A TRIDIAGONAL APPROACH TO MATRIX INTEGRALS

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Abstract. Physicists in the 70's starting with 't Hooft established that the number of suitably labeled planar maps with prescribed vertex degree distribution can be represented as the leading coefficient of the $\frac{1}{N}$-expansion of a joint cumulant of traces of powers of a standard N-by-N GUE matrix. Here we undertake the calculation of this leading coefficient in a different way, namely, after first tridiagonalizing the GUE matrix *à la* Trotter and Dumitriu-Edelman and then "de-symmetrizing," we apply the cluster expansion technique (specifically, the Brydges-Kennedy-Abdesselam-Rivasseau formula) from rigorous statistical mechanics. We thus arrive at our main result, which is an alternate combinatorial interpretation for the leading coefficient in terms of edge-labeled planar trees equipped with a vertex-four-coloring subject to certain simple rules. Objects of the latter type, without matching up exactly, bear a family resemblance to the well-labeled trees already in common use to enumerate planar maps, e.g., those of Cori-Vauquelin and of Schaeffer. By using a combinatorial insight of Goulden-Jackson, we can straightforwardly reconcile our main result with a formula of Tutte from the 60's counting rooted planar maps with a prescribed Eulerian (all degrees even) vertex degree distribution. (But our main result has no Eulerian hypotheses.) Ultimately the contribution of the paper is simply to demonstrate the close connection of tridiagonalized GUE matrices to well-labeled trees.

Contents

1. Introduction and main results

Physicists in the 70's starting with 't Hooft [27] developed a beautiful combinatorial interpretation for the limit

$$\lim_{N\to\infty} N^{2-\ell-\frac{n}{2}}\kappa\left(\operatorname{tr}\Xi_N^{\lambda_1},\dots,\operatorname{tr}\Xi_N^{\lambda_\ell}\right) \tag{1}$$

where λ is a partition, $n=|\lambda|$, $\ell=\ell(\lambda)$, Ξ_N is an N-by-N standard GUE matrix, and $\kappa(\cdot)$ is the joint cumulant functional. Namely, the limit (1) was interpreted as the number of suitably labeled planar maps with vertex degree distribution λ. We recall details of this interpretation, including notation and terminology, later in this introduction.

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The goal of this paper is to recalculate the limit (1) using a different toolbox to get a different combinatorial interpretation of the same number. Of course the physicists went much farther and developed an interpretation for all coefficients of the $\frac{1}{N}$-expansion of the joint cumulant appearing in the limit (1) in terms of higher genus maps. For simplicity we focus in this paper exclusively on the leading order, not venturing beyond genus zero. It seems, however, that our method is not intrinsically limited to genus zero.

Here is an outline of our recomputation of (1). The first stage is to replace the N-by-N GUE matrix Ξ_N by its (lightly modified) tridiagonalization *à la* Trotter [42] and Dumitriu-Edelman [21]. (We carry out this relatively easy step in §1.5 below. For background on tridiagonalization see [5, Section 4.5].) The second stage is to apply the Brydges-Kennedy-Abdesselam-Rivasseau (BKAR) formula from rigorous statistical mechanics to obtain a delicate expansion of the joint cumulant under the limit in (1). (We carry out this step in §2.6 below after setting up the needed machinery in the preceding part of §2. The formula in question is (54) below.) It is worth remarking that this is probably the first paper in which the BKAR formula has been used to perform an exact calculation; ordinarily one applies it only to obtain upper bounds on joint cumulants. We will discuss the BKAR formula in §2, including background, references and a short proof. We therefore omit (nearly all) further discussion of it in this introduction. The third and last stage of our re-calculation is to analyze formula (54) in order to obtain our main result. (The calculations in question are carried out in §3 and §4 below. The main result is stated as Theorem 1.2.5 below and further elaborated in graphical terms in §1.6 below.)

Our main result reinterprets the limit (1) in terms of vertex-four-colored edge-labeled planar trees subject to simple coloring rules. Objects of the latter type have the following notable properties:

- They bear a family resemblance to the well-labeled trees already commonly used to enumerate planar maps, e.g., those of Cori-Vauquelin [20] and of Schaeffer [36]. But they do not match up exactly.
- They are amenable to analysis by means of a combinatorial insight of Goulden-Jackson [25] concerning factorizations of the cycle $(12\cdots n) \in S_n$.
- They are simple examples of Grothendieck's *dessins d'enfants* [37], [39] albeit with extra colors.

Thus it is not surprising that we can straightforwardly reconcile our combinatorial interpretation of (1) with a formula of Tutte [43] for the number of Eulerian (all degrees even) rooted planar maps with prescribed vertex degree distribution.

It is striking that the tridiagonalization/BKAR process somehow "reprograms" the usual Gaussian calculus to yield well-labeled trees instead of planar maps. The contribution of this paper is simply to demonstrate this connection. Much further work remains to be done in order to exploit this connection fully.

In the remainder of this introduction we formulate our main result precisely and develop in some detail several themes briefly stated above. In particular, we reconcile our main result with the formula of Tutte mentioned above and we entertain (without being able to answer definitively) the question of which species of already-invented well-labeled tree is closest to the one arising here.

1.1. **Table of notation.** We briefly mention the most basic items of notation and terminology used throughout the paper. The reader should scan the table once quickly and then use it as a reference.

1.1.1. *General notation and terminology.* Let $|S|$ denote the cardinality of a finite set S. Let $\mathbb{1}\{\cdot\}$ be probabilist's indicator notation. The (i,j)-entry of a matrix A is invariably denoted by $A(i,j)$. Let $\langle n\rangle = \{1,\dots,n\}$ for positive integers n. Let Part_n denote the lattice of partitions of the set $\langle n\rangle$. (For further notation related to Part_n, see §2 below.)

1.1.2. *Numerical partitions.* A *numerical partition* (or simply *partition*, context permitting) is a monotone decreasing sequence $\lambda = \{\lambda_i\}_{i=1}^\infty$ of nonnegative integers such that $\lambda_i = 0$ for $i \gg 0$. The (nonzero) terms λ_i are called the *parts* of λ. By and large we follow notation of Macdonald [32]. Let $|\lambda| = \sum_i \lambda_i$. We also write $\lambda \vdash n \Leftrightarrow |\lambda| = n$. Let $m_i(\lambda) = |\{j \mid \lambda_j = i\}|$. Let $\ell(\lambda) = |\{i \mid \lambda_i > 0\}| = \sum_i m_i(\lambda)$. Let $z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)!$. Abusing notation we occasionally write $\lambda = \prod_i i^{m_i(\lambda)}$.

1.1.3. *Graphs.* For us a *graph* is a finite set of vertices and a finite set of edges along with the specification of an incidence relation which designates for each edge a set of one or two endpoints among the vertices. Furthermore, in the case of graphs without multiple edges, we simply identify edges with their endpoint sets.

1.1.4. *Permutations.* Let S_n denote the group of permutations of $\langle n\rangle$. For $\sigma \in S_n$, let $\mathrm{supp}\,\sigma = \{i \in \langle n\rangle \mid \sigma(i) \neq i\}$, which we call the *support* of σ. In other words, $\mathrm{supp}\,\sigma$ is the complement of the set of fixed points of σ. A *cycle* in S_n is a permutation with nonempty support on which it acts transitively. The *length* of a cycle is the cardinality of its support. Cycles are called *disjoint* if they have disjoint supports. A cycle of length m is called an *m-cycle*. (In our usage there are no 1-cycles.) A 2-cycle is called a *transposition*. Each $\sigma \in S_n$ has a factorization into disjoint cycles unique up to ordering of the factors, hereafter called simply the *canonical factorization* of σ. For $\sigma \in S_n$, let $\mathrm{Orb}_n(\sigma) \in \mathrm{Part}_n$ denote the finest partition consisting of σ-stable blocks. Blocks of $\mathrm{Orb}_n(\sigma)$ are called *σ-orbits*. (Whereas 1-cycles are disallowed here, σ-orbits may of course be singletons.) As usual we index the conjugacy class of a permutation $\sigma \in S_n$ by the numerical partition $\lambda \vdash n$ recording the cardinalities of the blocks of the set partition $\mathrm{Orb}_n(\sigma)$. We also write $\ell(\sigma) = |\mathrm{Orb}_n(\sigma)| = \ell(\lambda)$ and $\sigma \sim \lambda$.

1.2. **A statement of the main result.** We now formulate the main result of the paper in purely combinatorial terms completely hiding the method of proof.

1.2.1. *The set* Map_n. Let Map_n denote the set of ordered pairs $(\theta,\iota) \in S_n \times S_n$ of permutations satisfying the following conditions:

$$\iota \text{ is fixed-point-free and squares to the identity.} \tag{2}$$

$$\ell(\theta) - \ell(\iota) + \ell(\theta\iota) = 2, \text{ cf. Euler's formula } V - E + F = 2. \tag{3}$$

$$\theta \text{ and } \iota \text{ generate a subgroup of } S_n \text{ acting transitively on } \langle n\rangle. \tag{4}$$

For convenience we also define

$$\mathrm{Map}_n(\theta) = \{\iota \in S_n \mid (\theta,\iota) \in \mathrm{Map}_n\}.$$

The set Map_n is allied with planar maps in a fashion we recall briefly in §1.3 below.

1.2.2. *The set* GJ_n. Let GJ_n denote the set of ordered pairs $(\theta,\sigma) \in S_n \times S_n$ of permutations satisfying

$$\ell(\theta) + \ell(\sigma) = n+1 \text{ and } \ell(\theta\sigma) = 1. \tag{5}$$

Members of GJ_n will be called *Goulden-Jackson* pairs. For convenience we define

$$\mathrm{GJ}_n(\theta) = \{\sigma \in S_n \mid (\theta,\sigma) \in \mathrm{GJ}_n\}.$$

In §1.6 we recall the interpretation of elements of GJ_n in terms of planar trees.

1.2.3. *The set* $\mathrm{dMotz}_n(\theta,\sigma)$. Given $(\theta,\sigma) \in \mathrm{GJ}_n$, let $\mathrm{dMotz}_n(\theta,\sigma)$ denote the set of functions $g : \langle n\rangle \to \mathbb{Z}$ with the following properties:

$$|g| \le 1. \tag{6}$$

$$g \text{ averages to 0 on } \theta\text{-orbits.} \tag{7}$$

$$g \circ \sigma = g. \tag{8}$$

$$\{g = -1\} \cap \operatorname{supp}\sigma = \emptyset. \tag{9}$$

$$\{g = 0\} \cap \operatorname{supp}\sigma^2 = \emptyset. \tag{10}$$

$$\{g = 0\} \subset \operatorname{supp}\sigma. \tag{11}$$

The rationale for the (ungainly) notation dMotz is given in Proposition 3.5.5 below.

1.2.4. *The sets* GJdM_n *and* $\mathrm{GJdM}_n(\theta)$. Combining notions introduced above, we define the following more complicated sets:

$$\mathrm{GJdM}_n = \{(\theta,\sigma,g) \in \mathrm{GJ}_n \times \{0,\pm1\}^{\langle n\rangle} \mid g \in \mathrm{dMotz}_n(\theta,\sigma)\}. \tag{12}$$

$$\mathrm{GJdM}_n(\theta) = \{(\sigma,g) \in S_n \times \{0,\pm1\}^{\langle n\rangle} \mid (\theta,\sigma,g) \in \mathrm{GJdM}_n\}. \tag{13}$$

We briefly indicate in §1.6 below a graphical interpretation for members of GJdM_n in terms of vertex-four-colored edge-labeled planar trees.

Here is the main result of the paper.

Theorem 1.2.5. *For all* $\theta \in S_n$ *one has*

$$\left(\frac{n}{2} - \ell(\theta) + 2\right)|\mathrm{Map}_n(\theta)| = |\mathrm{GJdM}_n(\theta)|. \tag{14}$$

The proof of Theorem 1.2.5 commences in §2 and takes up the rest of the paper. Our proof of Theorem 1.2.5 is analytic. The problem of finding a constructive bijective proof is tantalizingly open; see §1.7 below for further comment on this point.

1.3. **Planar maps and Tutte's formula.** We recall intuitions guiding the study of the set Map_n, introduce notation needed throughout the paper and finally recall a famous result of Tutte.

1.3.1. *The link between planar maps and permutation pairs.* A *planar map* is a cellular decomposition of the 2-sphere with connected 1-skeleton. We call 0-cells (resp., 1-cells and 2-cells) *vertices* (resp., *edges* and *faces*). Each edge is viewed as two half-edges stuck together. The *degree* of a vertex is the number of half-edges incident upon it. A *half-edge-labeled* planar map of n half-edges is a planar map equipped with a numbering from 1 to n of its half-edges. Each half-edge-labeled planar map of n half-edges gives rise to a permutation pair $(\theta,\iota) \in \mathrm{Map}_n$ by the following procedure. Let θ be the permutation whose cycles record in counterclockwise order the labels of half-edges sprouting from the vertices of degree > 1; every label of a half-edge terminating in a vertex of degree 1 is a fixed point of θ. Let ι be the permutation which exchanges labels of half-edges belonging to the same edge. It is well-known that every element $(\theta, \iota) \in \mathrm{Map}_n$ arises from a half-edge-labeled planar map in the manner just specified. We regard two half-edge-labeled planar maps as equivalent if both give rise to the same element of Map_n. See the series of three survey papers [19] for an introduction to the point of view emphasizing permutation pairs. Generally our attitude is that permutation pairs are the objects of rigorous study in this paper, whereas we view planar maps and related sorts of graphs as (very appealing) heuristic devices.

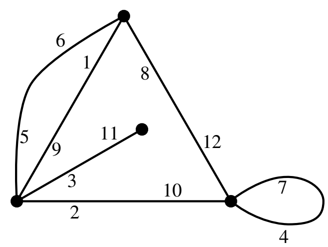

FIGURE 1. This drawing represents a half-edge-labeled planar map corresponding to the pair $(\theta, \iota) \in \mathrm{Map}_{12}$ where $\theta = (1, 8, 6)(2, 3, 9, 5)(4, 7, 12, 10)$ and $\iota = (1, 9)(2, 10)(3, 11)(4, 7)(5, 6)(8, 12)$. The vertex degree distribution of the planar map depicted here is the numerical partition $4^2 \cdot 3^1 \cdot 1^1$.

1.3.2. *Rooted planar maps.* A *rooted planar map* of n half-edges is (in effect) a half-edge-labeled planar map from which one erases all of the labels but n. Let us identify S_{n-1} with the subgroup of S_n consisting of permutations fixing the point n and let S_{n-1} act on Map_n by simultaneous conjugation, i.e., the action of $\rho \in S_{n-1}$ on $(\theta, \iota) \in \mathrm{Map}_n$ is $(\rho\theta\rho^{-1}, \rho\iota\rho^{-1}) \in \mathrm{Map}_n$. Then rooted planar maps (up to equivalence) are indexed by the orbit space Map_n/S_{n-1}.

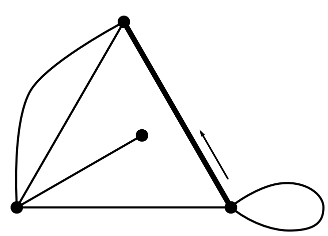

FIGURE 2. This drawing shows the rooted planar map arising by erasure of labels from the half-edge-labeled planar map depicted in Figure 1.

Lemma 1.3.3. S_{n-1} *acts simply transitively on* Map_n.

Proof. Fix $\rho \in S_{n-1}$ and $(\theta, \iota) \in \mathrm{Map}_n$ such that $(\theta, \iota) = (\rho\theta\rho^{-1}, \rho\iota\rho^{-1})$, i.e., such that ρ commutes with both θ and ι. It is enough to show that $\rho = 1$. In any case, the set of points of $\langle n \rangle$ fixed by ρ is not empty and moreover stable under the action of the group of permutations generated by θ and ι. But the latter group by definition of Map_n acts transitively on $\langle n \rangle$. Thus every point of $\langle n \rangle$ is fixed by ρ. $\square$

1.3.4. *The numbers $\mathfrak{M}_\lambda$ and $\mathfrak{M}^\star_\lambda$.* Let $\lambda \vdash n$ be a partition and let $\ell = \ell(\lambda)$. For any permutation $\theta \in S_n$ belonging to the conjugacy class indexed by λ let

$$\mathfrak{M}_\lambda = |\mathrm{Map}_n(\theta)|.$$

The number $\mathfrak{M}_\lambda$ is well-defined because the number on the right depends only on the conjugacy class of θ. Let $\mathrm{Map}_{\lambda\vdash n}$ denote the subset of Map_n consisting of pairs (θ, ι) such that θ belongs to the conjugacy class indexed by λ. Clearly $\mathrm{Map}_{\lambda\vdash n}$ is stable under the action of S_{n-1}. Let

$$\mathfrak{M}^\star_\lambda = |\mathrm{Map}_{\lambda\vdash n}/S_{n-1}|.$$

The number $\mathfrak{M}^\star_\lambda$ counts (equivalence classes of) rooted planar maps having vertex degree distribution λ. Since $\frac{n!}{z_\lambda}$ is the cardinality of the conjugacy class in S_n indexed by λ and S_{n-1} acts freely on Map_n by Lemma 1.3.3, one has the comparison formula

$$\mathfrak{M}_\lambda = \frac{z_\lambda}{n}\mathfrak{M}^\star_\lambda. \tag{15}$$

Thus the numbers $\mathfrak{M}_\lambda$ and $\mathfrak{M}^\star_\lambda$ carry the same information even if they have rather different connotations.

1.3.5. *Counting of Eulerian rooted planar maps, following Tutte.* Recall that planar maps with all vertex degrees even are called *Eulerian.* Now let $\lambda \vdash n$ and $\ell = \ell(\lambda)$, as above. Also let $m_i = m_i(\lambda)$. Assume that every part λ_i is even so that λ is a possible vertex-degree distribution of an Eulerian planar map. Tutte [43] has given in the Eulerian case a simple explicit formula for the number $\mathfrak{M}^\star_\lambda$, namely

$$\mathfrak{M}^\star_\lambda = \frac{2(\frac{n}{2})!}{(\frac{n}{2}-\ell+2)!}\cdot\prod_{i\geq 1}\frac{1}{m_{2i}!}\left(\begin{array}{c}2i-1\\ i\end{array}\right)^{m_{2i}}. \tag{16}$$

See also [36] for a more recent proof of this same formula by an elegant construction of a bijection. Using (15) above we can rewrite Tutte's formula (16) as

$$\mathfrak{M}_\lambda = \frac{(\frac{n}{2}-1)!}{(\frac{n}{2}-\ell+2)!}\cdot\prod_{i=1}^{\ell}\frac{\lambda_i}{2}\cdot\prod_{i=1}^{\ell}\left(\begin{array}{c}\lambda_i\\ \lambda_i/2\end{array}\right). \tag{17}$$

We will find the latter presentation of Tutte's result more convenient.

1.4. **Enumeration of planar maps via matrix integrals.** We turn next to the physicists' point of view on the numbers $\mathfrak{M}_\lambda$.

1.4.1. *Standard GUE matrices.* A random N-by-N hermitian matrix Ξ is called a *standard GUE matrix* if its law has the density $\exp\left(-\frac{1}{2}\mathrm{tr}\, H^2\right)$ with respect to Lebesgue measure, up to a normalization factor. Equivalently, one requires the family $\{\Xi(i,j)\}_{1\leq i\leq j\leq N}$ of matrix entries on or above the diagonal to be independent and to have a centered Gaussian joint distribution characterized by $\mathbf{E}\Xi(i,j)^2 = \delta_{ij}$ and $\mathbf{E}|\Xi(i,j)|^2 = 1$.

1.4.2. *The number $\mathfrak{M}_{\lambda,N}$ and its leading order behavior.* Let λ be a numerical partition and let $\ell = \ell(\lambda)$. Let N be a positive integer. Let Ξ_N be a standard N-by-N GUE matrix. Let

$$\mathfrak{M}_{\lambda,N} = \kappa\left(\mathrm{tr}\,\Xi_N^{\lambda_1}, \ldots, \mathrm{tr}\,\Xi_N^{\lambda_\ell}\right) \tag{18}$$

where $\kappa(\cdot)$ is the joint cumulant functional. (See §2.1 below to be reminded of the definition and first properties of joint cumulants.) Physicists in the 1970's obtained the limit formula

$$\mathfrak{M}_\lambda = \lim_{N\to\infty} N^{\ell-2-\frac{n}{2}}\mathfrak{M}_{\lambda,N}. \tag{19}$$

The right side here is precisely the limit (1) with which we began the introduction. More generally physicists derived for $\mathfrak{M}_{\lambda,N}$ an asymptotic expansion in powers of $\frac{1}{N}$ with coefficients counting diagrams of higher genus. But in this paper we will be content to study genus zero (leading order) behavior only.

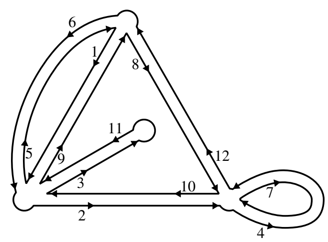

FIGURE 3. This drawing is a rendering of Figure 1 as a *fat graph* or *ribbon graph*, similar to diagrams used in the physics literature.

1.4.3. *Notes and references.* The paper [27] is recognized as the initiation of GUE enumeration of maps although no formula recognizable to a mathematician as (19) could be found there. Many hands subsequently developed the theory around formula (19). Without any pretension to completeness, we mention the references [23], [26], [30] and [44] as ways to enter this vast territory.

1.4.4. *Remark.* Using Weisner's theorem [41, Cor. 3.9.3, p. 275] in conjunction with the Isserlis-Wick formula (30) recalled below, it is possible to give a proof of formula (19) completely within the domain of algebraic combinatorics as, say, summed up in Stanley's text [41], using the permutation pair point of view. One is not obliged to use Feynman-like diagrams although it is hard to deny their intuitive appeal.

1.4.5. *Remark.* To prove Theorem 1.2.5 we will deal with the number $\mathfrak{M}_\lambda$ solely via formula (19). Without loss of comprehension, from §2 of the paper onward, the reader could take formula (19) as the definition of $\mathfrak{M}_\lambda$. The point of the paper really is to provide a re-interpretation "from scratch" of the limit, including a self-contained proof of the existence of the limit.

1.5. **The tridiagonal representation of $\mathfrak{M}_{\lambda,N}$.** We carry out the first step of our recalculation of the limit (1).

1.5.1. *Tridiagonalization of standard GUE matrices.* Let N be a positive integer which eventually we send to infinity. Let Ξ_N be an N-by-N standard GUE matrix. The result of applying to Ξ_N the well-known Givens-Householder tridiagonalization procedure (albeit starting at the lower right corner rather than the upper left) yields a random matrix with the same law as that of the random matrix

$$\left[\begin{array}{ccccccc} \xi_1 & \sqrt{\eta_1} & & & & & \\ \sqrt{\eta_1} & \xi_2 & \sqrt{\eta_2} & & & & \\ & \sqrt{\eta_2} & \xi_3 & \sqrt{\eta_3} & & & \\ & & \sqrt{\eta_3} & \ddots & \ddots & & \\ & & & \ddots & \xi_{N-1} & \sqrt{\eta_{N-1}} \\ & & & & \sqrt{\eta_{N-1}} & \xi_N \end{array}\right], \tag{20}$$

where the family

$$\{\xi_i\}_{i=1}^N \cup \{\eta_i\}_{i=1}^{N-1}$$

of real random variables is independent, each random variable ξ_i is standard normal and each random variable η_i has a distribution of Γ-type for which

$$\frac{e^{-x}x^{i-1}}{(i-1)!}\mathbb{1}\{x>0\}$$

is the probability density function. The tridiagonalization procedure does not change the law of the spectrum, and so in the definition (18) one could in principle replace Ξ_N by the random matrix (20).

1.5.2. *Notes and references on tridiagonalization.* The idea to approach the semicircle law for GOE matrices through tridiagonalization is due to Trotter [42]. The idea of tridiagonalization was later developed to yield tridiagonal matrix models for Gaussian β-ensembles by Dumitriu-Edelman [21] and developed further by these authors to yield a CLT [22]. The matrix (20) is the case $\beta = 2$ of the Dumitriu-Edelman model for the Gaussian β-ensemble. Here, for simplicity, we do not push our analysis beyond the case of $\beta = 2$. See [5, Section 4.5] for background on tridiagonalization.

1.5.3. *Another tridiagonal representation of $\mathfrak{M}_{\lambda,N}$.* Now the random variables $\sqrt{\eta_i}$ are not so nice for our purposes. Accordingly, we conjugate the matrix (20) suitably to get another tridiagonal matrix

$$\mathrm{Tri}_N = \left[\begin{array}{cccccc} \xi_1 & \eta_1 & & & & \\ 1 & \xi_2 & \eta_2 & & & \\ & 1 & \xi_3 & \eta_3 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \xi_{N-1} & \eta_{N-1} \\ & & & & 1 & \xi_N \end{array}\right] \tag{21}$$

the spectrum of which has the same law. Replacing Ξ_N in formula (18) by Tri_N we obtain the formula

$$\mathfrak{M}_{\lambda,N} = \kappa\left(\mathrm{tr}\,\mathrm{Tri}_N^{\lambda_1}, \ldots, \mathrm{tr}\,\mathrm{Tri}_N^{\lambda_\ell}\right) \tag{22}$$

holding for every partition λ, where as usual $\ell = \ell(\lambda)$.

1.5.4. *Plan of proof.* To prove Theorem 1.2.5 we will use formula (22) rather than formula (18) to evaluate the limit on the right side of (19), i.e., the limit (1). The perhaps unexpected extra ingredient in our calculation is the BKAR formula from rigorous statistical mechanics. (See Theorem 2.3.2 and its application Theorem 2.4.6 below.) The BKAR formula will permit us to control cancellation on the right side of formula (22) by means of repeated integration by parts. The evaluation of the limit on the right side of (19) by the tridiagonal/BKAR route will take up the rest of the paper from §2 onward.

1.6. **Recovery of Tutte's formula from Theorem 1.2.5.** Our goal here is to reconcile formulas (14) and (17) by showing directly that their right sides are equal. The calculations needed to do this are completed in §1.6.8 below after suitable preparation. In contrast to our essentially analytical *modus operandi* in the rest of the paper, under this heading we use a relatively informal "tree-surgical" approach to enumeration which is reminiscent of the method of [36] without matching up exactly. As a byproduct of our discussion under this heading we provide a simple graphical interpretation for each member of the set GJdM_n.

1.6.1. *Shabat-Voevodsky trees.* The simplest examples of the *dessins d'enfants* introduced by Grothendieck (see [37] for background) are the two-colored planar trees. These objects come to number-theoretic life in connection with the Shabat-Voevodsky polynomials [39]. See also the short note [10] for a beautiful if not entirely elementary construction of these polynomials. We will not delve into the theory of *dessins d'enfants* here, but we will acknowledge the tangential relationship of our work to this theory by calling a bipartite edge-labeled planar tree (vertices colored white and black, with no two adjacent vertices of the same color, and with edges numbered from 1 to n, where n is the number of edges) a *Shabat-Voevodsky tree*. Let SV_n denote the set of (equivalence classes of) Shabat-Voevodsky trees of n edges.

1.6.2. *Generalized definitions.* For technical flexibility we need to generalize several definitions given above in a harmless way. Let A be any finite set and let $n = |A|$. Let S_A denote the group of permutations of the set A. Let SV_A denote the set of (equivalence classes of) Shabat-Voevodsky trees with n edges labeled by distinct elements of the set A rather than by distinct elements of the set $\langle n\rangle$. In the same spirit, let $\mathrm{GJ}_A \subset S_A \times S_A$ denote the subset defined by evident analogy with the definition of GJ_n in the case $A = \langle n\rangle$.

Proposition 1.6.3. *For finite sets A, the sets SV_A and GJ_A are canonically in bijection.*

This is a commonplace both in the theory of *dessins d'enfants* and in combinatorics in relation to the problem of calculating connection coefficients for conjugacy classes of the symmetric group.

Proof. Given a Shabat-Voevodsky tree $\mathfrak{T}$ belonging to SV_A, by writing down for each white vertex of degree > 1 in counterclockwise order the labels of edges incident on the vertex, one obtains a permutation $\theta_{\mathfrak{T}} \in S_A$ canonically factored into cycles; the label of each edge terminating in a white leaf is a fixed point of $\theta_{\mathfrak{T}}$. Similarly one obtains a permutation $\sigma_{\mathfrak{T}} \in S_A$ by reversing the roles of white and black. One checks immediately that $(\theta_{\mathfrak{T}}, \sigma_{\mathfrak{T}}) \in \mathrm{GJ}_A$, and that every $(\theta, \sigma) \in \mathrm{GJ}_A$ so arises in an essentially unique way. See drawing (a) in Figure 4 below for an illustration of the passage from a Goulden-Jackson pair to a Shabat-Voevodsky tree. The notation $\theta_{\mathfrak{T}}$ and $\sigma_{\mathfrak{T}}$ introduced in this proof will be needed below to complete the job of reconciling Theorem 1.2.5 with Tutte's formula. □

Lemma 1.6.4. *For partitions $\lambda \vdash n$ and $\theta \in S_n$ such that $\theta \sim \lambda$ we have*

$$|\mathrm{GJ}_n(\theta)| = \frac{(n-1)!}{(n-\ell+1)!} \cdot \prod_{i=1}^{\ell} \lambda_i \tag{23}$$

where $\ell = \ell(\lambda)$.

Proof. Let μ be any partition such that $\ell(\lambda)+\ell(\mu) = n+1$ and $|\lambda| = |\mu| = n$. The result [25, Thm. 2.2] translated into the present setup says that

$$|\{(\rho,\sigma) \in S_n \times S_n \mid \rho\sigma = (1\cdots n),\ \rho \sim \lambda \text{ and } \sigma \sim \mu\}| = n\frac{(\ell(\lambda)-1)!(\ell(\mu)-1)!}{\prod_i m_i(\lambda)! \prod_j m_j(\mu)!}.$$

We note that the statement above is originally due to other authors (see [7]) and that it was originally proved by an inductive method. We note also that the main goal of [25] was to give a different bijective proof of the same result. The idea animating the latter proof we have recapitulated as Proposition 1.6.3 above. It follows that

$$|\{(\rho,\sigma) \in \mathrm{GJ}_n \mid \rho \sim \lambda \text{ and } \sigma \sim \mu\}| \;=\; n!\frac{(\ell(\lambda)-1)!(\ell(\mu)-1)!}{\prod_i m_i(\lambda)! \prod_j m_j(\mu)!}, \quad \text{hence}$$

$$|\{\sigma \in \mathrm{GJ}_n(\theta) \mid \sigma \sim \mu\}| \;=\; z_\lambda \cdot \frac{(\ell(\lambda)-1)!(\ell(\mu)-1)!}{\prod_i m_i(\lambda)! \prod_j m_j(\mu)!} \quad \text{and finally}$$

$$|\mathrm{GJ}_n(\theta)| = (\ell(\lambda)-1)! \cdot (n-\ell(\lambda))! \cdot \prod_{i=1}^{\ell(\lambda)} \lambda_i \cdot \sum_{\substack{\mu \text{ s.t. } |\mu|=n \text{ and} \\ \ell(\mu)=n+1-\ell(\lambda)}} \frac{1}{\prod_j m_j(\mu)!}.$$

The sum at extreme right can then be evaluated with the help of the formal power series identity

$$\sum_\nu \frac{x^{\ell(\nu)} y^{|\nu|}}{\prod_i m_i(\nu)!} = \exp\left(\frac{xy}{1-y}\right) = \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} \frac{(n-1)!}{(n-\ell)!\ell!(\ell-1)!} x^\ell y^n,$$

where the sum on the extreme left is extended over all numerical partitions ν. The proof is complete. □

1.6.5. *Graphical interpretation of the set* GJdM_n. Fix $(\theta,\sigma) \in \mathrm{GJ}_n(\theta,\sigma)$ and an element $g \in \mathrm{dMotz}_n(\theta,\sigma)$. Let $\mathfrak{T}_n(\theta,\sigma) \in \mathrm{SV}_n$ be a Shabat-Voevodsky tree from which one recovers the pair (θ,σ). By (8), the function g factors through $\mathrm{Orb}_n(\sigma)$ and thus may be construed as a function defined on the set of black vertices of $\mathfrak{T}_n(\theta,\sigma)$. In other words, g can be interpreted as a "painting over" of the black vertices of $\mathfrak{T}_n(\theta,\sigma)$ using three new colors, say blue, green, and red, corresponding to 0, -1 and 1, respectively. Let $\mathfrak{T}_n(\theta,\sigma;g)$ denote the resulting four-colored edge-labeled planar tree of n edges. Thus we have constructed a bijection identifying GJdM_n with the set of (equivalence classes of) edge-labeled vertex-colored planar trees of n edges where the vertex-coloring has to obey the following rules:

- Only four colors (blue, white, green and red) are used altogether.
- In any pair of adjacent vertices, exactly one is white.
- Every white vertex has as many red neighbors as green, cf. (7).

- Every green vertex has degree one, i.e., is a leaf, cf. (9).
- Every blue vertex has degree two, cf. (10) and (11).

If we restrict attention to the set $\mathrm{GJdM}_n(\theta)$, then the representing trees $\mathfrak{T}$ are required to satisfy the further condition that $\theta = \theta_{\mathfrak{T}}$.

Lemma 1.6.6. *Fix* $(\theta, \sigma, g) \in \mathrm{GJdM}_n$ *such that every* θ*-orbit has even cardinality. (i) Then* $|\{g = -1\} \cap A| = |A|/2$ *for every block* $A \in \mathrm{Orb}_n(\theta)$*. (ii) Furthermore, every element of* $\{g = -1\}$ *is a fixed point of* σ*.*

Proof. Let $\mathfrak{T} = \mathfrak{T}(\theta, \sigma; g)$. In graphical language, the claim being made here is that for every white vertex of $\mathfrak{T}$ exactly half of its neighbors are green leaves. In view of the coloring rules we have only to rule out the existence of blue vertices. In any case, every white vertex of $\mathfrak{T}$ has an even number of blue neighbors. Were $\mathfrak{T}$ to have at least one blue vertex, the coloring rules would force a circuit to exist, which is a contradiction. □

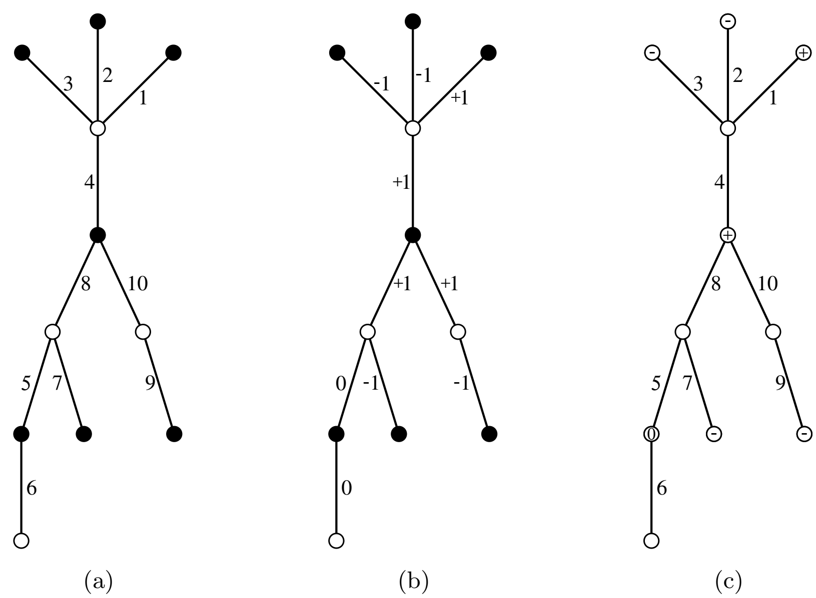

FIGURE 4. Let $\theta = (1, 2, 3, 4)(5, 7, 8)(9, 10)$ and $\sigma = (4, 8, 10)(5, 6)$. Then $(\theta, \sigma) \in \mathrm{GJ}_{10}$. The corresponding Shabat–Voevodsky tree in SV_{10} is (a), and (b) illustrates an element g from $\mathrm{dMotz}_{10}(\theta, \sigma)$. The triple $(\theta, \sigma; g) \in \mathrm{GJdM}_{10}$ is encoded by $\mathfrak{T}_{10}(\theta, \sigma; g)$ in (c).

1.6.7. *The cancellation construction.* Let $X \subset A$ be an inclusion of finite sets. Let $\tau \in S_A$ be a permutation. For $i \in A \setminus X$, let $\mu(\tau, A, X, i)$ be the least of the positive integers m such that $\tau^m(i) \in A \setminus X$. We define $\tau \backslash X \in S_{A \backslash X}$ by the formula $(\tau \backslash X)(i) = \tau^{m(\tau, A, X, i)}(i)$ for $i \in A \setminus X$. A more intuitively accessible if less precise description of $\tau \backslash X$ is as follows. Firstly, one writes out the canonical factorization of τ. Secondly, one strikes all elements of X from the factorization. Thirdly and finally, one discards all cycles reduced to length ≤ 1 by the operation of striking elements of X. The resulting expression is then the canonical factorization of $\tau \backslash X$.

1.6.8. *Snipping off green leaves.* Fix an Eulerian partition λ (all parts even) along with some $\theta \in S_n$ such that $\theta \sim \lambda$. As usual let $\ell = \ell(\lambda) = \ell(\theta)$. Fix any set $X \subset \langle n \rangle$ intersecting each block $A \in \mathrm{Orb}_n(\theta)$ in a set of cardinality $|A|/2$. Let $\mathrm{GJdM}_n(\theta, X)$ denote the subset of $\mathrm{GJdM}_n(\theta)$ consisting of (σ, g) such that $\{g = -1\} = X$. In order to reconcile the expression on the right side of (17) with the expression on the right side of (14), it will be enough by Lemma 1.6.6 to prove that

$$|\mathrm{GJdM}_n(\theta, X)| = \frac{(\frac{n}{2}-1)!}{(\frac{n}{2}-\ell+1)!}\prod_{i=1}^{\ell}\frac{\lambda_i}{2}. \tag{24}$$

Now pick $(\sigma, g) \in \mathrm{GJdM}_n(\theta, X)$ arbitrarily and let $\mathfrak{T} = \mathfrak{T}(\theta, \sigma; g)$. In turn, let $\mathfrak{T}'$ be the object obtained from $\mathfrak{T}$ by snipping off each green leaf and attached "stem," while leaving the white vertex at the other end in place, and blackening all red vertices. We emphasize that the sets of white vertices of $\mathfrak{T}$ and $\mathfrak{T}'$ are exactly the same. Then (the equivalence class of) the object $\mathfrak{T}'$ belongs to $\mathrm{SV}_{\langle n\rangle\setminus X}$ and satisfies $\theta_{\mathfrak{T}'} = \theta\setminus X$. Now on the one hand, given $\mathfrak{T}'$, it is clear how to reconstruct $\mathfrak{T}$ by reattaching the green leaves and labeled stems that were snipped off. On the other hand, the possible objects $\mathfrak{T}'$ are counted (up to equivalence) by Lemma 1.6.4, and thus (24) indeed holds. In this way the right sides of (14) and (17) are reconciled.

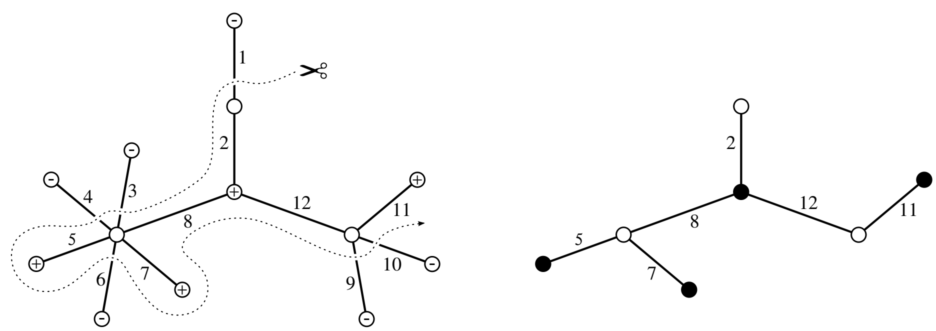

FIGURE 5. Snipping green leaves off the (left) tree $\mathfrak{T}(\theta, \sigma; g)$ in $\mathrm{GJdM}_{12}(\theta, X)$ where $\theta = (1,2)(3,4,5,6,7,8)(9,10,11,12)$, $X = \{1,3,4,6,9,10\}$, $\theta \setminus X = (5,7,8)(11,12)$, and $\sigma = (2,8,12)$, then blackening red vertices, gives the (right) Shabat–Voevodsky tree $\mathfrak{T}'$ in $\mathrm{SV}_{\langle 12\rangle\setminus X}$ for which $\theta_{\mathfrak{T}'} = \theta \setminus X$.

1.7. **Well-labeled trees, the Brownian map and a bijective proof of Theorem 1.2.5.** A constructive bijective proof of Theorem 1.2.5 is much to be desired. To obtain such a thing would be to add to the already large stock of known bijections between sets of well-labeled planar trees and sets of planar maps. Lately research into such bijections is being driven by intense activity in probability and physics in connection with the *Brownian map* [9], [31], [33]. As possible sources of insights which could be used to prove Theorem 1.2.5 bijectively, without attempting to be complete, we mention the papers [6], [8], [12], [13], [14], [20], and [36]. The *mobiles* considered in [8] and [14] seem closest to the well-labeled trees of the class GJdM_n. Detailed comparisons, yet to be worked out, may yield the desired proof. Finally, we remark that our analytic method of proof of Theorem 1.2.5 has the merit of naturally generalizing to higher genus and could conceivably be used as a tool to generate conjectures about bijections in higher genus.

2. Joint cumulants of functions of a Gaussian random vector

Our goal in this section is to derive a delicate expansion of the right side of formula (22). (See Proposition 2.6.3 below.) We obtain this expansion by specializing a general representation for the joint cumulant of several polynomial functions of a given Gaussian random vector. (See Theorem 2.4.6 below). We obtain the latter representation by applying the BKAR formula from rigorous statistical mechanics. (See Theorem 2.3.2 below.) We have written this section anticipating that the reader would be unfamiliar with the BKAR formula but otherwise familiar with common tools from combinatorics and probability.

2.1. **Joint cumulants and related apparatus.** We briefly recall the formalism of set partitions, Möbius inversion and joint cumulants. For an overview see [34] or the beginning of [35]. (Caution: we do not follow the notation of these sources too closely.) For a probability textbook treatment of joint cumulants see [40, II.12.8]. See [29, Section 8.6] for an accessible short introduction to the general theory of Möbius functions of finite posets.

2.1.1. *Set partitions.* Let n be a positive integer. Recall our abbreviated notation $\langle n\rangle = \{1, \dots, n\}$. A *set partition* of $\langle n\rangle$ (or, context permitting, simply a *partition*) is by definition a disjoint family of nonempty subsets of $\langle n\rangle$ the union of which equals $\langle n\rangle$. The family of partitions of $\langle n\rangle$ is denoted by Part_n. Given $\Pi \in \mathrm{Part}_n$, each member of Π is called a *block*. Given $\Pi_1, \Pi_2 \in \mathrm{Part}_n$ we write $\Pi_1 \leq \Pi_2$ and say that Π_1 is a *refinement* of Π_2 if for every block $A \in \Pi_1$ there exists some block $B \in \Pi_2$ such that $A \subset B$. We also write $\Pi_1 < \Pi_2$ if $\Pi_1 \leq \Pi_2$ but $\Pi_1 \neq \Pi_2$. Thus partially ordered by refinement, Part_n becomes a *lattice*, i.e., a poset in which every family F of elements has a greatest lower bound $\wedge F$ and a least upper bound $\vee F$. The least partition $\{\{i\} \mid i \in \langle n\rangle\} = \wedge \mathrm{Part}_n = \vee\emptyset$ will be denoted by $\mathbf{0}_n$. The greatest partition $\{\langle n\rangle\} = \vee\mathrm{Part}_n = \wedge\emptyset$ will be denoted by $\mathbf{1}_n$. For $\Pi_1, \Pi_2 \in \mathrm{Part}_n$ such that $\Pi_1 \leq \Pi_2$, let

$$[\Pi_1 : \Pi_2] = \{\Pi \in \mathrm{Part}_n \mid \Pi_1 \leq \Pi \leq \Pi_2\},$$

which one calls the *interval* bounded below by Π_1 and above by Π_2.

2.1.2. *The Möbius function of* Part_n. The *Möbius function*

$$\mu = \mu_{\mathrm{Part}_n} = ((\Pi_1, \Pi_2) \mapsto \mu(\Pi_1 : \Pi_2)) : \mathrm{Part}_n \times \mathrm{Part}_n \to \mathbb{Z}$$

is that function which, when viewed as a Part_n-by-Part_n matrix, is inverse to the *incidence matrix*

$$((\Pi_1, \Pi_2) \mapsto \mathbb{1}\{\Pi_1 \leq \Pi_2\}) : \mathrm{Part}_n \times \mathrm{Part}_n \to \{0, 1\}.$$

Since the incidence matrix is upper unitriangular, so also is the matrix μ, i.e.,

$$\mu(\Pi_1 : \Pi_2) = \delta_{\Pi_1,\Pi_2} \text{ unless } \Pi_1 < \Pi_2.$$

By definition of μ one has

$$\sum_{\Pi\in[\Pi_1:\Pi_2]} \mu(\Pi : \Pi_2) = \sum_{\Pi\in[\Pi_1:\Pi_2]} \mu(\Pi_1 : \Pi) = \delta_{\Pi_1,\Pi_2} \text{ for } \Pi_1 \leq \Pi_2. \tag{25}$$

This is the *Möbius inversion formula* for the lattice Part_n. The Möbius function is given explicitly for $\Pi_1 \leq \Pi_2$ by the expression

$$\mu(\Pi_1 : \Pi_2) = \prod_{B \in \Pi_2} (-1)^{|\{A \in \Pi_1 | A \subset B\}|-1} (|\{A \in \Pi_1 \mid A \subset B\}| - 1)!. \tag{26}$$

Note that $\mu(\Pi_1 : \Pi_2)$ depends only on the isomorphism class of the poset $[\Pi_1 : \Pi_2]$. (This last remark holds for the Möbius function of any finite poset.)

2.1.3. *The joint cumulant functional.* Let S be a finite index set. Let $\{X_i\}_{i \in S}$ be a family of real-valued random variables each member of which has absolute moments of all orders. The *joint cumulant* of these variables is defined by the formula

$$\kappa\left(\{X_i\}_{i \in S}\right) = \left(\prod_{i \in S} \frac{\partial}{\partial t_i}\right) \log \mathbf{E} \exp\left(\sum_{i \in S} t_i X_i\right)\Bigg|_{t_i\,=\,0 \text{ for } i\,\in\,S}, \tag{27}$$

where the variables t_i are treated formally. Hereafter suppose for simplicity that $\{X_i\}_{i \in S} = \{X_i\}_{i=1}^n$. Via formula (26) one has an equivalent expression

$$\kappa(X_1, \dots, X_n) = \sum_{\Pi \in \mathrm{Part}_n} \mu(\Pi : \mathbf{1}_n) \prod_{A \in \Pi} \mathbf{E} \prod_{i \in A} X_i \tag{28}$$

for the joint cumulant functional. By the Möbius inversion formula (25) one then has an expansion

$$\mathbf{E} \prod_{i=1}^n X_i = \sum_{\Pi \in \mathrm{Part}_n} \prod_{A \in \Pi} \kappa\left(\{X_i\}_{i \in A}\right). \tag{29}$$

2.1.4. *The Isserlis-Wick formula.* Suppose now $X_1, \dots, X_n$ are real random variables with a centered Gaussian joint distribution. Using (27) one can show that the joint cumulant of three or more random variables with a Gaussian joint distribution vanishes identically. Thus, after substituting into (29), one obtains the relation

$$\mathbf{E} \prod_{i=1}^n X_i = \sum_{\substack{\Pi \in \mathrm{Part}_n \text{ s.t.} \\ \text{all blocks are} \\ \text{of cardinality } 2}} \prod_{\{i,j\} \in \Pi} \mathbf{E} X_i X_j, \tag{30}$$

known as the *Wick formula* among physicists but in fact due to Isserlis [28].

2.1.5. *Trivial generalization of* (28). Suppose that for some partition $\Theta \in \mathrm{Part}_n$ one is given a family of real random variables $\{Y_A\}_{A \in \Theta}$ with absolute moments of all orders. Then by (26) one has for formula (28) a trivial generalization

$$\kappa\left(\{Y_A\}_{A \in \Theta}\right) = \sum_{\Pi \in [\Theta : \mathbf{1}_n]} \mu(\Pi : \mathbf{1}_n) \prod_{B \in \Pi} \mathbf{E} \prod_{\substack{A \in \Theta \\ \text{s.t. } A \subset B}} Y_A \tag{31}$$

which will be especially important in the sequel.

2.2. **The probability measures $\mathbb{P}_\Gamma^\Theta$.** Under this heading we make the key definition figuring in the BKAR formula and (hence) in the statement of the refined expansion of the right side of (22) we are aiming to obtain. We actually give a couple of equivalent definitions, each of which has its uses. The formalism we set up here will be in use throughout the paper.

2.2.1. *The set* Bond_n. Let

$$\mathrm{Bond}_n = \{\{i,j\} \subset \langle n\rangle \mid i,j \in \langle n\rangle,\ i \neq j\} \subset 2^{\langle n\rangle}.$$

For each subset $\Gamma \subset \mathrm{Bond}_n$ and partition $\Theta \in \mathrm{Part}_n$, with some abuse of notation, let $\Gamma \vee \Theta$ denote the greatest lower bound of the family of partitions $\Psi \in [\Theta : \mathbf{1}_n]$ such that each member of Γ is contained in some block of Ψ. Roughly speaking $\Gamma \vee \Theta$ arises from Θ by coalescing pairs of blocks whenever they are "bonded" by some member of Γ.

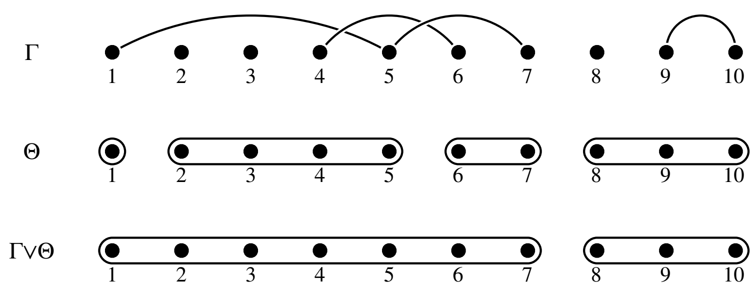

FIGURE 6. The set of bonds $\Gamma = \{\{1,5\},\{4,6\},\{5,7\},\{9,10\}\}$, the partition $\Theta = \{\{1\},\{2,3,4,5\},\{6,7\},\{8,9,10\}\}$, and the partition $\Gamma \vee \Theta = \{\{1,2,3,4,5,6,7\},\{8,9,10\}\}$.

2.2.2. *The graphs* $\mathfrak{G}(\Theta,\Gamma)$. Given $\Theta \in \mathrm{Part}_n$ and $\Gamma \subset \mathrm{Bond}_n$, we define a graph $\mathfrak{G}(\Theta,\Gamma)$ by the following conventions:

- Each member of Θ is interpreted as a vertex.
- Each member of Γ is interpreted as an edge.
- For all edges $e = \{i,j\} \in \Gamma$ and vertices $A, B \in \Theta$ such that $i \in A$ and $j \in B$, the set of endpoints of e is declared to be $\{A, B\}$.

The graph $\mathfrak{G}(\Theta,\Gamma)$ has in general multiple edges and loops joining a vertex to itself. Most graphs we need to consider in this paper arise naturally in the form $\mathfrak{G}(\Theta,\Gamma)$. Note that the family of connected components of the graph $\mathfrak{G}(\Theta,\Gamma)$ is canonically in bijection with the set $\Gamma \vee \Theta$.

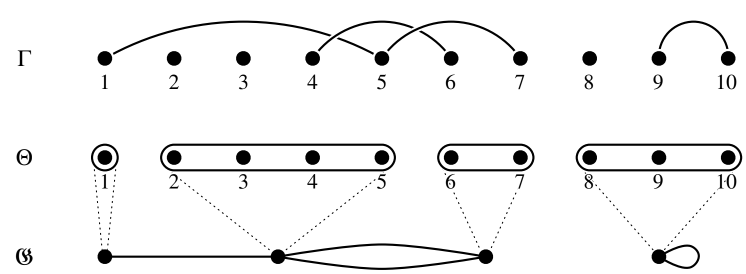

FIGURE 7. The set of bonds $\Gamma = \{\{1,5\},\{4,6\},\{5,7\},\{9,10\}\}$, the partition $\Theta = \{\{1\},\{2,3,4,5\},\{6,7\},\{8,9,10\}\}$, and the graph $\mathfrak{G} = \mathfrak{G}(\Theta,\Gamma)$.

2.2.3. *The set* $\mathrm{Tree}_n(\Theta)$. For $\Theta \in \mathrm{Part}_n$, let $\mathrm{Tree}_n(\Theta)$ denote the set of $\Gamma \subset \mathrm{Bond}_n$ such that $\mathbf{1}_n = \Gamma \vee \Theta$ and $|\Gamma| + 1 = |\Theta|$. Equivalently, $\mathrm{Tree}_n(\Theta)$ is the set whose members are sets $\Gamma \subset \mathrm{Bond}_n$ such that the graph $\mathfrak{G}(\Theta, \Gamma)$ with vertex set Θ and edge set Γ is connected and has Euler characteristic 1, i.e., is a tree.

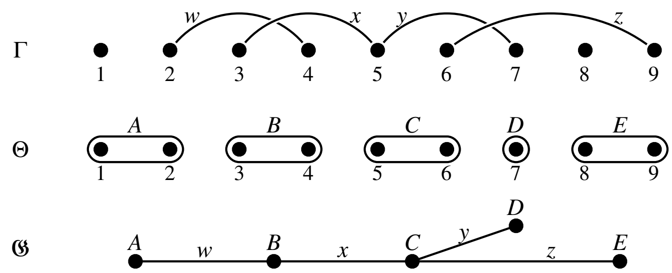

FIGURE 8. A member $\Gamma = \{\{2,4\},\{3,5\},\{5,7\},\{6,9\}\}$ of $\mathrm{Tree}_n(\Theta)$ for $\Theta = \{\{1,2\},\{3,4\},\{5,6\},\{7\},\{8,9\}\}$, and the tree $\mathfrak{G} = \mathfrak{G}(\Theta, \Gamma)$.

2.2.4. *The Schur Product Theorem and related notation.*

- Let Mat_n denote the space of n-by-n matrices with real entries
- Let $\mathrm{Sym}_n \subset \mathrm{Mat}_n$ denote the space of symmetric matrices.
- Let $\mathrm{Sym}_n^+ = \{Q \in \mathrm{Sym}_n \mid Q \text{ is positive semidefinite}\}$.
- For $A, B \in \mathrm{Mat}_n$, recall that the *Hadamard product* (alternatively and arguably more correctly: the *Schur product*) $A \star B \in \mathrm{Mat}_n$ is defined by the formula $(A \star B)(i,j) = A(i,j)B(i,j)$ (entry-by-entry multiplication).

According to the *Schur Product Theorem* [38] if $A, B \in \mathrm{Sym}_n^+$, then $A \star B \in \mathrm{Sym}_n^+$. The latter fact is of extreme importance in the sequel.

2.2.5. *The set* $\mathfrak{Q}_n$. Let $\mathfrak{Q}_n$ denote the set consisting of all matrices $Q \in \mathrm{Sym}_n^+$ with the following properties:

- All entries of Q belong to the closed unit interval $[0, 1]$.
- All diagonal entries of Q are equal to 1.

Since the n-by-n identity matrix belongs to $\mathfrak{Q}_n$, the latter set is not empty. It is easy to see that the set $\mathfrak{Q}_n$ is closed, convex, bounded and hence compact. For $\Theta \in \mathrm{Part}_n$ and $\Gamma \in \mathrm{Tree}_n(\Theta)$, the probability measure $\mathbb{P}_\Gamma^\Theta$ we aim to define will be defined on the set $\mathfrak{Q}_n$.

2.2.6. *The matrix representation of partitions.* Given $\Pi \in \mathrm{Part}_n$, we define the matrix $[\Pi] \in \mathrm{Mat}_n$ to have entries

$$[\Pi](i,j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ belong to the same block of } \Pi, \\ 0 & \text{otherwise.} \end{cases}$$

We say that the matrix $[\Pi]$ thus defined *represents* Π. For example, one has

$$[\{\{1,2,3\},\{4,5\}\}] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In particular, $[\mathbf{0}_n]$ is the n-by-n identity matrix and $[\mathbf{1}_n]$ is the n-by-n matrix with all entries equal to 1. Finally and crucially, note that $[\Pi] \in \mathfrak{Q}_n$ for $\Pi \in \mathrm{Part}_n$.

2.2.7. *Definition of* $\mathbb{P}_\Gamma^\Theta$. Fix $\Theta \in \mathrm{Part}_n$ arbitrarily and let $k = |\Theta|$. We define a family

$$\{\mathbb{P}_\Gamma^\Theta\}_{\Gamma \in \mathrm{Tree}_n(\Theta)}$$

of probability measures on $\mathfrak{Q}_n$ by requiring the integration formula

$$\sum_{\Gamma \in \mathrm{Tree}_n(\Theta)} \int f_\Gamma \, \mathrm{d}\mathbb{P}_\Gamma^\Theta \tag{32}$$

$$= \sum_{\substack{(e_1,\dots,e_{k-1}) \in \mathrm{Bond}_n^{k-1} \\ \text{s.t. } \{e_1,\dots,e_{k-1}\} \in \mathrm{Tree}_n(\Theta)}} \int \cdots \int_{1=t_0>t_1>\cdots>t_{k-1}>t_k=0} f_{\{e_1,\dots,e_{k-1}\}} \left(\sum_{\alpha=0}^{k-1} (t_\alpha - t_{\alpha+1}) [\{e_1,\dots,e_\alpha\} \vee \Theta] \right) \prod_{\alpha=1}^{k-1} \mathrm{d}t_\alpha$$

$$= \sum_{\substack{(e_1,\dots,e_{k-1}) \in \mathrm{Bond}_n^{k-1} \\ \text{s.t. } \{e_1,\dots,e_{k-1}\} \in \mathrm{Tree}_n(\Theta)}} \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{k-2}} \mathrm{d}t_{k-1} \, f_{\{e_1,\dots,e_{k-1}\}} \left([\Theta] + \sum_{\alpha=1}^{k-1} t_\alpha ([\{e_1,\dots,e_\alpha\} \vee \Theta] - [\{e_1,\dots,e_{\alpha-1}\} \vee \Theta]) \right)$$

to hold for every family

$$\{f_\Gamma : \mathfrak{Q}_n \to \mathbb{R}\}_{\Gamma \in \mathrm{Tree}_n(\Theta)}$$

of continuous functions.

Lemma 2.2.8 (Alternate characterization of $\mathbb{P}_\Gamma^\Theta$)**.** *Fix $\Theta \in \mathrm{Part}_n$ and $\Gamma \in \mathrm{Tree}_n(\Theta)$. Let $k = |\Theta|$. Let $X \in \mathfrak{Q}_n$ be a random matrix with law $\mathbb{P}_\Gamma^\Theta$. For $\{i,j\} \in \mathrm{Bond}_n$ and blocks $A, B \in \Theta$ such that $i \in A$ and $j \in B$, let $\Gamma(i,j) \subset \Gamma$ be the subset consisting of edges visited by the unique geodesic walk in the tree $\mathfrak{G}(\Theta,\Gamma)$ joining A to B. Then the following statements concerning the random matrix X hold:*

(i) *The family of matrix entries*

$$\{X(i,j) \mid 1 \le i < j \le n \text{ and } \{i,j\} \in \Gamma\}$$

is i.i.d.uniformly distributed in $(0,1)$.

(ii) *For all $i,j \in \langle n \rangle$ one has*

$$X(i,j) = \min\left(\{1\} \cup \{X(i',j') \mid \{i',j'\} \in \Gamma(i,j)\}\right)$$

almost surely.

The lemma reconciles the definition (32) of $\mathbb{P}_\Gamma^\Theta$ given above with the form of the definition typical in the literature.

Proof. We begin by building an explicit random matrix with law $\mathbb{P}_\Gamma^\Theta$. Let

$$T = (T_1, \dots, T_{k-1})$$

be a real random vector uniformly distributed in the simplex

$$\{(t_1,\dots,t_{k-1}) \in \mathbb{R}^{k-1} \mid 1 > t_1 > \cdots > t_{k-1} > 0\}.$$

For convenience let $T_0 = 1$ and $T_k = 0$. Write $\Gamma = \{e_1, \ldots, e_{k-1}\}$. Let $\rho \in S_{k-1}$ be a uniformly distributed random permutation independent of T. Consider the random matrix

$$\begin{aligned}(33)\qquad & [\Theta] + \sum_{\alpha=1}^{k-1} T_\alpha([\{e_{\rho(1)}, \ldots, e_{\rho(\alpha)}\} \vee \Theta] - [\{e_{\rho(1)}, \ldots, e_{\rho(\alpha-1)}\} \vee \Theta]) \\ = & \sum_{\alpha=0}^{k-1} (T_\alpha - T_{\alpha+1})[\{e_{\rho(1)}, \ldots, e_{\rho(\alpha)}\} \vee \Theta]\end{aligned}$$

which clearly takes its values in $\mathfrak{Q}_n$. It is a trivial matter to confirm that the law on $\mathfrak{Q}_n$ of the random matrix (33) is $\mathbb{P}_\Gamma^\Theta$. Without loss of generality we may identify the given random matrix X with the random matrix (33).

Fix $\{i, j\} \in \mathrm{Bond}_n$ and blocks $A, B \in \Theta$ such that $i \in A$ and $j \in B$. It will be enough to evaluate the matrix entry $X(i, j)$ in terms of T and ρ. We begin by observing that $X(i, j) = T_\beta$ where the (random) index β is the least index α such that

$$[\{e_{\rho(1)}, \ldots, e_{\rho(\alpha)}\} \vee \Theta](i, j) = 1.$$

Equivalently, β is the least index α such that A and B are connected by some walk in the (random) forest

$$\mathfrak{G}(\Theta, \{e_{\rho(1)}, \ldots, e_{\rho(\alpha)}\}).$$

Now $e \in \Gamma$ satisfies $e \in \Gamma(i, j)$ if and only if A and B are NOT joined by a walk in the forest $\mathfrak{G}(\Theta, \Gamma \setminus \{e\})$. Thus β is the least index α such that

$$\{e_{\rho(1)}, \ldots, e_{\rho(\alpha)}\} \supset \Gamma(i, j).$$

By this reasoning we arrive at the formula

$$X(i, j) = \min(\{1\} \cup \{T_{\rho^{-1}(\alpha)} \mid \alpha = 1, \ldots, k-1 \text{ s.t. } e_\alpha \in \Gamma(i, j)\}).$$

Now write $e_i = \{a_i, b_i\}$ where $a_i < b_i$ for $i = 1, \ldots, k-1$. It is clear that the random vector

$$(X(a_1, b_1), \ldots, X(a_{k-1}, b_{k-1})) = (T_{\rho^{-1}(1)}, \ldots, T_{\rho^{-1}(k-1)})$$

is uniformly distributed in the cube $(0, 1)^{k-1}$. Statements (i) and (ii) follow. □

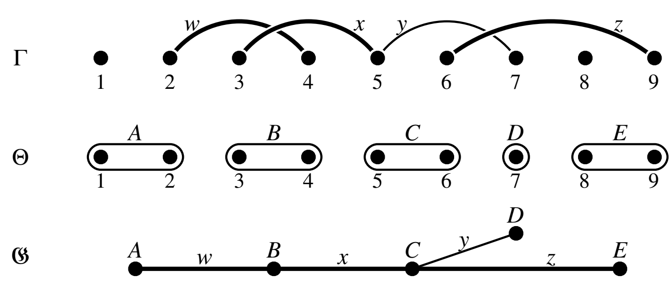

FIGURE 9. For $\Gamma = \{w, x, y, z\} \subset \mathrm{Bond}_9$ and $\Theta = \{A, B, C, D, E\} \in \mathrm{Part}_9$ as drawn above, $\Gamma(1, 8) = \Gamma(2, 8) = \Gamma(1, 9) = \Gamma(2, 9) = \{w, x, z\}$ corresponds to the geodesic path joining A with E in the tree $\mathfrak{G} = \mathfrak{G}(\Theta, \Gamma)$.

2.3. **A variant of the BKAR formula.**

2.3.1. *Differentiation of functions on* Sym_n. For short, we say that a function $f : \mathrm{Sym}_n \to \mathbb{R}$ is *polynomial* if $f(Q)$ is a polynomial with real coefficients in the entries of Q. For $e = \{i,j\} \in \mathrm{Bond}_n$, $Q \in \mathrm{Sym}_n$ and polynomial functions $f : \mathrm{Sym}_n \to \mathbb{R}$, let

$$(\partial_e f)(Q) = \frac{d}{dt} f(Q + t(e_{ij} + e_{ji}))\bigg|_{t=0} \quad (\{e_{\alpha\beta}\}_{\alpha,\beta=1}^n : \text{ standard basis of } \mathrm{Mat}_n),$$

thus defining a first order linear differential operator ∂_e acting on polynomial functions defined on Sym_n. More generally, for each $\Gamma \subset \mathrm{Bond}_n$ let

$$\partial^\Gamma = \prod_{e\in\Gamma} \partial_e.$$

We have then the following fundamental integration identity.

Theorem 2.3.2 (Variant of the BKAR formula)**.** *For set partitions* $\Theta \in \mathrm{Part}_n$ *and polynomial functions* $f : \mathrm{Sym}_n \to \mathbb{R}$ *we have*

$$\sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi : \mathbf{1}_n) f([\Pi]) = \sum_{\Gamma\in\mathrm{Tree}_n(\Theta)} \int \partial^\Gamma f \, \mathrm{d}\mathbb{P}_\Gamma^\Theta. \tag{34}$$

Formula (34) is true for more general functions f than polynomials ones, but here, for simplicity, we stick to the polynomial case. No greater generality will be needed. In any case, extension of (34) to larger classes of functions can easily enough be accomplished by polynomial approximation. For the reader's convenience we supply a short proof of (34) in §2.3.4 below; the effort of the setup above renders the proof more or less trivial.

2.3.3. *Background and references concerning the BKAR formula.* We mention first of all the paper [16] of Brydges and Kennedy. Next we mention the papers [3] and [4] of Abdesselam and Rivasseau. This explains the abbreviation BKAR. The notes [1] give an accessible introduction to the BKAR formula and many further references. The paper [2] is a typical application of the BKAR formula wherein the latter is used to bound joint cumulants. The recent paper [24] generalizes the BKAR formula in a natural way to the setting of matroids. The BKAR formula is a relatively recent development in a very old and well-established line of research in statistical mechanics focused on *cluster expansions.* Concerning the latter, we refer the reader to [15] and [17] as possible entry points to that vast literature.

2.3.4. *A proof of Theorem 2.3.2.* We may assume that f takes the form

$$f(Q) = \prod_{1\le i\le j\le n} Q(i,j)^{\nu(i,j)} \text{ for } Q \in \mathrm{Sym}_n$$

where

$$\nu = \{\nu(i,j) \mid 1 \le i \le j \le n\}$$

is a family of nonnegative integers. Let

$$\mathrm{supp}\,\nu = \{\{i,j\} \in \mathrm{Bond}_n \mid 1 \le i < j \le n \text{ s.t. } \nu(i,j) > 0\}.$$

We have

$$\Pi \in [\Theta : \mathbf{1}_n] \Rightarrow f([\Pi]) = \mathbb{1}\{(\mathrm{supp}\,\nu) \vee \Theta \le \Pi\}$$

and hence by the Möbius inversion formula (25) we have

(35)
$$(\text{LHS of (34)}) = \mathbb{1}\{(\operatorname{supp}\nu)\vee\Theta = \mathbf{1}_n\} = \begin{cases} 1 & \text{if } \mathfrak{G}(\Theta,\operatorname{supp}\nu) \text{ is connected,}\\ 0 & \text{otherwise.}\end{cases}$$

For $\Pi \in \mathrm{Part}_n$ let

$$N(\Pi) = \sum_{\substack{1\leq i<j\leq n\\ \{i,j\} \text{ is contained}\\ \text{in no block of } \Pi}} \nu(i,j).$$

Note that for $1 \leq i' < j' \leq n$ we have

$$(\partial_{\{i',j'\}}f)(Q) = \begin{cases} \nu(i',j') \displaystyle\prod_{1\leq i<j\leq n} Q(i,j)^{\nu(i,j)-\delta_{ii'}\delta_{jj'}} & \text{if } \{i',j'\}\in\operatorname{supp}\nu,\\ 0 & \text{if } \{i',j'\}\notin\operatorname{supp}\nu.\end{cases}$$

Substituting directly into the definition (32) of $\mathbb{P}_\Gamma^\Theta$ we then have

(36)
$$\begin{aligned}
&(\text{RHS of (34)})\\
&= \sum_{\substack{(e_1,\dots,e_{k-1})\in\mathrm{Bond}_n^{k-1}\\ \text{s.t. } \{e_1,\dots,e_{k-1}\}\in\mathrm{Tree}_n(\Theta)}} \int_0^1 \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{k-2}}\mathrm{d}t_{k-1}\\
&\qquad \partial_{e_1}\cdots\partial_{e_{k-1}} f\left([\Theta]+\sum_{\alpha=1}^{k-1} t_\alpha([\{e_1,\dots,e_\alpha\}\vee\Theta]-[\{e_1,\dots,e_{\alpha-1}\}\vee\Theta])\right)\\
&= \sum_{\substack{e=(e_1,\dots,e_{k-1})\in(\operatorname{supp}\nu)^{k-1}\\ \text{s.t. } \{e_1,\dots,e_{k-1}\}\in\mathrm{Tree}_n(\Theta)}} \int_0^1 \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\cdots\int_0^{t_{k-2}}\mathrm{d}t_{k-1}\\
&\qquad\qquad \prod_{\alpha=1}^{k-1}\nu(e_\alpha)t_\alpha^{N(\{e_1,\dots,e_{\alpha-1}\}\vee\Theta)-N(\{e_1,\dots,e_\alpha\}\vee\Theta)-1}\\
&= \sum_{\substack{e=(e_1,\dots,e_{k-1})\in(\operatorname{supp}\nu)^{k-1}\\ \text{s.t. } \{e_1,\dots,e_{k-1}\}\in\mathrm{Tree}_n(\Theta).}} \prod_{\alpha=1}^{k-1}\frac{\nu(e_\alpha)}{N(\{e_1,\dots,e_{\alpha-1}\}\vee\Theta)}.
\end{aligned}$$

Now the right sides of (35) and (36) both vanish if $(\operatorname{supp}\nu)\vee\Theta \neq \mathbf{1}_n$. Otherwise, the right side of (35) equals 1 and thus equals the right side of (36) by the lemma recalled immediately below. □

Lemma 2.3.5. *As above, let $\Theta \in \mathrm{Part}_n$ be a set partition and let $k = |\Theta|$. For every sequence $(e_1,\dots,e_{k-1}) \in \mathrm{Bond}_n^{k-1}$ one has $\{e_1,\dots,e_{k-1}\} \in \mathrm{Tree}_n(\Theta)$ if and only if for $\alpha = 1,\dots,k-1$ the set e_α is contained in no block of the set partition $\{e_1,\dots,e_{\alpha-1}\}\vee\Theta$.*

We can safely omit the proof.

2.4. **Formulation of the main technical result.**

2.4.1. *Variables.* Let n and N be positive integers. Let

$$z = \{\{z_{ij}\}_{i=1}^n\}_{j=0}^{2N}$$

be a family of independent (commutative) algebraic variables. Let $\mathbb{R}[z]$ be the polynomial algebra these variables generate over the real numbers. We remark that the index j runs here from 0 to $2N$ rather than, say, from 1 to N in order to accommodate the intended application with little adjustment of notation.

2.4.2. *Differential operators.* For $e = \{i, i'\} \in \mathrm{Bond}_n$ we define a partial differential operator

$$D_e = \sum_{j=0}^{2N} \frac{\partial^2}{\partial z_{ij} \partial z_{i'j}} \tag{37}$$

acting on the polynomial algebra $\mathbb{R}[z]$. Given $\Gamma \subset \mathrm{Bond}_n$, we in turn define

$$D^\Gamma = \prod_{e \in \Gamma} D_e. \tag{38}$$

2.4.3. *A family of polynomials.* Fix $\Theta \in \mathrm{Part}_n$. For each $A \in \Theta$ fix a polynomial

$$f_A \in \mathbb{R}[\{\{z_{ij}\}_{i \in A}\}_{j=0}^{2N}] \subset \mathbb{R}[z]$$

and let

$$f = \prod_{A \in \Theta} f_A \in \mathbb{R}[z].$$

2.4.4. *Gaussian random variables.* Let

$$\zeta = \{\{\zeta_{ij}\}_{i=1}^n\}_{j=0}^{2N}$$

be a family of real random variables with a centered Gaussian joint distribution such that

$$\mathbf{E}\zeta_{ij}\zeta_{i'j'} = \delta_{jj'}\mathbf{E}\zeta_{i0}\zeta_{i'0}. \tag{39}$$

Note that the random vector ζ has the structure of a family of $2N+1$ i.i.d. copies of the random vector $\{\zeta_{i0}\}_{i=1}^n$. But also note that we do not place any restrictions on the covariances $\mathbf{E}\zeta_{i0}\zeta_{i'0}$. The latter freedom is crucial for the intended application.

2.4.5. *The Q-recoupling construction.* For each $Q \in \mathfrak{Q}_n$ let

$$\zeta \star Q = \left\{\{(\zeta \star Q)_{ij}\}_{i=1}^n\right\}_{j=0}^{2N}$$

be a family of real random variables with a centered Gaussian joint distribution characterized by the covariances

$$\mathbf{E}(\zeta \star Q)_{ij}(\zeta \star Q)_{i'j'} = Q(i,i')\mathbf{E}\zeta_{ij}\zeta_{i'j'} = \delta_{jj'}Q(i,i')\mathbf{E}\zeta_{i0}\zeta_{i'0}. \tag{40}$$

Such a family $\zeta \star Q$ exists and has a uniquely determined law because the requisite positive-semidefiniteness is guaranteed by the Schur Product Theorem. We say that $\zeta \star Q$ arises from ζ by *Q-recoupling*. The probability space on which $\zeta \star Q$ is defined is allowed to depend on Q. It is of no concern to us. Note that $\zeta \star Q$ has the structure of $2N+1$ i.i.d. of copies of the random vector $\{(\zeta \star Q)_{i0}\}_{i=1}^n$. Note also that for $i = 1, \ldots, n$ the subfamily $\left\{(\zeta \star Q)_{ij}\right\}_{j=0}^{2N}$ of $\zeta \star Q$ consists of $2N+1$ i.i.d copies the random variable ζ_{i0}. Finally, note that $\zeta \star [\mathbf{1}_n]$ is a copy of ζ.

Here is the main technical result of the paper.

Theorem 2.4.6. *Notation and assumptions are as above. We have*
(41)

$$\kappa\left(\{f_A(\zeta)\}_{A\in\Theta}\right) = \sum_{\Gamma\in\mathrm{Tree}_n(\Theta)}\left(\prod_{\{i,i'\}\in\Gamma}\mathbf{E}\zeta_{i0}\zeta_{i'0}\right)\int \mathbf{E}\left[(D^\Gamma f)(\zeta\star Q)\right]\ \mathbb{P}_\Gamma^\Theta(\mathrm{d}Q).$$

The proof of (41) will be given in §2.5 below. Now to make sense of the right side of (41) it is necessary to give a consistent interpretation to expressions of the form

$$\int \mathbf{E}\left[g(\zeta\star Q)\right]\ \mathbb{P}_\Gamma^\Theta(\mathrm{d}Q)\quad (g\in\mathbb{R}[z],\Theta\in\mathrm{Part}_n,\Gamma\in\mathrm{Tree}_n(\Theta)). \tag{42}$$

Our convention is invariably to interpret these expressions as iterated integrals. This interpretation makes sense and indeed yields a well-defined numerical value because the inner integral $\mathbf{E}\left[g(\zeta\star Q)\right]$ by the Isserlis-Wick formula (30) depends polynomially on Q.

2.4.7. *Notes and references.* Formula (41) hypergeneralizes explicit identities used to prove the Poincaré inequality for Gaussian random variables. For an accessible discussion of identities of the latter type see [11]. In turn the latter paper suggests the problem of developing an analogue of (41) for Bernoulli random variables.

2.5. **Proof of Theorem 2.4.6.** The next lemma rewrites the Isserlis-Wick formula in a more convenient form involving differential operators.

Lemma 2.5.1. *Let S be a finite index set. Let $t=\{t_i\}_{i\in S}$ be a family of independent commuting algebraic variables. Each variable t_i is assigned the degree 1, and we consider the polynomial ring $\mathbb{R}[t]$ graded by degree. Let $\tau=\{\tau_i\}_{i\in S}$ be a family of real random variables with a centered Gaussian joint distribution. For polynomials $f=f(t)\in\mathbb{R}[t]$ homogeneous of degree k we have*

$$\mathbf{E}f(\tau) = \begin{cases} 0 & \textit{if } k \textit{ is odd,}\\ \dfrac{1}{m!}\left(\dfrac{1}{2}\displaystyle\sum_{i,j\in S}(\mathbf{E}\tau_i\tau_j)\frac{\partial^2}{\partial t_i\partial t_j}\right)^m f(t) & \textit{if } k=2m \textit{ is even.}\end{cases} \tag{43}$$

Proof. If the family $\{\tau_i\}_{i\in S}$ is i.i.d. standard normal and $f(t)=(\sum_{i\in S}a_it_i)^k$ for a family of real constants $\{a_i\}_{i\in S}$ such that $\sum_{i\in S}a_i^2=1$, formula (43) holds. Indeed, in that case the left side equals $\mathbf{E}T^k$ for a standard normal random variable T, and one can straightforwardly verify that the right side takes the same value. But the polynomials $(\sum_{i\in S}a_it_i)^k$ span the subspace of $\mathbb{R}[t]$ consisting of polynomials homogeneous of degree k, and moreover, formula (43) is stable under homogeneous linear change of variable. Thus formula (43) holds in general. □

2.5.2. *The $\natural$-construction.* For any polynomial $g\in\mathbb{R}[z]$ it is convenient to define a function $g^\natural:\mathrm{Sym}_n\to\mathbb{R}$ which depends linearly on g and which for g homogeneous of degree k is given by the formula
(44)

$$g^\natural(Q) = \begin{cases} 0 & \text{if } k \text{ is odd,}\\ \dfrac{1}{m!}\left(\dfrac{1}{2}\displaystyle\sum_{i,i'=1}^{n}\sum_{J=0}^{2N}Q(i,i')(\mathbf{E}\zeta_{i0}\zeta_{i'0})\frac{\partial^2}{\partial z_{iJ}\partial z_{i'J}}\right)^m g(z) & \text{if } k=2m \text{ is even.}\end{cases}$$

It is clear that $g^\natural(Q)$ depends polynomially on the matrix entries of Q.

Lemma 2.5.3. *For $g \in \mathbb{R}[z]$ and $Q \in \mathfrak{Q}_n$ one has*

$$\mathbf{E} g(\zeta \star Q) = g^\natural(Q). \tag{45}$$

Furthermore, given also $\Gamma \subset \mathrm{Bond}_n$,

$$\partial^\Gamma(g^\natural) = \left(\prod_{\{i,i'\}\in\Gamma} \mathbf{E}\zeta_{i0}\zeta_{i'0}\right)(D^\Gamma g)^\natural. \tag{46}$$

Formula (46) is an algebraic variant of the *heat equation.*

Proof. Formula (45) follows immediately from Lemma 2.5.1. To prove (46) we may proceed by induction on $|\Gamma|$. The decisive case is then clearly that in which $\Gamma = \{e\}$ for some $e \in \mathrm{Bond}_n$. In the latter special case differentiation on both sides of (44) immediately proves formula (46). □

2.5.4. *The independent copies trick.* We have

$$\begin{aligned}
\kappa\left(\{f_A(\zeta)\}_{A\in\Theta}\right) &= \sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi:\mathbf{1}_n) \prod_{B\in\Pi} \mathbf{E} \prod_{\substack{A\in\Theta\\ \text{s.t. } A\subset B}} f_A(\zeta) \\
&= \sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi:\mathbf{1}_n) \prod_{B\in\Pi} \mathbf{E} \prod_{\substack{A\in\Theta\\ \text{s.t. } A\subset B}} f_A(\zeta\star[\Pi]) \\
&= \sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi:\mathbf{1}_n)\mathbf{E} f(\zeta\star[\Pi]).
\end{aligned} \tag{47}$$

The first step of the calculation is an application of formula (31) and the remaining steps exploit the covariance structure of the family $\zeta\star Q$ in a straightforward way. The last step of the calculation is an instance of the commonly used "independent copies trick" whereby one writes a product of expectations of random variables as the expectation of a product of independent copies of the variables.

2.5.5. *Application of the BKAR formula.* We have

$$\begin{aligned}
(\text{LHS of } (41)) &= \sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi:\mathbf{1}_n)\mathbf{E} f(\zeta\star[\Pi]) = \sum_{\Pi\in[\Theta:\mathbf{1}_n]} \mu(\Pi:\mathbf{1}_n) f^\natural([\Pi]) \\
&= \sum_{\Gamma\in\mathrm{Tree}_n(\Theta)} \int (\partial^\Gamma f^\natural)(Q)\mathbb{P}_\Gamma^\Theta(dQ) \\
&= \sum_{\Gamma\in\mathrm{Tree}_n(\Theta)} \left(\prod_{\{i,i'\}\in\Gamma} (\mathbf{E}\zeta_{i0}\zeta_{i'0})\right) \int (D^\Gamma f)^\natural(Q)\mathbb{P}_\Gamma^\Theta(dQ) \\
&= (\text{RHS of } (41)).
\end{aligned}$$

The steps are justified as follows.

Step 1. Formula (47) (independent copies trick).
Step 2. Formula (45) (Wick formula in terms of differential operators).
Step 3. Theorem 2.3.2 (BKAR formula).
Step 4. Formula (46) (heat equation).
Step 5. Formula (45) (Wick formula again).

The proof of Theorem 2.4.6 is complete. □

2.6. **Refinement of formula** (22)**.** Under this heading we apply Theorem 2.4.6 to expand the right side of (22) in a refined way.

2.6.1. *Opening the brackets.* Fix a numerical partition λ and a positive integer N. Let $n = |\lambda|$ and $\ell = \ell(\lambda)$. Fix $\theta \in S_n$ such that $\theta \sim \lambda$ and let $\Theta = \mathrm{Orb}_n(\theta) \in \mathrm{Part}_n$. We then have

$$\mathfrak{M}_{\lambda,N} = \sum_{h:\langle n\rangle \to \langle N\rangle} \kappa\left(\left\{\prod_{i\in A} \mathrm{Tri}_N(h(i), h(\theta(i)))\right\}_{A\in\Theta}\right)$$

after opening the brackets in formula (22) in evident fashion. Let

$$\mathrm{Motz}_n^N(\theta) = \left\{h : \langle n\rangle \to \langle N\rangle \,\middle|\, \max_{i\in\langle n\rangle} |h(\theta(i)) - h(i)| \le 1\right\}. \tag{48}$$

Now a sequence of integers with increments in the set $\{0, \pm 1\}$ is often called a

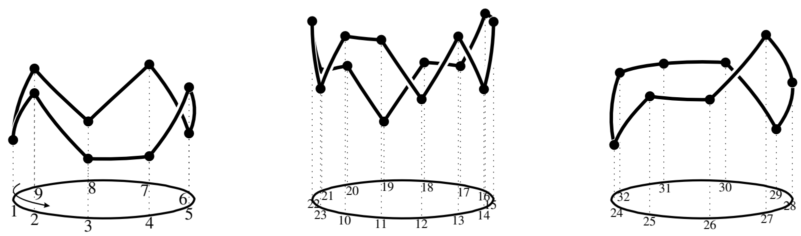

FIGURE 10. An $h \in \mathrm{Motz}_{32}^3(\theta)$ for a permutation $\theta \in S_{32}$ with three disjoint cycles.

Motzkin path. And it is evident that members of $\mathrm{Motz}_n^N(\theta)$ are collections of closed Motzkin paths indexed by the blocks of the set partition $\mathrm{Orb}_n(\theta)$. This is the rationale for our notation $\mathrm{Motz}_n^N(\theta)$. Figure 10 provides an intuitive picture. Given any function $h : \langle n\rangle \to \mathbb{Z}$, let

$$J_h(\epsilon) = \{i \in \langle n\rangle \mid h(\theta(i)) = h(i) + \epsilon\} \text{ for } \epsilon \in \{0, \pm 1\}. \tag{49}$$

We then have

$$\mathfrak{M}_{\lambda,N} = \sum_{h\in\mathrm{Motz}_n^N(\theta)} \kappa\left(\left\{\prod_{i\in J_h(0)\cap A} \xi_{h(i)} \cdot \prod_{i\in J_h(1)\cap A} \eta_{h(i)}\right\}_{A\in\Theta}\right) \tag{50}$$

since

$$\mathrm{Tri}_N(i+1, i) = 0, \ \ \mathrm{Tri}_N(i,i) = \xi_i, \ \ \mathrm{Tri}_N(i, i+1) = \eta_i,$$

and otherwise for $|i-j| > 1$ one has $\mathrm{Tri}_N(i,j) = 0$.

2.6.2. *Specialization of Theorem 2.4.6.* Fix $h \in \mathrm{Motz}_n^N(\theta)$ arbitrarily. We will be considering not just one instance of Theorem 2.4.6 but rather a family of such indexed by h. For $A \in \Theta$ let

$$f_A^h = \prod_{i\in J_h(0)\cap A} z_{i0} \cdot \prod_{i\in J_h(1)\cap A} \sum_{j=1}^{2h(i)} \frac{z_{ij}^2}{2} \in \mathbb{R}[\{\{z_{ij}\}_{i\in A}\}_{j=0}^{2N}].$$

In turn let

$$f^h = \prod_{A\in\Theta} f_A^h \ = \prod_{i\in J_h(0)} z_{i0} \cdot \prod_{i\in J_h(1)} \sum_{j=1}^{2h(i)} \frac{z_{ij}^2}{2} \in \mathbb{R}[\{\{z_{ij}\}_{i=1}^n\}_{j=0}^{2N}] = \mathbb{R}[z]. \tag{51}$$

Let

$$\{\{\xi_{ij}\}_{i=1}^{\infty}\}_{j=0}^{\infty}$$

be an i.i.d. family of standard normal random variables. In turn consider the centered Gaussian family

$$\zeta^h = \left\{\left\{\zeta^h_{ij}\right\}_{i=1}^{n}\right\}_{j=0}^{2N} = \left\{\left\{\xi_{h(i),j}\right\}_{i=1}^{n}\right\}_{j=0}^{2N}. \tag{52}$$

Note that by definition of ζ^h we have

$$\mathbf{E}\zeta^h_{i_1j_1}\zeta^h_{i_2j_2} = \mathbf{E}\xi_{h(i_1),j_1}\xi_{h(i_2),j_2} = \delta_{h(i_1),h(i_2)}\delta_{j_1j_2}. \tag{53}$$

For each $Q \in \mathfrak{Q}_n$, let $\zeta^h \star Q$ denote the family arising from ζ^h by Q-recoupling.

Here is the promised refined expansion of the right side of (22).

Proposition 2.6.3. *Notation and assumptions are as above. We have*

$$\mathfrak{M}_{\lambda,N} = \sum_{(\Gamma,h)\in\mathrm{Tree}_n(\Theta)\times\mathrm{Motz}_n^N(\theta)} \left(\prod_{\{i_1,i_2\}\in\Gamma}\delta_{h(i_1),h(i_2)}\right)\int \mathbf{E}\left[(D^\Gamma f^h)(\zeta^h\star Q)\right]\,\mathbb{P}^\Theta_\Gamma(\mathrm{d}Q). \tag{54}$$

Furthermore, the summand on the right side of (54) *indexed by* (Γ, h) *vanishes unless the following four conditions hold:*

(55) *h is constant on each $e \in \Gamma$.*

(56) *$e \subset J_h(0)$ or $e \subset J_h(1)$ for each $e \in \Gamma$.*

(57) *For each $i \in J_h(0)$ there exists at most one $e \in \Gamma$ such that $i \in e$.*

(58) *For each $i \in \langle n\rangle$ there exist at most two $e \in \Gamma$ such that $i \in e$.*

Condition (55) is trivially necessary for nonvanishing. We record it for the sake of completeness. But more significantly, condition (58) is not so obvious and it is imposed on Γ independently of h.

Proof. We begin by proving formula (54). To do so it suffices by (50) to fix $h \in \mathrm{Motz}_n^N(\theta)$ arbitrarily and to prove that

$$\kappa\left(\left\{\prod_{i\in J_h(0)\cap A}\xi_{h(i)}\cdot\prod_{i\in J_h(1)\cap A}\eta_{h(i)}\right\}_{A\in\Theta}\right) = \sum_{\Gamma\in\mathrm{Tree}_n(\Theta)}\left(\prod_{\{i_1,i_2\}\in\Gamma}\delta_{h(i_1),h(i_2)}\right)\int\mathbf{E}\left[(D^\Gamma f^h)(\zeta^h\star Q)\right]\,\mathbb{P}^\Theta_\Gamma(\mathrm{d}Q). \tag{59}$$

Now we are free to replace the family $\{\xi_i\}_{i=1}^N \cup \{\eta_i\}_{i=1}^{N-1}$ appearing in the definition of the tridiagonal matrix Tri_N by any other family with the same joint law. Thus we may assume without loss of generality that

$$\xi_i = \xi_{i0} \text{ and } \eta_i = \sum_{j=1}^{2i}\frac{\xi_{ij}^2}{2}. \tag{60}$$

Relations (53) and (60) taken into account, it is clear that (59) is a specialization of Theorem 2.4.6 and thus holds. Thus in turn formula (54) indeed holds.

Now fix a pair (Γ, h) such that (55) holds and moreover $D^\Gamma f^h \neq 0$. It will be enough to show that for this pair (Γ, h) statements (56)—(58) hold. By opening the brackets in the definition of f^h we infer the existence of a function $g : J_h(1) \to \mathbb{Z}$ such that $0 < g(i) \leq 2h(i)$ for $i \in J_h(1)$ and such that the monomial

$$Z = \prod_{i \in J_h(0)} z_{i0} \cdot \prod_{i \in J_h(1)} z_{i,g(i)}^2$$

satisfies $D^\Gamma Z \neq 0$. Recall also that

$$D_{\{i,i'\}} = \sum_{j=0}^{2N} \frac{\partial^2}{\partial z_{ij} \partial z_{i'j}} \text{ for } \{i, i'\} \in \mathrm{Bond}_n.$$

Failure of (56) would entail existence of $e \in \Gamma$ such that either e meets $J_h(-1)$ or else e meets both $J_h(0)$ and $J_h(1)$. In either case we would have $D_e Z = 0$, which is a contradiction. Thus (56) holds. Failure of (57) would entail existence of distinct $e, e' \in \Gamma$ such that $e \cap e' \cap J_h(0) \neq \emptyset$. In this case we would have $D_e D_{e'} Z = 0$, which is again a contradiction. Thus (57) holds. Failure of (58) would entail existence of distinct $e_1, e_2, e_3 \in \Gamma$ such that $e_1 \cap e_2 \cap e_3 \neq \emptyset$, in which case $D_{e_1} D_{e_2} D_{e_3} Z = 0$, which is yet again a contradiction. Thus (58) holds. The proof of Proposition 2.6.3 is complete. □

3. Linear forests, cycle-cut permutations and Goulden-Jackson pairs

In this section for the sake of clarity we hold ourselves somewhat aloof from the proof of Theorem 1.2.5 and develop some simple concepts on their own terms. All these concepts are motivated by Proposition 2.6.3 and they will be deployed in §4 below to clinch the proof of Theorem 1.2.5.

3.1. **Linear forests.** The notion developed under this heading is directly motivated by statement (58) of Proposition 2.6.3 above.

3.1.1. *Definition.* Let $\Gamma \subset \mathrm{Bond}_n$ be a subset. We call Γ a *linear forest* if the graph $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ is circuitless (i.e., a forest) and every vertex of the graph $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ has degree at most 2.

3.1.2. *The set* $\mathrm{Tree}_n^{\mathrm{LF}}(\Theta)$. For $\Theta \in \mathrm{Part}_n$, let $\mathrm{Tree}_n^{\mathrm{LF}}(\Theta)$ denote the subset of $\mathrm{Tree}_n(\Theta)$ consisting of linear forests. Now suppose $\Gamma \in \mathrm{Tree}_n(\Theta)$ is given. Then Γ necessarily has the property that the graph $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ is circuitless. Thus for $\Gamma \in \mathrm{Tree}_n(\Theta)$, one has $\Gamma \in \mathrm{Tree}_n^{\mathrm{LF}}(\Theta)$ if and only if every vertex of $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ has degree at most 2.

3.1.3. *Decomposition of linear forests into connected components.* We call a linear forest $\Gamma \subset \mathrm{Bond}_n$ *connected* if the graph $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ has exactly one connected component not reducing to an isolated vertex. Every connected linear forest $\Gamma \subset \mathrm{Bond}_n$ is of the form

(61) $\qquad \Gamma = \{\{i_1, i_2\}, \ldots, \{i_{m-1}, i_m\}\}$ for $m \geq 2$ and distinct $i_1, \ldots, i_m \in \langle n \rangle$.

Note furthermore that the sequence $i_1, \ldots, i_m$ is uniquely determined by Γ up to a reversal of the order of the sequence. It is clear that every linear forest Γ has a disjoint union decomposition

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$$

unique up to ordering of the sets in the decomposition, where each set Γ_i is a connected linear forest and the union

$$\cup\Gamma = (\cup\Gamma_1) \cup \cdots \cup (\cup\Gamma_k)$$

is also disjoint. Each set Γ_i is called a *connected component* of Γ. We say that $\Gamma = \bigcup \Gamma_i$ is the *decomposition* of Γ into its connected components.

3.1.4. *The boundary of a linear forest.* Let $\Gamma \subset \mathrm{Bond}_n$ be a linear forest. We define the *boundary* $\partial\Gamma$ to be the set of unordered pairs of the form $\{a, b\}$ where $\{a\}$ and $\{b\}$ are distinct degree one vertices of the forest $\mathfrak{G}(\mathbf{0}_n, \Gamma)$ joined by some walk. The members of $\partial\Gamma$ are in evident bijection with the connected components of Γ.

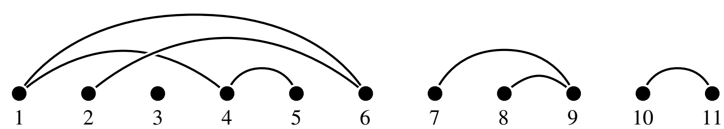

FIGURE 11. The graph $\mathfrak{G}(\mathbf{0}_{11}, \Gamma)$ is depicted above for the linear forest $\Gamma = \{\{1,4\},\{1,6\},\{2,6\},\{4,5\},\{7,9\},\{8,9\},\{10,11\}\} \subset \mathrm{Bond}_{11}$. The connected components of Γ are $\Gamma_1 = \{\{1,4\},\{1,6\},\{2,6\},\{4,5\}\}$, $\Gamma_2 = \{\{7,9\},\{8,9\}\}$, and $\Gamma_3 = \{\{10,11\}\}$. The boundary of Γ is $\partial\Gamma = \{\{2,5\},\{7,8\},\{10,11\}\}$.

3.2. **Cycle-cut permutations.** We next introduce a notion which is nearly equivalent to that of a linear forest.

3.2.1. *Cycle-cuttings.* Let $\sigma \in S_n$ be any permutation. A subset of $\mathrm{supp}\,\sigma$ intersecting each σ-orbit contained in $\mathrm{supp}\,\sigma$ in exactly one point will be called a *cycle-cutting.* A pair (σ, A) consisting of $\sigma \in S_n$ and a cycle-cutting $A \subset \langle n\rangle$ of σ will be called a *cycle-cut permutation.* Now let $\lambda \vdash n$ index the conjugacy class of σ. We define

$$\mathbf{m}(\sigma) = \mathbf{m}(\lambda) = \prod_i i^{m_i(\lambda)} = \prod_{i=1}^{\ell(\lambda)} \lambda_i. \tag{62}$$

Note that σ has exactly $\mathbf{m}(\sigma)$ cycle-cuttings. Given also $i \in \langle n\rangle$, let $\mathbf{m}(\sigma, i)$ denote the cardinality of the σ-orbit to which i belongs. Note that

$$\mathbf{m}(\sigma) = \prod_{a\in A} \mathbf{m}(\sigma, a) \tag{63}$$

for any cycle-cutting A of σ.

3.2.2. *Construction of linear forests from cycle-cut permutations.* Given a cycle-cut permutation (σ, A) of $\langle n\rangle$, let

$$\mathrm{LF}(\sigma, A) = \{\{i, \sigma(i)\} \mid i \in (\mathrm{supp}\,\sigma) \setminus A\} \subset \mathrm{Bond}_n,$$

which is clearly a linear forest. For each cycle-cut permutation (σ, A) of $\langle n\rangle$ and associated linear forest $\Gamma = \mathrm{LF}(\sigma, A)$ it is furthermore clear that

$$\cup\Gamma = \mathrm{supp}\,\sigma,\ |\Gamma| = n - \ell(\sigma),\ \partial\Gamma = \{\{a, \sigma(a)\} \mid a \in A\},\ \text{and} \tag{64}$$

$$\begin{aligned} |\partial\Gamma| &= |A| = \ell(\sigma) - |\{i \in \langle n\rangle \mid \sigma(i) = i\}| \\ &= \text{the number of connected components of } \Gamma. \end{aligned} \tag{65}$$

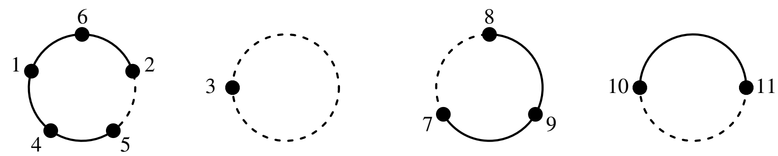

FIGURE 12. For the cycle-cut permutation (σ, A) depicted above where $\sigma = (1,4,5,2,6)(7,9,8)(10,11) \in S_{11}$ and $A = \{5,8,10\}$, the linear forest $\Gamma = \mathrm{LF}(\sigma, A)$ is $\Gamma = \{\{1,4\},\{1,6\},\{2,6\},\{4,5\},\{7,9\},\{8,9\},\{10,11\}\}$ and the boundary is $\partial\Gamma = \{\{2,5\},\{7,8\},\{10,11\}\}$.

The notions of cycle-cut permutation and of linear forest are equivalent up to some manageable powers of 2, as the next lemma explains.

Lemma 3.2.3. *For each linear forest* $\Gamma \subset \mathrm{Bond}_n$*, the set of cycle-cut permutations* (σ, A) *of* $\langle n\rangle$ *such that* $\Gamma = \mathrm{LF}(\sigma, A)$ *has cardinality* 2^k *where* k *is the number of connected components of* Γ*.*

Proof. Suppose that $\Gamma = \mathrm{LF}(\sigma, A)$ for some cycle-cut permutation (σ, A) of $\langle n\rangle$. Let $\sigma = \sigma_1 \cdots \sigma_k$ be the canonical factorization of σ into disjoint cycles. For $i = 1, \dots, k$ let a_i be the unique element of $A \cap \mathrm{supp}\,\sigma_i$. Then

$$\mathrm{LF}(\sigma, A) = \bigcup_{i=1}^{k} \mathrm{LF}(\sigma_i, \{a_i\})$$

is the canonical decomposition of Γ into connected components. Thus we may assume without loss of generality that Γ is connected and more specifically of the form on line (61), in which case

$$((i_1, \dots, i_m), \{i_m\}) \text{ and } ((i_m, \dots, i_1), \{i_1\})$$

are the two cycle-cut permutations associated with Γ. □

3.2.4. *Remark.* Given a linear forest Γ, we call a choice of point from each member of $\partial\Gamma$ an *orientation*, and we call Γ an *oriented* linear forest if it is equipped with an orientation. The notion of cycle-cut permutation is precisely equivalent to the notion of oriented linear forest, with cycle-cuttings corresponding one-to-one with orientations. See Figure 13 for an illustration. We will not use the definition of an oriented linear forest in the sequel (we have already made enough definitions) but we mention it because it is a natural way to understand the powers of 2 coming up in Lemma 3.2.3.

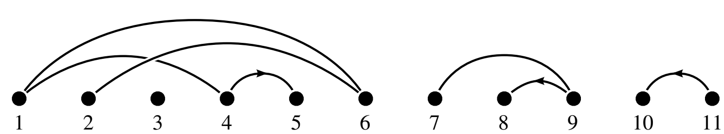

FIGURE 13. The oriented linear forest associated with the permutation $\sigma = (1,4,5,2,6)(7,9,8)(10,11) \in S_{11}$ and the cycle-cutting $A = \{5,8,10\}$.

3.3. **Relationship between linear forests and Goulden-Jackson pairs.** We now make the decisive linkage between on the one hand combinatorial objects related to the BKAR formula and on the other hand Goulden-Jackson pairs.

Proposition 3.3.1. *Fix* $\theta \in S_n$ *and let* $\Theta = \mathrm{Orb}_n(\theta)$. *Fix a cycle-cut permutation* (σ, A) *of* $\langle n\rangle$ *and let* $\Gamma = \mathrm{LF}(\sigma, A) \subset \mathrm{Bond}_n$. *Then* $\sigma \in \mathrm{GJ}_n(\theta)$ *iff* $\Gamma \in \mathrm{Tree}_n^{\mathrm{LF}}(\Theta)$.

Proof. Consider the *Cayley graph*

$$\mathfrak{C} = \mathfrak{G}(\mathbf{0}_n, \{\{i, \theta(i)\}, \{i, \sigma(i)\} \mid i \in \langle n\rangle\}).$$

Then the subgroup of S_n generated by θ and σ acts transitively on $\langle n\rangle$ iff $\mathfrak{C}$ is connected. It is easy to check in turn that $\mathfrak{C}$ is connected iff $\mathfrak{G}(\Theta, \Gamma)$ is connected.

Suppose now that we have $\sigma \in \mathrm{GJ}_n(\theta)$, i.e., $(\theta, \sigma) \in \mathrm{GJ}_n$. Then

$$|\Gamma| = n - \ell(\sigma) = \ell(\theta) - 1 = |\Theta| - 1.$$

Furthermore $\mathfrak{C}$ is connected since $\ell(\sigma\theta) = 1$ and hence $\mathfrak{G}(\Theta, \Gamma)$ is connected. Thus $\mathfrak{G}(\Theta, \Gamma)$ is a tree and hence $\Gamma \in \mathrm{Tree}^{\mathrm{LF}}(\Theta)$.

Suppose now rather that $\Gamma \in \mathrm{Tree}^{\mathrm{LF}}(\Theta)$ and hence that $\mathfrak{G}(\Theta, \Gamma)$ is a tree. Then we have

$$n - \ell(\sigma) = |\Gamma| = |\Theta| - 1 = \ell(\theta) - 1$$

and hence $\ell(\sigma) + \ell(\theta) = n + 1$ holds. It remains only verify $\ell(\sigma\theta) = 1$. In any case, $\mathfrak{G}(\Theta, \Gamma)$ is connected, hence $\mathfrak{C}$ is connected and hence σ and θ generate a subgroup of S_n acting transitively on $\langle n\rangle$. Lemma 3.3.2 immediately below then yields the bound $\ell(\sigma\theta) \leq 1$, which finishes the proof. □

Lemma 3.3.2. *Let* $\sigma, \theta \in S_n$ *be permutations together generating a subgroup of* S_n *acting transitively on* $\langle n\rangle$. *Then* $\ell(\sigma) + \ell(\theta) + \ell(\sigma\theta) \leq n + 2$.

Proof. The lemma reiterates [19, Thm. 3.6, p. 421] in different notation. For readers familiar with the theory of compact Riemann surfaces, we supplement this reference with the following brief explanation. From the permutations θ and σ one knows how to construct a compact Riemann surface of genus g (= number of handles) presented as an n-sheeted covering of the Riemann sphere branched only at 0, 1 and ∞ such that the *Riemann-Hurwitz formula*

$$2g - 2 = -2n + (n - \ell(\theta)) + (n - \ell(\sigma)) + (n - \ell(\theta\sigma))$$

holds. The desired inequality follows simply from the fact that $g \geq 0$. □

The preceding theory provides valuable information about integrals against the measure $\mathbb{P}_\Gamma^\Theta$ of certain simple functions.

Proposition 3.3.3. *Fix* $(\theta, \sigma) \in \mathrm{GJ}_n$. *Let* $\Theta = \mathrm{Orb}_n(\theta)$. *Let* A *be a cycle-cutting for* σ. *Let* $\Gamma = \mathrm{LF}(\sigma, A) \in \mathrm{Tree}_n^{\mathrm{LF}}(\Theta)$. *For* $i, j \in \langle n\rangle$ *let* $\Gamma(i, j) \subset \Gamma$ *be as defined in Lemma 2.2.8. Let* $X \in \mathfrak{Q}_n$ *be a random matrix with law* $\mathbb{P}_\Gamma^\Theta$. *(i) The set* Γ *is the disjoint union of the sets* $\Gamma(a, \sigma(a))$ *for* $a \in A$. *(ii) The family* $\{X(a, \sigma(a))\}_{a\in A}$ *of random variables is independent. (iii)* $\mathbf{E}X(a, \sigma(a)) = \frac{1}{\mathbf{m}(\sigma,a)}$ *for* $a \in A$.

Consequently one has

$$\mathbf{E}\prod_{b\in B} X(b, \sigma(b)) = 1 \Big/ \prod_{b\in B} \mathbf{m}(\sigma, b) \tag{66}$$

for any subset $B \subset A$.

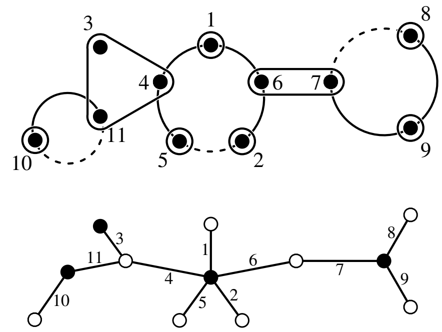

FIGURE 14. The top figure shows the cycle-cut permutation (σ, A) and the set partition $\Theta = \mathrm{Orb}_n(\theta)$ for $\theta = (3, 11, 4)(6, 7)$, $\sigma = (1, 4, 5, 2, 6)(7, 9, 8)(10, 11)$ in S_{11}, and $A = \{5, 8, 10\}$. The corresponding Shabat–Voevodsky tree for (θ, σ) is the bottom figure.

Proof. Let $\Gamma = \bigcup_{a\in A} \Gamma_a$ be the unique decomposition of Γ into its connected components labeled so that $\partial\Gamma_a = \{a, \sigma(a)\}$ for $a \in A$. It is not hard to see that for each $a \in A$ one has $\Gamma_a = \Gamma(a, \sigma(a))$. Thus statement (i) holds. Statement (ii) follows via Lemma 2.2.8 from statement (i). Statement (iii) follows from Lemma 2.2.8 and the undergraduate-level remark that for random variables $U_1, \ldots, U_k$ i.i.d. uniform in $(0, 1)$ one has $\mathbf{E} \min_{i=1}^k U_i = \frac{1}{k+1}$. □

3.4. **Structures related to** $\mathrm{dMotz}_n(\theta, \sigma)$**.** Having developed above an interpretation of statement (58) of Proposition 2.6.3 in group-theoretical terms, we turn next to the task of providing an analogous interpretation of statements (55)—(57).

3.4.1. *The set* $\mathrm{Motz}_n(\theta, \sigma)$. For $(\theta, \sigma) \in \mathrm{GJ}_n$, let

$$\mathrm{Motz}_n(\theta, \sigma) = \{h : \langle n\rangle \to \mathbb{Z} \mid h \circ \sigma = h \text{ and } h \circ \theta - h \in \mathrm{dMotz}_n(\theta, \sigma)\}. \tag{67}$$

In this setting we think of the function $h \circ \theta - h$ as a sort of derivative of h. From this definition one immediately deduces the following statement:

$$h \in \mathrm{Motz}_n(\theta, \sigma) \Leftrightarrow h + c \in \mathrm{Motz}_n(\theta, \sigma) \text{ for constants } c \in \mathbb{Z}. \tag{68}$$

For each positive integer N we also define

$$\mathrm{Motz}_n^N(\theta, \sigma) = \mathrm{Motz}_n(\theta, \sigma) \cap \mathrm{Motz}_n^N(\theta). \tag{69}$$

3.4.2. *"Tilde versions" of the preceding definitions.* Let $(\theta, \sigma) \in \mathrm{GJ}_n$. Let

$$\widetilde{\mathrm{dMotz}}_n(\theta, \sigma) \supset \mathrm{dMotz}_n(\theta, \sigma)$$

be the superset consisting of $g : \langle n\rangle \to \mathbb{Z}$ satisfying (6)–(10) but perhaps not satisfying (11). In turn, let

$$\widetilde{\mathrm{Motz}}_n(\theta, \sigma) = \{h : \langle n\rangle \to \mathbb{Z} \mid h \circ \sigma = h \text{ and } h \circ \theta - h \in \widetilde{\mathrm{dMotz}}_n(\theta, \sigma)\}. \tag{70}$$

Note that the variant of (68) with $\widetilde{\mathrm{Motz}}_n(\theta, \sigma)$ in place of $\mathrm{Motz}_n(\theta, \sigma)$ still holds. Note the trivial but important relation

$$\mathrm{Motz}_n(\theta, \sigma) = \{h \in \widetilde{\mathrm{Motz}}_n(\theta, \sigma) \mid J_h(0) \subset \mathrm{supp}\, \sigma\} \tag{71}$$

where $J_h(\epsilon)$ is as defined on line (49). We also define

$$\widetilde{\mathrm{Motz}}_n^N(\theta, \sigma) = \mathrm{Motz}_n^N(\theta) \cap \widetilde{\mathrm{Motz}}_n(\theta, \sigma). \tag{72}$$

Proposition 3.4.3. *Fix* $(\theta,\sigma)\in \mathrm{GJ}_n$*, a cycle-cutting* A *of* σ *and* $h\in \mathrm{Motz}_n^N(\theta)$*. Let* $\Gamma = \mathrm{LF}(\sigma,A)$*. If the pair* (Γ,h) *satisfies statements* (55)—(57)*, then* $h\in \widetilde{\mathrm{Motz}}_n^N(\theta,\sigma)$*.*

Proof. Statement (55) implies $h\circ\sigma = h$. Let $g = h\circ\theta - h$. It remains only to show that $g\in \widetilde{\mathrm{dMotz}}_n^N(\theta)$. The definition of $\mathrm{Motz}_n^N(\theta)$ implies that g satisfies (6). Clearly, g satisfies (7). Statement (56) implies that g satisfies (8) and (9). Statement (57) implies that g satisfies (10). Thus we indeed have $g\in \widetilde{\mathrm{dMotz}}_n^N(\theta)$ and hence $h\in \widetilde{\mathrm{Motz}}_n^N(\theta,\sigma)$. □

3.5. **A limit calculation.** Our main result under this heading explains the left factor on the left side of formula (14). (See Proposition 3.5.1 below.) Along the way we explain the sense in which each element of $\mathrm{dMotz}(\theta,\sigma)$ has an antiderivative. (See Proposition 3.5.5 below.)

Proposition 3.5.1. *Let* $(\theta,\sigma)\in \mathrm{GJ}_n$*. Let* A *be a cycle-cutting of* σ*. Let* N *be a positive integer. Let* $g\in \mathrm{dMotz}_n(\theta,\sigma)$*. Then we have*

$$\left|\frac{N^{\frac{n}{2}-\ell+2}}{\frac{n}{2}-\ell+2} - \sum_{\substack{h\in \mathrm{Motz}_n^N(\theta,\sigma)\\ \text{s.t. } h\circ\theta-h=g}} \prod_{i\in J_h(1)\setminus((\mathrm{supp}\,\sigma)\setminus A)} h(i)\right| \le cN^{\frac{n}{2}-\ell+1}, \tag{73}$$

where the constant c *depends only on* n*.*

The proof will be completed in §3.5.6 below after suitable preparation.

Lemma 3.5.2. *For* $(\theta,\sigma)\in \mathrm{GJ}_n$*, a cycle-cutting* A *of* σ *and* $h\in\widetilde{\mathrm{Motz}}_n(\theta,\sigma)$ *we have*

$$\frac{1}{\mathbf{m}(\sigma)} \quad = \quad 2^{-|A|}\prod_{a\in J_h(1)\cap A}\frac{2}{\mathbf{m}(\sigma,a)} \quad \text{and} \tag{74}$$

$$\frac{n}{2}-\ell(\theta)+1 \quad = \quad |J_h(1)\setminus((\mathrm{supp}\,\sigma)\setminus A)| + \frac{|J_h(0)\setminus \mathrm{supp}\,\sigma|}{2}. \tag{75}$$

The lemma is overkill for proving Proposition 3.5.1, but will later be needed at full strength to finish the proof of Theorem 1.2.5.

Proof. By (6) and (7) we evidently have

$$\frac{n}{2} = \frac{|J_h(0)|+|J_h(1)|+|J_h(-1)|}{2} = \frac{|J_h(0)|}{2}+|J_h(1)|.$$

By (8) each of the sets $J_h(-1)$, $J_h(0)$ and $J_h(1)$ is σ-stable, i.e., each is a union of σ-orbits. To abbreviate notation let $\Sigma = \mathrm{supp}\,\sigma$. We have

$$\ell(\theta)-1 = n-\ell(\sigma) = |\Sigma\setminus A| = |J_h(0)\cap(\Sigma\setminus A)| + |J_h(1)\cap(\Sigma\setminus A)|,$$

at the first step by definition of a Goulden-Jackson pair, at the second step as a consequence of the definition of a cycle-cutting and at the last step by (9). By (10) we have $\mathbf{m}(\sigma,a)=2$ for $a\in A\cap J_h(0)$, whence (74) and furthermore we have

$$|J_h(0)\cap A| = \frac{|J_h(0)\cap\Sigma|}{2} = |J_h(0)\cap(\Sigma\setminus A)|.$$

Formula (75) can then be obtained by combining the displayed lines above. □

Lemma 3.5.3. *Fix $(\theta,\sigma)\in \mathrm{GJ}_n$ and a function $h:\langle n\rangle\to\mathbb{Z}$ such that $h\circ\sigma=h$. Then we have*

$$\max_{i,j\in\langle n\rangle}|h(i)-h(j)|\leq (n-1)\max_{i\in\langle n\rangle}|h(\theta(i))-h(i)|. \tag{76}$$

In particular, for $(\theta,\sigma)\in\mathrm{GJ}_n$ we have crude bounds

$$\left|\left\{h\in\widetilde{\mathrm{Motz}}_n(\theta,\sigma)\middle|h(1)=0\right\}\right|\leq 3^n\ \text{ and }\ \left|\widetilde{\mathrm{Motz}}_n^N(\theta,\sigma)\right|\leq 3^nN. \tag{77}$$

Proof. By hypothesis $|h\circ(\sigma\theta)-h|=|h'|$ and $\ell(\sigma\theta)=1$, whence the bound. □

Lemma 3.5.4. *Fix $(\theta,\sigma)\in\mathrm{GJ}_n$. Let $g:\langle n\rangle\to\mathbb{Z}$ be a function averaging to 0 on each θ-orbit. Then there exists a function $h:\langle n\rangle\to\mathbb{Z}$ such that $h\circ\sigma=h$ and $h'=g$.*

Proof. Consider again the Cayley graph

$$\mathfrak{C}=\mathfrak{G}(\mathbf{0}_n,\{\{i,\theta(i)\},\{i,\sigma(i)\}\mid i\in\langle n\rangle\})$$

introduced in the proof of Proposition 3.3.1. Since $\ell(\theta\sigma)=1$, it is clear that $\mathfrak{C}$ is connected. Let A (resp., B) be a cycle-cutting for θ (resp., σ). Let

$$\mathfrak{T}=\mathfrak{G}(\mathbf{0}_n,\{\{i,\theta(i)\}\mid i\in(\operatorname{supp}\theta)\setminus A\}\cup\{\{j,\sigma(j)\}\mid j\in(\operatorname{supp}\sigma)\setminus B\}).$$

It is clear that any two distinct vertices of $\mathfrak{T}$ joined by a walk in $\mathfrak{C}$ remain joined by some walk in $\mathfrak{T}$. Thus $\mathfrak{T}$ is connected. Furthermore, $\mathfrak{T}$ has no more than $n-1$ edges because

$$|(\operatorname{supp}\theta)\setminus A|+|(\operatorname{supp}\sigma)\setminus B|=(n-\ell(\theta))+(n-\ell(\sigma))=n-1.$$

Thus $\mathfrak{T}$ is a tree spanning $\mathfrak{C}$. In particular, $\mathfrak{T}$ has exactly $n-1$ edges. Now (so to speak) every vector field on a tree is the gradient of a potential and this statement holds over $\mathbb{Z}$. Thus there exists some function $h:\langle n\rangle\to\mathbb{Z}$ such that

$$h(\theta(i))-h(i)=g(i)\ \text{ for } i\in A \text{ and }\ h(\sigma(j))-h(j)=0\ \text{ for } j\in B.$$

Clearly, we have $h\circ\sigma=h$. Finally, since g averages to zero on each θ-orbit, we have $h\circ\theta-h=g$. Thus h exists. □

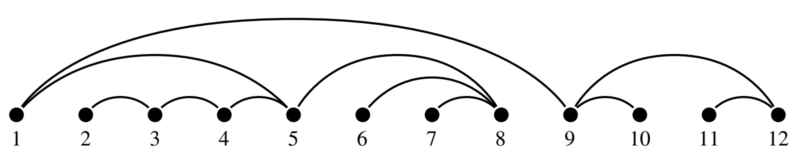

FIGURE 15. The tree $\mathfrak{T}$ from Lemma 3.5.4 for the cycle-cut permutations (θ,A) and (σ,B) where $(\theta,\sigma)\in\mathrm{GJ}_{12}$, $\theta=(1,9)(2,3,4,5)(6,7,8)$, $\sigma=(1,5,8)(9,10,11,12)$, $A=\{5,6,9\}$, and $B=\{8,10\}$.

Proposition 3.5.5. *For $(\theta,\sigma)\in\mathrm{GJ}_n$ and $g\in\mathrm{dMotz}_n(\theta,\sigma)$ there exists unique $h\in\mathrm{Motz}_n(\theta,\sigma)$ such that $h'=g$ and $h(1)=0$.*

This finally is the rationale for the peculiar notation $\mathrm{dMotz}(\theta,\sigma)$.

Proof. Lemmas 3.5.3 and 3.5.4 together prove this. □

3.5.6. *Proof of Proposition 3.5.1.* Let

$$H = \sum_{\substack{h \in \mathrm{Motz}_n^N(\theta,\sigma) \\ \mathrm{s.t.}\, h\circ\theta - h = g}} \frac{1}{N} \prod_{i \in J_h(1)\setminus((\mathrm{supp}\,\sigma)\setminus A)} \left(\frac{h(i)}{N}\right) \text{ and } \nu = \frac{n}{2} - \ell + 1.$$

It will be enough to prove that

$$\left| H - \frac{1}{\nu+1} \right| \leq \frac{(n+1)^2}{N}. \tag{78}$$

By (71) and (75) we have

$$h \in \mathrm{Motz}_n^N(\theta,\sigma) \Rightarrow \nu = |J_h(1) \setminus ((\mathrm{supp}\,\sigma) \setminus A)|.$$

By Lemma 3.5.3 and the definitions we have

$$h \in \mathrm{Motz}_n(\theta,\sigma) \Rightarrow \max_{i,j \in \langle n \rangle} |h(i) - h(j)| < n.$$

By Proposition 3.5.5 there exists unique $h_0 \in \mathrm{Motz}_n(\theta,\sigma)$ such that $h_0' = g$ and $h_0(1) = 0$. Let

$$\widehat{H} = \sum_{\substack{h \in \mathrm{Motz}_n^N(\theta,\sigma) \\ \mathrm{s.t.}\, h' = g}} \frac{1}{N} \left(\frac{h(1)}{N}\right)^\nu = \sum_{\substack{k \in \langle N \rangle \text{ s.t.} \\ 1 \leq k + \min h_0 \text{ and} \\ k + \max h_0 \leq N}} \frac{1}{N} \left(\frac{k}{N}\right)^\nu$$

where the second equality is justified by (68) and Proposition 3.5.5. Then we have

$$|H - \widehat{H}| \leq \frac{n^2}{N} \text{ and } \left| -\widehat{H} + \sum_{k=1}^{N} \frac{1}{N} \left(\frac{k}{N}\right)^\nu \right| \leq \frac{2n}{N}.$$

Finally, for any integer $\nu \geq 0$ we have evident inequalities

$$\sum_{k=0}^{N-1} \frac{1}{N} \left(\frac{k}{N}\right)^\nu \leq \int_0^1 t^\nu \,\mathrm{d}t \leq \sum_{k=1}^{N} \frac{1}{N} \left(\frac{k}{N}\right)^\nu.$$

Estimate (78) follows from the inequalities on the last two displayed lines. □

4. Proof of Theorem 1.2.5

4.1. Refinement of expansion (54).

Proposition 4.1.1. *In the setup of Proposition 2.6.3 we have the yet more refined expansion*

$$\mathfrak{M}_{\lambda,N} = \sum_{(\sigma,A,h,\Gamma)} \int 2^{-|A|} \mathbf{E}\left[(D^\Gamma f^h)(\zeta^h \star Q)\right] \mathbb{P}_\Gamma^\Theta(\mathrm{d}Q) \tag{79}$$

where the sum is extended over quadruples (σ, A, h, Γ) where $\sigma \in \mathrm{GJ}_n(\theta)$, A is a cycle-cutting of σ, $h \in \widetilde{\mathrm{Motz}}_n^N(\theta,\sigma)$ and $\Gamma = \mathrm{LF}(\sigma, A)$.

This expansion has interest beyond the scope of this paper. Conceivably one could work out the $\frac{1}{N}$-expansion of the right side and derive an alternate interpretation of the coefficients of the $\frac{1}{N}$-expansion of $\mathfrak{M}_{\lambda,N}$.

Proof. By Proposition 2.6.3 combined with we have

$$\begin{aligned}\mathfrak{M}_{\lambda,N} &= \sum_{\substack{\Gamma\in\mathrm{Tree}_n(\Theta)\text{ and }h\in\mathrm{Motz}_n^N(\theta)\\ \text{s.t. (55)---(58) hold.}}} \int \mathbf{E}\left[(D^\Gamma f^h)(\zeta^h\star Q)\right]\,\mathbb{P}_\Gamma^\Theta(\mathrm{d}Q)\\ &= \sum_{\Gamma\in\mathrm{Tree}_n^{\mathrm{LF}}(\Theta)} \sum_{\substack{h\in\mathrm{Motz}_n^N(\theta)\\ \text{s.t. (55)---(57) hold.}}} \int \mathbf{E}\left[(D^\Gamma f^h)(\zeta^h\star Q)\right]\,\mathbb{P}_\Gamma^\Theta(\mathrm{d}Q).\end{aligned}$$

By Proposition 3.3.1 the formula (79) holds with the sum is extended over quadruples (σ, A, h, Γ) such that $\sigma \in \mathrm{GJ}_n(\theta)$, A is a cycle-cutting of σ, $h \in \mathrm{Motz}_n^N(\theta)$ satisfies (55)—(57) and $\Gamma = \mathrm{LF}(\sigma, A)$. Note that Lemma 3.2.3 justifies the correction factor $2^{-|A|}$. The formula (79) then holds as stated by Proposition 3.4.3. □

Lemma 4.1.2. *For a quadruple (σ, A, h, Γ) indexing a term on the right side of* (79) *we have*

$$D^\Gamma f^h = \prod_{i\in J_h(0)\setminus\mathrm{supp}\,\sigma} z_{i0}\cdot \prod_{i\in J_h(1)\setminus\mathrm{supp}\,\sigma}\sum_{j=1}^{2h(i)}\frac{z_{ij}^2}{2}\cdot \prod_{i\in J_h(1)\cap A}\sum_{j=1}^{2h(i)} z_{ij}z_{\sigma(i),j}. \tag{80}$$

Proof. Let

$$\sigma = \sigma_1\cdots\sigma_p\tau_1\cdots\tau_q$$

be the canonical factorization of σ into disjoint cycles, with the factors sorted so that

$$\bigcup_{\alpha=1}^p \mathrm{supp}\,\sigma_\alpha \subset J_h(1)\ \text{ and }\ \bigcup_{\beta=1}^q \mathrm{supp}\,\tau_\beta \subset J_h(0).$$

Such a sorting is possible because $h\circ\sigma = h$. Note that each permutation τ_β is necessarily a transposition since $i \in J_h(0) \Rightarrow \sigma^2(i) = i$. For $\alpha = 1,\dots,p$ and $\beta = 1,\dots,q$ let

$$\{a_\alpha\} = A\cap\mathrm{supp}\,\sigma_\alpha,\ M_\alpha = 2h(a_\alpha),\ \Gamma_\alpha = \mathrm{LF}(\sigma_\alpha,\{a_\alpha\})\ \text{ and }\ e_\beta = \mathrm{supp}\,\tau_\beta.$$

Then

$$\Gamma = \bigcup_{\alpha=1}^p \Gamma_\alpha \cup \bigcup_{\beta=1}^q \{e_\beta\}$$

is the decomposition of Γ into connected components and we have a factorization

$$D^\Gamma f^h = \prod_{i\in S_0} z_{i0}\cdot\prod_{i\in S_1}\sum_{j=1}^{2h(i)}\frac{z_{ij}^2}{2}\cdot\prod_{\alpha=1}^p D^{\Gamma_\alpha}\prod_{i\in\mathrm{supp}\,\sigma_\alpha}\sum_{j=1}^{M_\alpha}\frac{z_{ij}^2}{2}\cdot\prod_{\beta=1}^q D_{e_\beta}\prod_{i\in e_\beta} z_{i0}.$$

It is easy to see that $D_{e_\beta}\prod_{i\in e_\beta} z_{i0} = 1$. To finish the proof we need only evaluate $D^{\Gamma_\alpha}\prod_{i\in\mathrm{supp}\,\sigma_\alpha}\sum_{j=1}^{M_\alpha}\frac{z_{ij}^2}{2}$. For the latter purpose we note the formula

$$\left(\sum_{j=0}^{2N}\frac{\partial^2}{\partial z_{i_2 j}\partial z_{i_3 j}}\right)\left[\left(\frac{|\{i_1,i_2\}|}{2}\sum_{j=1}^M z_{i_1 j}z_{i_2 j}\right)\left(\sum_{j=1}^M\frac{z_{i_3 j}^2}{2}\right)\right] = \sum_{j=1}^M z_{i_1 j}z_{i_3 j}$$

holding for $i_1, i_2, i_3 \in \langle n\rangle$ such that $i_3 \notin \{i_1, i_2\}$ and $1 \le M \le 2N$. Using this and induction one can finish the proof. We omit the remaining details. □

4.2. **Application of the Marcinkiewicz-Zygmund inequality.**

4.2.1. *Variant of the Marcinkiewicz-Zygmund inequality.* For a real random variable Z and $p \in [1,\infty)$, let $\|Z\|_p = (\mathbf{E}|Z|^p)^{1/p}$. Now let $T_1,\dots,T_n$ be independent real random variables with absolute moments of all orders. Then for every positive integer k one has a bound

$$\left\|\sum_{i=1}^n T_i - \sum_{i=1}^n \mathbf{E}T_i\right\|_{2k} \leq kn^{k/2}\max_{i=1}^n \|T_i - \mathbf{E}T_i\|_{2k} \leq 2kn^{k/2}\max_{i=1}^n \|T_i\|_{2k} \tag{81}$$

which is a variant of the Marcinkiewicz-Zygmund inequality. We remark that this particular variant (which is at some remove from the full-strength version of the M.-Z. inequality) can be given a simple combinatorial proof.

Lemma 4.2.2. *Let (σ, A, h, Γ) be a quadruple indexing a summand on the right side of (79). Let $Q \in \mathfrak{Q}_n$. Let $\ell = \ell(\theta)$. Let*

$$S_0 = J_h(0)\setminus \operatorname{supp}\sigma,\quad S_1 = J_h(1)\setminus\operatorname{supp}\sigma,\quad A_1 = J_h(1)\cap A.$$

We have

$$\left|\mathbf{E}[(D^\Gamma f^h)(\zeta^h \star Q)] - \mathbb{1}\{S_0 = \emptyset\}\cdot \prod_{i\in A_1} 2Q(i,\sigma(i))\cdot \prod_{i\in S_1\cup A_1} h(i)\right| \leq cN^{\frac{n}{2}-\ell+\frac{1}{2}} \tag{82}$$

for a constant c depending only on n.

Proof. Let $S = S_0\cup S_1\cup A_1$, noting that the union is disjoint. For $i\in S$ let

$$Z_i = \begin{cases} z_{i0}\Big|_{\zeta^h\star Q} & \text{if } i\in S_0,\\ \displaystyle\sum_{j=1}^{2h(i)} \frac{z_{ij}^2}{2}\Big|_{\zeta^h\star Q} & \text{if } i\in S_1,\\ \displaystyle\sum_{j=1}^{2h(i)} z_{ij}z_{\sigma(i),j}\Big|_{\zeta^h\star Q} & \text{if } i\in A_1.\end{cases} \tag{83}$$

By the definitions and Lemma 4.1.2 we have

$$\mathbf{E}\prod_{i\in S} Z_i = \mathbf{E}[(D^\Gamma f)(\zeta^h\star Q)]. \tag{84}$$

We also have

$$\mathbf{E}Z_i = \begin{cases} 0 & \text{if } i\in S_0,\\ h(i) & \text{if } i\in S_1,\\ 2Q(i,\sigma(i))h(i) & \text{if } i\in A_1\end{cases} \tag{85}$$

by using the fact that by definition $\zeta^h\star Q$ is a centered Gaussian random vector with covariances

$$\mathbf{E}(\zeta\star Q)_{ij}(\zeta\star Q)_{i'j'} = \delta_{h(i),h(i')}\delta_{jj'}Q(i,i').$$

Using this same covariance information, the general bound (81) recalled above and the fact that ζ^h is a Gaussian random vector, we also have

$$\|Z_i - \mathbf{E}Z_i\|_{2n} \leq \gamma\begin{cases} 1 & \text{if } i\in S_0,\\ \sqrt{N} & \text{if } i\in S_1\cup A_1,\end{cases} \tag{86}$$

where the constant $\gamma \geq 1$ depends only on n. Finally we have

$$\left\| \prod_{i\in S} Z_i - \prod_{i\in S} \mathbf{E} Z_i \right\|_2 \leq \sum_{i\in S} \|Z_i - \mathbf{E}Z_i\|_{2n} \prod_{i'\in S\setminus\{i\}} \|Z_{i'}\|_{2n} \leq cN^{\min(|S\setminus S_0|,|S|-\frac{1}{2})}$$

where c depends only on n, whence estimate (82) by Lemma 3.5.2. □

4.3. **Closing arguments to prove Theorem 1.2.5.** From (82), by integrating on both sides against $\mathbb{P}_\Gamma^\Theta$ and using Jensen's inequality, along with Proposition 3.3.3 and formula (74), we deduce the inequality

(87)
$$\left| 2^{-|A|} \int \mathbf{E}[(D^\Gamma f^h)(\zeta^h \star Q)] \mathrm{d}\mathbb{P}_\Gamma^\Theta(Q) - \frac{\mathbb{1}\{S_0 = \emptyset\}}{\mathbf{m}(\sigma)} \prod_{i\in S_1\cup A_1} h(i) \right| \leq c_1 N^{\frac{n}{2}-\ell+\frac{1}{2}}.$$

After using (77) and (87) to approximate the right side of (79), we obtain the approximation

$$\left| \mathfrak{M}_{\lambda,N} - \sum_{(\sigma,A,h,\Gamma)} \frac{\mathbb{1}\{J_h(0) \subset \operatorname{supp}\sigma\}}{\mathbf{m}(\sigma)} \prod_{i\in J_h(1)\setminus((\operatorname{supp}\sigma)\setminus A)} h(i) \right| \leq c_2 N^{\frac{n}{2}-\ell+\frac{3}{2}}$$

where c_2 depends only on n and the sum is extended over the same family of quadruples (σ, A, h, Γ) as in (79). Using (71), (73), and again (77) we then get a further approximation

$$\left| \mathfrak{M}_{\lambda,N} - \frac{N^{\frac{n}{2}-\ell+2}}{\frac{n}{2}-\ell+2} \cdot \sum_{\sigma\in \mathrm{GJ}_n(\theta)} \sum_{\substack{\text{cycle-cuttings} \\ A \text{ of } \sigma}} \frac{|\mathrm{dMotz}_n(\theta,\sigma)|}{\mathbf{m}(\sigma)} \right| \leq c_3 N^{\frac{n}{2}-\ell+\frac{3}{2}}$$

where c_3 depends only on n. Note that the inner sum over cycle-cuttings A of σ is canceled by the factor $1/\mathbf{m}(\sigma)$. Thus the last estimate in conjunction with limit formula (19) proves Theorem 1.2.5. □

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