x+x^2)^2 + (x+x^3)^3 + \cdots$$

$$= \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2}$$

(5½)

Better yet

$$A(x,v) := \sum_{n,m\geq 0} a_{m,n} x^n v^m = \left[\frac{1}{1-(A+B)}\right]_{\substack{A=vx \\ B=x^2}} \in \mathbb{C}[[v,x]]$$

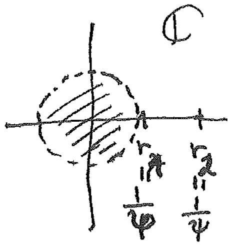
(and actually, even in $\mathbb{C}[v][[x]]$)

$$= 1 + (vx+x^2)^1 + (vx+x^2)^2 + (vx+x^2)^3 + \cdots$$

$$= \frac{1}{1-(vx+x^2)} = \frac{1}{1-vx-x^2}$$

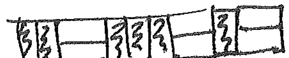
(5)

(c) The asymptotic $a_n \approx \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n+1}$ was controlled by (i.e. $\approx c\gamma^n$ for some c) the reciprocal of the pole r_1 of $A(x) = \frac{1}{1-x-x^2}$ nearest the origin in $\mathbb{C}$



(we'll say a little more about this later; or see Wilf §2.4 now; §5.2 later)

9/14/15 ⁒ The generating function can often be refined to keep track of more statistics, e.g. what if we wanted to compute $a_{m,n} = \#\{$tilings of $2\times n$ rectangle by dominoes with m vertical dominoes$\}$



, say to get a sense of what fraction of the n tiles are vertical, as $n\to\infty$.

Later we'll see how to immediately write down

$$\sum_{n,m\geq0} a_{m,n}x^n v^m = \frac{1}{1-vx^1-x^2}$$

This lets us find out the asymptotics of the expected number of vertical dominoes in a large random tiling, which should be $\frac{\sum_m a_{mn}m}{a_n}$: ($\frac{\sum_{\text{tiling } T} \#\text{verticals}}{\#\text{tilings}}$)

$$\sum_{n\geq0}\left(\sum_{m\geq0} a_{mn}m\right)x^n = \left[\frac{\partial}{\partial v}\sum_{n,m\geq0} a_{mn}x^n v^m\right]_{v=1}$$

$$= \left[\frac{\partial}{\partial v}\frac{1}{1-vx-x^2}\right]_{v=1}$$

$$= \left[\frac{x}{(1-vx-x^2)^2}\right]_{v=1} = \frac{x}{(1-x-x^2)^2}$$

or the fraction of the n tiles that are vertical $\frac{\sum_m a_{mn}m}{n\cdot a_n}$

$$\frac{x}{a(x-r_1)^2(x-r_2)^2} = \frac{A_1x+B_1}{(x-r_1)^2} + \frac{A_2x+B_2}{(x-r_2)^2} + \frac{C_1}{x-r_1} + \frac{D_1}{x-r_2}$$

Using partial fractions on this, one can show $\sum_{m\geq0} a_{mn}m \approx \frac{n}{5}\left(\frac{1+\sqrt5}{2}\right)^{n+1} \approx \frac{1}{\sqrt5}n\,a_n$ since $a_n \approx \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{n+1}$

Thus the expectation $\approx \frac{n}{\sqrt5}$, so out of the n tiles, expect roughly $\frac{1}{\sqrt5}$ are vertical, asymptotically.

(6)

The ring of formal power series $R[[x]]$

(where $R = \mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{C}[v]$ or any commutative ring with 1)
or $\mathbb{F}_q$

DEFIN: $R[[x]] := \{ a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n = A(x)$ with $(a_0, a_1, a_2, \dots) \in R \}$

is a ring having coefficientwise $+$: if $B(x) = \sum_{n=0}^{\infty} b_n x^n$
(commutative)
then $A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

and multiplication $\times$ via convolution:

$$C(x) := A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n$$

with $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$

so its $0 = 0 + 0 \cdot x + 0 \cdot x^2 + \dots$

$1 = 1 + 0 \cdot x + 0 \cdot x^2 + \dots$

and one can check ...

PROP: $A(x) = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$ is a unit, i.e. $\exists\, B(x)$ with $1 = A(x)B(x)$

$\iff$ a_0 is a unit of R, i.e. $\exists\, b_0 \in R$ with $1 = a_0 b_0$

proof: $1 = A(x)B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$
$\quad ||$
$1 + 0 \cdot x + 0 x^2 + \dots$

$\iff$ $a_0 b_0 = 1$, i.e. $b_0 = a_0^{-1}$ in R (so need a_0 to be a unit if there's a hope of $A(x)$ being a unit)

and then

$a_0 b_1 + a_1 b_0 = 0$ means $b_1 = -\frac{a_1 b_0}{a_0}$

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$ means $b_2 = \frac{-(a_1 b_1 + a_2 b_0)}{a_0}$ (b_1, b_0 already determined)

$\vdots$

∎

e.g. $A(x)(1 - x - x^2) = 1$ ($1 - x - x^2$: unit! in $\mathbb{C}$)
$\Downarrow$
$A(x) = \frac{1}{1 - x - x^2}$ exists in $\mathbb{C}[[x]]$

$\left(= 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots\right)$

(7)

DEF'N: A sequence $A_1(x), A_2(x), \ldots$ in $R[[x]]$ converges, i.e. $A(x) = \lim_{j\to\infty} A_j(x)$ exists

if $\forall n \geq 0$ the coefficient of x^n in $A_j(x)$ stabilizes for $j >> 0$,

$\underbrace{\text{coefficient of } x^n \text{ in } A_j(x)}_{[x^n]\, A_j(x)}$

i.e. $\forall n \geq 0$ $\exists N > 0$ and $a_n \in R$ such that $[x^n] A_j(x) = a_n$ $\forall n \geq N$.

e.g. $A(x) = \frac{1}{1-x-x^2} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \ldots$

$A_0(x) = 1$, $A_1(x) = 1 + (x+x^2)$, $A_2(x) = 1 + (x+x^2) + (x+x^2)^2$, $A_3(x) = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3$

converges in $\mathbb{C}[[x]]$, e.g. $[x^3] A(x) = [x^3] A_3(x) = [x^3] A_4(x) = \ldots = a_3 = 3$

e.g. $e^{x+1} := 1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \ldots$ does not (while $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \ldots$ does)

Alternatively $\{A_j(x)\}_{j=0,1,\ldots}$ converges in $R[[x]]$

if $\lim_{j\to\infty} \operatorname{mindeg}\left(A_j(x) - A_{j-1}(x)\right) = \infty$ where

DEF'N: $\operatorname{mindeg} \underbrace{A(x)}_{\sum_{n\geq 0} a_n x^n} := \{\text{smallest } d \text{ with } a_d \neq 0\}$

e.g. above $A_j(x) - A_{j-1}(x) = (x+x^2)^j$, having mindeg $= j \to \infty$ as $j \to \infty$

COR: $\sum_{j=0}^{\infty} B_j(x) = B_0(x) + B_1(x) + B_2(x) + \ldots$ converges in $R[[x]]$ $\iff$ $\operatorname{mindeg} B_j(x) = \infty$

$A_0(x) = B_0(x)$, $A_1(x) = B_0(x) + B_1(x)$, $A_2(x) = B_0(x) + B_1(x) + B_2(x)$

$= \lim_{n\to\infty} A_n(x)$ (with $B_j := A_j - A_{j-1}$)

(8)

COR: Infinite products of the form

$$\prod_{j=1}^{\infty} (1+B_j(x)) \text{ with } \operatorname{mindeg} B_j \geq 1 \ \forall j$$

converge in $R[[x]]$ $\iff$ $\lim_{j\to\infty} \operatorname{mindeg} B_j(x)$

proof: $A_j(x) = (1+B_1)(1+B_2)\cdots(1+B_j)$

has $A_j - A_{j-1} = (1+B_1)\cdots(1+B_{j-1})(1+B_j) - (1+B_1)\cdots(1+B_{j-1}) = B_j(1+B_1)\cdots(1+B_{j-1}) = B_j(1+\ldots\ldots)$

$B_j(1+\ldots\ldots)$ has mindeg = mindeg B_j ∎

EXAMPLE(S): Partition generating functions (see Stanley §1.8)

DEF'N: A (number) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of n

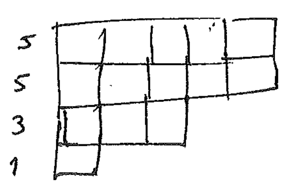
is a weakly decreasing $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ with $\lambda_1 + \lambda_2 + \ldots = n$
eventually 0
sequence
of nonnegative integers (i.e. $\lambda_i \in \mathbb{N} = \{0,1,2,\ldots\}$) and we write $\lambda \vdash n$ and $n = |\lambda|$

e.g. $\lambda = (5,5,3,1,0,0,\ldots) = (5,5,3,1,0)$
$= (5,5,3,1)$ is a partition of $n = 14$ $= 5+5+3+1$

Its length $\ell(\lambda) := \#\{i : \lambda_i > 0\}$ = # of nonzero parts λ_i

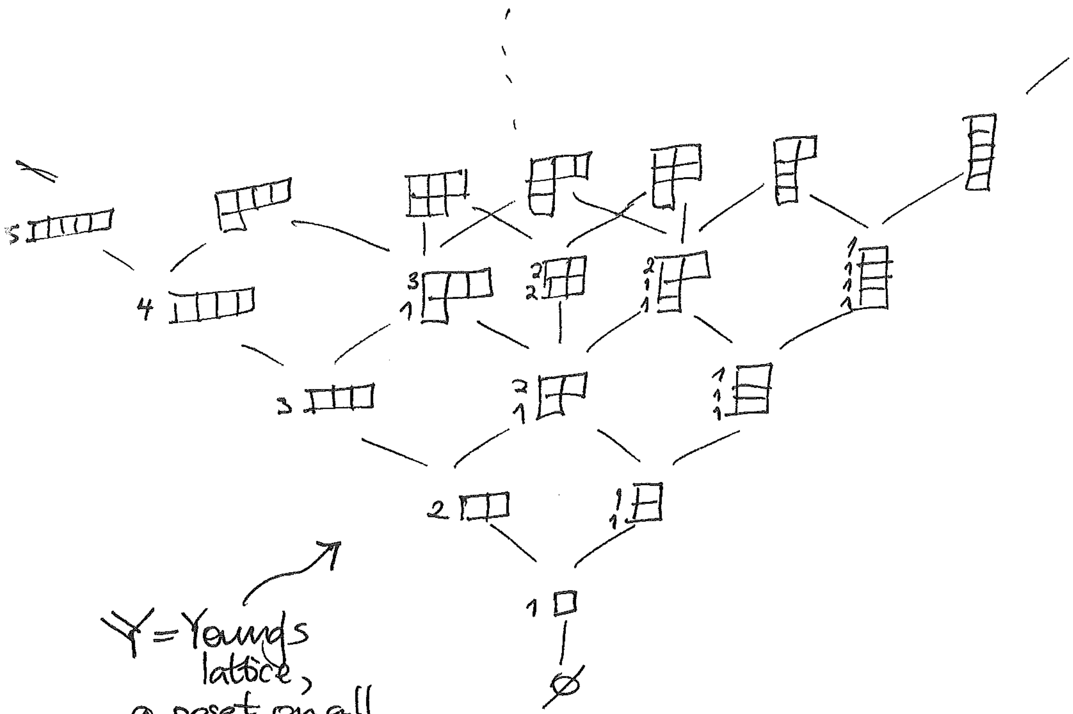
Its Ferrers diagram is a left & top justified array of unit squares with λ_i in row i from the top

e.g. $\lambda = (5,5,3,1) \longleftrightarrow$



(9)

Let $p(n) :=$ # of partitions $\lambda \vdash n$



$p(n)$	n
⋮	⋮
	6
7	5
5	4
3	3
2	2
1	1
1	0

$\mathbb{Y}$ = Young's lattice, a poset on all partitions.

① $\displaystyle\sum_{n\geq 0} p(n)q^n = \sum_{\text{all partitions }\lambda} q^{|\lambda|}$

$\cap$

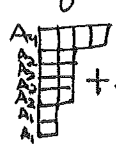
$\mathbb{C}[[q]]$

$(1+A_1+A_1^2+A_1^3+\dots)(1+A_2+A_2^2+A_2^3+\dots)(1+A_3+A_3^2+A_3^3+\dots)(1+A_4+A_4^2+\dots)\cdots$

$(q^0+q^1+q^2+q^3+\dots)(q^0+q^2+q^4+q^6+\dots)(q^0+q^3+q^6+q^9+\dots)(q^0+q^4+q^8+\dots)\cdots$

$A_i \mapsto q^i$

$A_1^2 A_2^4 A_4^1 \mapsto q^{14}$

$= + \dots +$  $+ \dots$

$= (\underbrace{1+q+q^2+q^3+\dots}_{B_1})(\underbrace{1+q^2+(q^2)^2+\dots}_{B_2})(\underbrace{1+q^3+(q^3)^2+\dots}_{B_3})\cdots$ a convergent product!

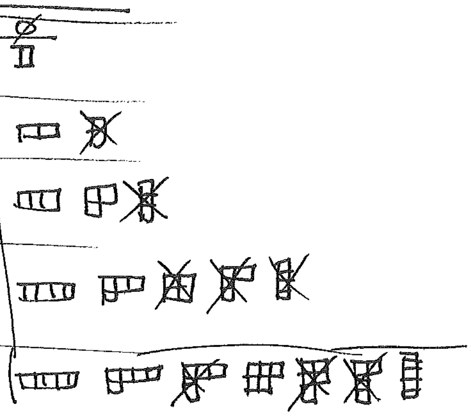
$= \dfrac{1}{1-q}\,\dfrac{1}{1-q^2}\,\dfrac{1}{1-q^3}\cdots$

$= \dfrac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$ ← denominator is also a convergent products (and invertible)

(10)

② Let $q(n) :=$ # of partitions of n into <u>distinct</u> parts

n	$q(n)$	
0	1	$\emptyset$
1	1	
2	1	
3	2	
4	2	
5	3	

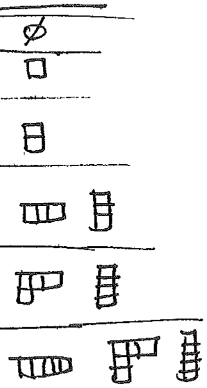


$$Q(x) := \sum_{n\geq 0} q(n)\cdot q^n = (1+q^1)(1+q^2)(1+q^3)(1+q^4)\cdots = \underbrace{\prod_{j\geq 1}(1+q^j)}_{\text{convergent! (in } \mathbb{C}[[q]])}$$

9/16/15 →

③ Let $p_{odd}(n) :=$ # of partitions of n into <u>odd</u> parts

n	$p_{odd}(n)$	
0	1	$\emptyset$
1	1	
2	1	
3	2	
4	2	
5	3	



Looks the <u>same</u>, i.e. CONJECTURE: $p_{odd}(n) = q(n)\ \forall n\geq 0$. Why?

The gen. fns. will explain it:

$$P_{odd}(x) = (1+q^1+q^2+\dots)(1+q^3+(q^3)^2+\dots)(1+q^5+(q^5)^2+\dots)\cdots$$

$$= \frac{1}{(1-q^1)(1-q^3)(1-q^5)\cdots} = \underbrace{\frac{1}{\prod_{j\geq 0}(1-q^{2j+1})}}_{\text{convergent}} \quad \left(\overset{\text{not clear yet}}{\overset{??}{=}} \prod_{j\geq 1}(1+q^j) \right)$$

(11)

Well,

$$Q(q) = (1+q^1)(1+q^2)(1+q^3)\cdots$$

$$= \frac{1-q^2}{1-q}\cdot\frac{1-(q^2)^2}{1-q^2}\cdot\frac{1-(q^3)^2}{1-q^3}\cdots$$

$$= \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})(1-q^{12})\cdots}{(1-q^1)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)\cdots}$$

$$= \frac{1}{(1-q^1)(1-q^3)(1-q^5)\cdots} = P_{odd}(q) \quad !$$

Was that cancellation legal? Yes; let's justify it differently...

Let $R(q) := (1-q^1)(1-q^3)(1-q^5)\cdots = \frac{1}{P_{odd}(q)}$ in $\mathbb{C}[[q]]$

(convergent!)

It suffices to show $1 \overset{?}{=} Q(q)R(q)$ in $\mathbb{C}[[q]]$ (since mult. inverses are unique)

$1 + 0\cdot q + 0\cdot q^2 + \cdots$

$$= \Big((1+q^1)\underbrace{(1+q^2)(1+q^3)\cdots}_{\text{convergent}}\Big)\Big((1-q^1)\underbrace{(1-q^3)(1-q^5)\cdots}\Big)$$

$$= \underbrace{(1+q^1)(1-q^1)}_{1-q^2}\Big((1+q^2)(1+q^3)\cdots\Big)\Big((1-q^3)(1-q^5)\cdots\Big)$$

$$= (1-q^4)\cdot\Big((1+q^3)(1+q^4)(1+q^5)\cdots\Big)\Big((1-q^3)(1-q^5)\cdots\Big)$$

$$= (1-q^4)(1-q^6)\cdot\Big((1+q^4)(1+q^5)\cdots\Big)\Big((1-q^5)(1-q^7)\cdots\Big)$$

$$= \underbrace{(1-q^8)\cdot(1-q^6)\cdot\Big((1+q^5)(1+q^6)\cdots\Big)\Big((1-q^5)(1-q^7)\cdots\Big)}_{\text{starts } 1+0\cdot q^1+0\cdot q^2+0\cdot q^3+0\cdot q^4+(??)}$$

⋮

etc.

Bijective proof: Given λ a partition with odd parts $2j-1$ of multiplicity r_j, write $r_j = 2^{i_1} + 2^{i_2} + \cdots$ in its binary expansion and create μ having parts $(2j-1)2^{i_1}, (2j-1)2^{i_2}, \ldots$

Stanley gives two! (see p. 64)

e.g. $\lambda = (9^5, 5^{12}, 3^2, 1^3) = (9^{2^0+2^2}, 5^{2^2+2^3}, 3^{2^1}, 1^{2^0+2^1}) \leftrightarrow \mu = (9\cdot 2^0, 9\cdot 2^2, 5\cdot 2^2, 5\cdot 2^3, 3\cdot 2^1, 1\cdot 2^0, 1\cdot 2^1)$

$= (9,36,20,40,6,1,2)$

$= (40,36,20,9,6,2,1)$ ■

Reversible: $\mu = (20,10,7,6,4) = (5\cdot 2^2, 5\cdot 2^1, 7\cdot 2^0, 3\cdot 2^1, 1\cdot 2^2) \leftrightarrow \lambda = (5^{2^2+2^1}, 7^{2^0}, 3^{2^1}, 1^{2^2}) = (7^1, 5^6, 3^2, 1^4)$

(12)

A peek into...

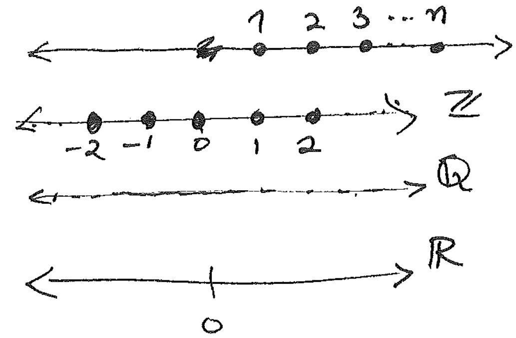
Posets (Stanley Ch. 3)

DEF'N: A poset $(P, \le_P)$ is a set P with a binary relation $x \le_P y$ satisfying

- $x \le_P x$ (reflexive)
- $x \le_P y,\ y \le_P x \implies x = y$ (antisymmetric)
- $x \le_P y,\ y \le_P z \implies x \le_P z$ (transitive)

EXAMPLES: ($[n] := \{1,2,\ldots,n\}$, $\le_{[n]}$)

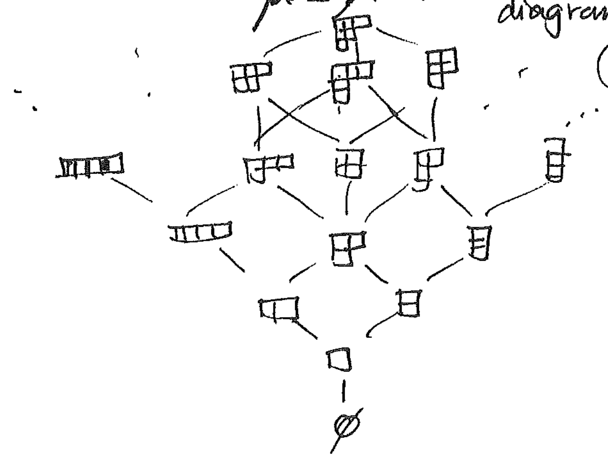
① $(\mathbb{Z}, \le_{\mathbb{Z}})$ $\cap$ $(\mathbb{Q}, \le_{\mathbb{Q}})$ $\cap$ $(\mathbb{R}, \le_{\mathbb{R}})$



totally/linearly ordered; a chain

② $Y :=$ Young's lattice on {all partitions λ}

$\mu \le \lambda$ if Ferrers diagram of $\mu \subseteq$ Ferrers diagram of λ ("$\lambda \subseteq \mu$")



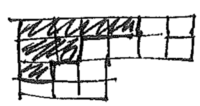
$p(4)=5$

$p(3)=3$

$p(2)=2$

$p(1)=1$

$p(0)=1$



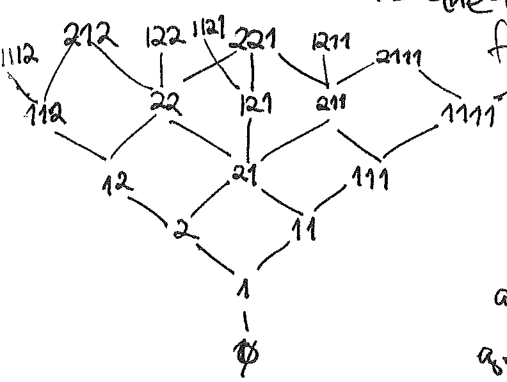
$\mu = (4,2,1) \le \lambda = (6,6,3,3)$

9/18/15 ④ $YF :=$ Young-Fibonacci lattice (see §3.21 Example 3.21.2 #4)

"Z_1" = 1-Fibonacci differential poset on {strings of 1's & 2's}

:= the transitive closure of the relation $x \lessdot y$ if y is obtained from x either by

- replacing leftmost 1 by 2, or
- adding 1 anywhere left of all other 1's



$a_4 = F_5 = 5$

$a_3 = F_4 = 3$

$a_2 = F_3 = 2$

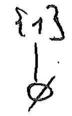
$a_1 = F_2 = 1$

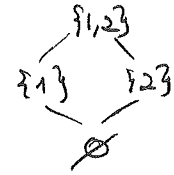
$a_0 = F_1 = 1$

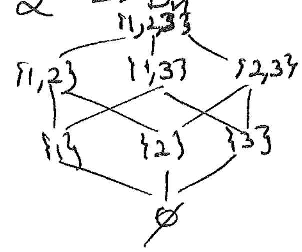
Both Y & YF are differential posets (§3.21) which makes them very special, and helps solve certain path-counting problems for them...

② For S a set, $(2^S, \leq)$ = Boolean algebra = {all subsets of S} with $S \leq T$ if $S \subseteq T$

When $S = [n] := \{1,2,\ldots,n\}$ we will call $2^S =: B_n$

e.g. $B_1 =$ 

| $B_2 =$ 

| $B_3 =$ 

Some common poset properties

	acc = ascending chain condition (no ∞ ascending chains $x_1 \lneq x_2 \lneq x_3 \lneq \ldots$)	dcc = descending chain condition (no $x_1 \gneq x_2 \gneq x_3 \gneq \ldots$)	chain finite = acc + dcc (no ∞ chains)	locally finite, i.e. all intervals $[x,y] := \{z \in P : x \leq z \leq y\}$ are finite	∃ a bottom element $\hat{0}$	∃ a top element $\hat{1}$
$\mathbb{Z}$	no	no	no	yes	no	no
$\mathbb{Q}, \mathbb{R}$	no	no	no	no	no	no
$[n]$	yes	yes	yes	yes	yes $\hat{0}=1$	yes $\hat{1}=n$
2^S for $\lvert S\rvert=\infty$	no	no	no	no	yes; $\hat{0}=\emptyset$	yes; $\hat{1}=S$
$B_n = 2^{\{1,2,\ldots,n\}}$	yes	yes	yes	yes	$\hat{0}=\emptyset$	$\hat{1}=S=\{1,2,\ldots,n\}$
$\mathbb{Y}$	no	yes	no	yes	$\hat{0}=\emptyset$	no
$\mathbb{YF}$	no	yes	yes	yes	$\hat{0}=\emptyset$	no

When P is locally-finite (or even locally chain-finite, i.e. all intervals $[x,y]$ are chain-finite)

then $\leq_P$ is the transitive closure of the covering relation $x \lessdot_P y$ defined by $x \leq_P y$ and $\nexists z$ in P with $x \lneq_P z \lneq_P y$

Then one can represent P by its Hasse diagram:

draw P as nodes in the plane with edges x / y whenever $x \lessdot_P y$

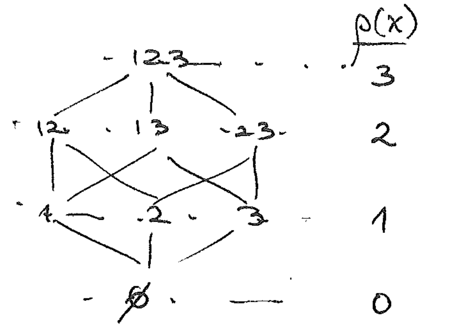
(and y higher in the plane)

(14)

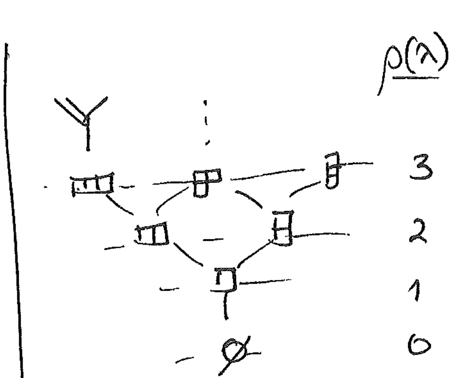
DEF'N: If P is finite, (resp. if it is locally chain-finite and has a bottom element $\hat{0}$,) say P is *graded* if every maximal *chain* (= totally ordered subset) has same size (resp. if $\forall x \in P$, every maximal chain in $[\hat{0}, x]$ has same size).

In this case, there is a unique rank function $\rho : P \longrightarrow \{0,1,2,\ldots\}$ satisfying $\rho(x) = 0$ if only if x is minimal in P and $\rho(y) = \rho(x)+1$ if $y \gtrdot_P x$.

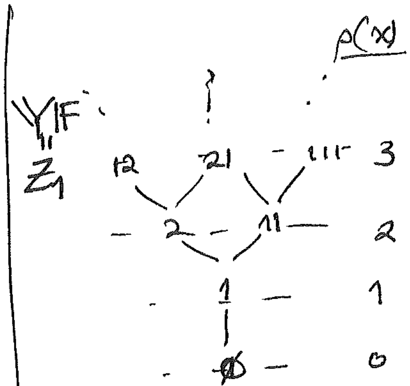
e.g. B_3



$\rho(T) = |T|$



$\rho(\lambda) = |\lambda|$



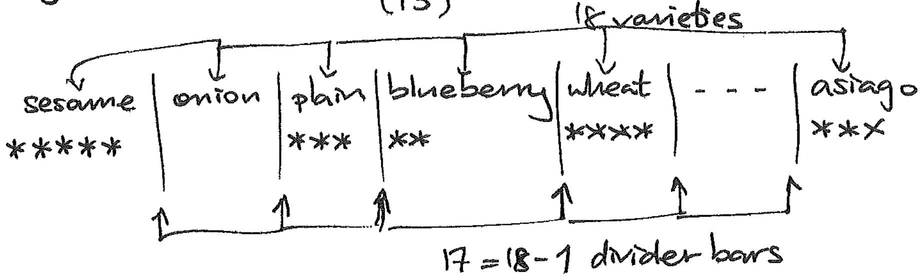
$\rho(x) = x_1 + \ldots + x_\ell$, $x = (x_1, \ldots, x_\ell)$

We sneaked into an extra lecture day this material (because we had brought bagels...)

$\#\{$ways to choose a baker's dozen (13) bagels from 18 varieties of bagel$\} = \left(\!\binom{18}{13}\!\right)$

$= \binom{18+13-1}{13}$

$= \binom{30}{13}$



$\rightsquigarrow$ ***** || *** | ** | **** |||||||||||| ***

18-1+13 positions, in which to choose 13 *'s

$n=30$, $k=13$: $= \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $= \#k$-subsets of $[n] = \{1,2,\ldots,n\}$ $= \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_k \times \mathfrak{S}_{n-k}|}$

since $\mathfrak{S}_n$ = symm. group on n letters acts transitively (with 1 orbit) on k-subsets of $[n]$

and Orbit-Stabilizer LEMMA says if G acts on X, any orbit $\mathcal{O} \subseteq X$ has $|\mathcal{O}| = \frac{|G|}{|G_x|}$ where $G_x := \{g \in G : g(x) = x\}$

(15) Back to formal power series for a bit...

We'll have use for these elements of $\mathbb{C}[[x]]$:

DEF'N: $e^x := \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\log(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\forall \lambda \in \mathbb{C}$, $(1+x)^\lambda := \sum_{k\geq 0} \binom{\lambda}{k} x^k$

$\binom{\lambda}{k} \overset{DEF}{:=} \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(k-1))}{k!} \in \mathbb{C}$

(just like $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$ if $n \in \mathbb{N} = \{0,1,2,\ldots\}$)

They do have all the usual properties you might expect,

EXAMPLES ① $(1+x)^\lambda (1+x)^\mu = (1+x)^{\lambda+\mu}$ in $\mathbb{C}[[x]]$

② $e^{\log(1+x)} = 1 + x$. ③ $e^x e^y = e^{x+y}$, etc...

$e^{\log(1+x)}$ defined to be $= 1 + \log(1+x) + \frac{(\log(1+x))^2}{2!} + \cdots$

$= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)^2}{2!} + \cdots$

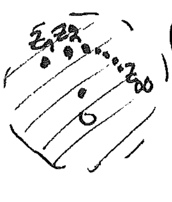
Why does this even converge in $\mathbb{C}[[x]]$? Because $\log(1+x) = \underline{0} + x - \frac{x^2}{2} + \cdots$

(In fact, PROP: If $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n \geq 0} b_n x^n$ and $b_0 = 0$, then $A(B(x)) := \sum_{n\geq 0} a_n B(x)^n$ converges in $\mathbb{C}[[x]]$.)

How to justify ①, ②, ③ etc...? ② is Laborious without a cheat from calc (Taylor series) or complex analysis; (applied to $e^{\log(1+x)} - (1+x) = f(x)$ analytic for $|x|<1$)

THM: If $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic for $|z| < R$ and f vanishes on $|z| < R$ (or even on ∞ many points $z_1, z_2, \ldots \to z_\infty$ approaching a limit point in $|z|<R$)

(J.B. Conway, Functions of one complex variable, Thm 3.7)



then $f(z) \equiv 0$ i.e. $a_0 = a_1 = \cdots = 0$.

(16)

③ $(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$ for $n \in \mathbb{N}$

but also

$$\frac{1}{(1-x)^n} = (1+(-x))^{-n} = \sum_{k\geq 0} \binom{-n}{k}(-x)^k = \sum_{k\geq 0} \frac{(-n)(-n-1)(-n-2)\cdots(-n-(k-1))}{k!}(-1)^k x^k$$

$$= \sum_{k\geq 0} \frac{n(n+1)(n+2)(n+k-1)}{k!} x^k$$

$$= \sum_{k\geq 0} \underbrace{\binom{n+k-1}{k}}_{\left(\binom{n}{k}\right)} x^k$$

$\left(\binom{n}{k}\right)$ = # k-element multisubsets of $[n]$

$\parallel$

onion, sesame, asiago

$(1+x+x^2+\dots)(1+x+x^2+\dots)\dots(1+x+x^2+\dots)$ — n parentheses $=$ 18 flavors of bagels

$**|*||\cdots|*|**$

$\left(\binom{18}{13}\right) = \binom{18+13-1}{13}$

④ $$\frac{1}{1-4x} = \sum_{k\geq 0} \binom{-1}{k}(-4x)^k = \sum_{k\geq 0} \underbrace{\binom{1+k-1}{k}}_{=\binom{k}{k}=1} 4^k x^k = \sum_{k\geq 0} 4^k x^k \checkmark$$

$\parallel$

$(1+(-4x))^{-1}$

but also $$\frac{1}{(1-4x)^2} = \sum_{k\geq 0} \underbrace{\binom{2+k-1}{k}}_{\binom{k+1}{k}=k+1} 4^k x^k = \sum_{k\geq 0} (k+1)4^k x^k$$

$$\frac{1}{(1-4x)^3} = \sum_{k\geq 0} \binom{k+2}{2} 4^k x^k$$

$\vdots$

useful for coefficient extraction after partial fraction expansions

9/21/15 → ⑤ $$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-\frac{1}{2}} = \sum_{k\geq 0} \binom{-\frac{1}{2}}{k}(-4)^k x^k$$

$$= \sum_{k\geq 0} \left(\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots -\frac{(2k-1)}{2}}{k!} (-4)^k \right) x^k$$

$$= \sum_{k\geq 0} \frac{2^k \cancel{4^k}(1)(3)(5)\cdots(2k-1)}{\cancel{2^k}\cdot k!} x^k$$

$$= \sum_{k\geq 0} \frac{\overbrace{2^k \cdot k!}^{2\cdot 4\cdot 6\cdots 2k}(1)(3)(5)\cdots(2k-1)}{k!\cdot k!} x^k = \sum_{k\geq 0} \frac{(2k)!}{k!\,k!} x^k$$

$$= \sum_{k\geq 0} \binom{2k}{k} x^k \quad (?)$$

(17) Another calculus tool in $R[[x]]$...

DEF'N: For $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$,

the formal derivative $A'(x) := \sum_{n\geq 1} \underbrace{n a_n}_{= a_n + a_n + \dots + a_n \text{ (n times)}} x^{n-1} \in R[[x]]$

It satisfies the usual rules from calculus:

$$(A(x)+B(x))' = A'(x) + B'(x)$$

$$(AB)' = (A')B + A\cdot(B')$$

$$\left(\frac{1}{A}\right)' = \frac{-A'}{A^2}$$

$$A(B(x))' = A'(B(x))\cdot B'(x)$$

More on sets, binomial, multinomial (Stanley §1.2)

Binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ has several (easy) interpretations

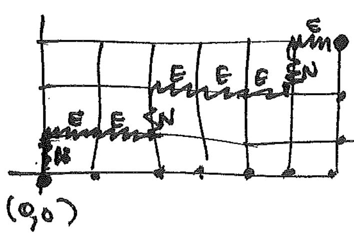
$\binom{n}{k} = \binom{n}{k,\, n-k}$ = # words with k 1's, $n-k$ 2's

i.e. rearrangements of $\underbrace{11\cdots1}_{k}\underbrace{22\cdots2}_{n-k}$

e.g. $\binom{4}{2} = 6 = \#\{2211, 2121, 1221, 2112, 1212, 1122\}$

= # lattice paths in $\mathbb{Z}^2$ taking east (E) or north (N) unit steps from $(0,0)$ to $(k, n-k)$

e.g. $n=9$

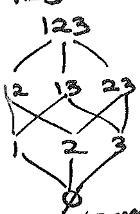


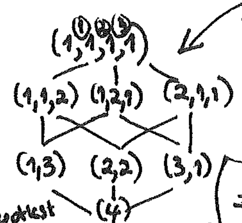
$k, n-k$: $(6,3)$ $\binom{9}{6,3}$ E's N's

1 2 3 4 5 6 7 8 9
N E E N E E E N E

= # {(ordered) compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$, $\alpha_i \in \mathbb{P}$ of $n+1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{k+1}$ with $k+1$ parts}

e.g. $n=3$



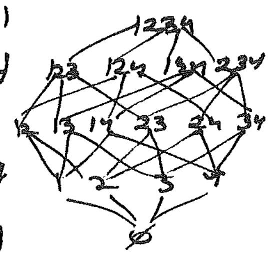


refinement order on compositions of $n+1$

partial sums except last $\leftarrow$ $\alpha = (\alpha_1, \ldots, \alpha_{k+1})$, $\sum \alpha_i = n+1$
$\{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_k\} \subset [n]$

= size of k^{th} rank in $B_n = 2^{\{1,2,\ldots,n\}}$

e.g. $n=4$: $\binom{4}{4}=1$, $\binom{4}{3}=4$, $\binom{4}{2}=6$, $\binom{4}{1}=4$, $\binom{4}{0}=1$



$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{binomial theorem}$$

Multinomials

EXAMPLE: How many rearrangements of BANANAS, i.e. of 3 A's, 1 B's, 2 N's, 1 S's ?

or of $\overset{1}{A}\overset{2}{A}\overset{3}{A}\overset{4}{B}\overset{5}{N}\overset{6}{N}\overset{7}{S}$

There is again a transitive $\mathfrak{S}_7$-action:

$\sigma = (12)(34)(567) = \begin{pmatrix} 1&2&3&4&5&6&7 \\ 2&1&4&3&6&7&5 \end{pmatrix}$ sends AAABNNS $\longmapsto$ AABASNN

The stabilizer is $\mathfrak{S}_3 \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1 \le \mathfrak{S}_7$, so $|\mathcal{O}| = \frac{|\mathfrak{S}_7|}{|\mathfrak{S}_3 \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1|} = \frac{7!}{3!\,1!\,2!\,1!} = \binom{7}{3,1,2,1}$

where $\mathfrak{S}_3 = \mathfrak{S}_{\{1,2,3\}}$, $\mathfrak{S}_1 = \mathfrak{S}_{\{4\}}$, $\mathfrak{S}_2 = \mathfrak{S}_{\{5,6\}}$, $\mathfrak{S}_1 = \mathfrak{S}_{\{7\}}$, and $|\mathcal{O}|$ = # rearrangements

(19)

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\,k_2! \cdots k_m!} \quad \text{for } k_1 + \cdots + k_m = n$$

$= \#$ words with k_1 1's, k_2 2's, $\vdots$, k_m m's, i.e. rearrangements of $\underbrace{11\cdots1}_{k_1}\underbrace{22\cdots2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m}$

$= \#$ lattice paths in $\mathbb{Z}^m$ taking $e_1, e_2, \ldots, e_m$ steps from $\underline{0} = (\underbrace{0, 0, \ldots, 0}_{m})$ to $(k_1, k_2, \ldots, k_m)$

e.g. BANANAS (positions 1 2 3 4 5 6 7) = 2 1 3 1 3 1 4 $\leftrightarrow$ $e_2\, e_1\, e_3\, e_1\, e_3\, e_1\, e_4$

$(0,0,0,0) \to \to \to \to \to \to \to (3,1,2,1)$

$= \#$ { chains / ~~flags~~ in B_n

$\{ \emptyset = S_0 \subset S_{k_1} \subset S_{k_1+k_2} \subset S_{k_1+k_2+k_3} \subset \cdots \subset S_{k_1+k_2+\cdots+k_{m-1}} \subset [n] = S_n$

passing through ranks $0, k_1, k_1+k_2, \ldots, k_1+k_2+\cdots+k_{m-1}, n \}$

$\binom{7}{3,1,2,1}$ e.g. BANANAS (positions 1 2 3 4 5 6 7) = 2 1 3 1 3 1 4 $\leftrightarrow$ $\emptyset \overset{3}{\subset} S_3 \overset{1}{\subset} S_4 \overset{2}{\subset} S_6 \overset{1}{\subset} [7]$

$S_3 = \{2,4,6\}$, $S_4 = \{1,2,4,6\}$, $S_6 = \{1,2,3,4,5,6\}$

Note $\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-(k_1+k_2)}{k_3}\cdots\binom{n-(k_1+k_2+\cdots+k_{m-2})}{k_{m-1}}\binom{k_m}{k_m}$

Also $(x_1 + x_2 + \cdots + x_m)^n = \sum_{(k_1, \ldots, k_m):\ \sum_{i=1}^m k_i = n} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} \cdots x_m^{k_m}$

$\parallel$

$(x_1 + \cdots + x_m)(x_1 + \cdots + x_m) \cdots (x_1 + \cdots + x_m)$

take x_1 from k_1 parentheses