# THE MATRIX MODEL FOR DESSINS D'ENFANTS

JAN AMBJØRN[†] AND LEONID CHEKHOV[*]

Abstract. We present the matrix models that are the generating functions for branched covers of the complex projective line ramified over 0, 1, and $\infty$ (Grotendieck's dessins d'enfants) of fixed genus, degree, and the ramification profile at infinity. For general ramifications at other points, the model is the two-logarithm matrix model with the external field studied previously by one of the authors (L.Ch.) and K.Palamarchuk. It lies in the class of the generalised Kontsevich models (GKM) thus being the Kadomtsev–Petviashvili (KP) hierarchy $\tau$-function and, upon the shift of times, this model is equivalent to a Hermitian one-matrix model with a general potential whose coefficients are related to the KP times by a Miwa-type transformation. The original model therefore enjoys a topological recursion and can be solved in terms of shifted moments of the standard Hermitian one-matrix model at all genera of the topological expansion. We also derive the matrix model for clean Belyi morphisms, which turns out to be the Kontsevich–Penner model introduced by the authors and Yu. Makeenko. Its partition function is also a KP hierarchy tau function, and this model is in turn equivalent to a Hermitian one-matrix model with a general potential. Finally we prove that the generating function for general two-profile Belyi morphisms is a GKM thus proving that it is also a KP hierarchy tau function in proper times.

## 1. Introduction

In a nice recent paper [23] Zograf provided recursion relations for the generating function of Grothendieck's *dessins d'enfants* enumerating the Belyi pairs $(C, f)$, where $C$ is a smooth algebraic curve and $f$ a meromorphic function $f : C \to \mathbb{C}P^1$ ramified only over the points $0, 1, \infty \in \mathbb{C}P^1$.

We recall some mathematical results relating Belyi pairs to Galois groups and begin with

**Theorem 1.1.** (Belyi, [5]) *A smooth complex algebraic curve $C$ is defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if it exists a nonconstant meromorphic function $f$ on $C$ ($f : C \to \mathbb{C}P^1$) ramified only over the points $0, 1, \infty \in \mathbb{C}P^1$.*

For a Belyi pair $(C, f)$ let $g$ be the genus of $C$ and $d$ the degree of $f$. If we take the inverse image $f^{-1}([0, 1]) \subset C$ of the real line segment $[0, 1] \in \mathbb{C}P^1$ we obtain a connected bipartite fat graph with $d$ edges with vertices being preimages of 0 and 1 and with the cyclic ordering of edges entering a vertex coming from the orientation of the curve $C$. This led Grothendieck to formulating the following lemma:

**Lemma 1.2.** (Grothendieck, [18]) *There is a one-to-one correspondence between the isomorphism classes of Belyi pairs and connected bipartite fat graphs.*

We define a Grothendieck *dessin d'enfant* to be a connected bipartite fat graph representing a Belyi pair.

It is well known that we can naturally extend the dessin $f^{-1}([0, 1]) \subset C$ corresponding to a Belyi pair $(C, f)$ to a bipartite triangulation of the curve $C$. For this, we cut the complex plane

[†]Niels Bohr Institute, Copenhagen University, Denmark, and IMAPP, Radboud University, Nijmengen, The Netherlands. Email: ambjorn@nbi.dk.

[*]Steklov Mathematical Institute and Laboratoire Poncelet, Moscow, Russia; Niels Bohr Institute, Copenhagen University, Denmark. Email: chekhov@mi.ras.ru.

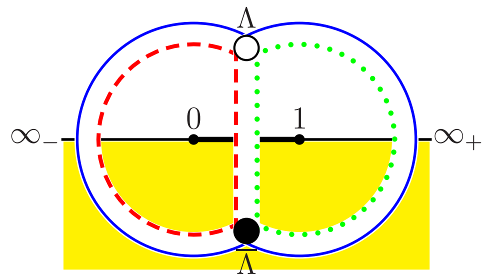



FIGURE 1. The Belyi graph $\Gamma_1$ corresponding to the Belyi pair $(\mathbb{C}P^1, \mathrm{id})$; $\infty_\pm$ indicate directions of approaching the infinite point in $\mathbb{C}P^1$. By $\Lambda$, $\overline{\Lambda}$ we indicate the insertions of the external field in the matrix-model formalism of Sec. 2. For example, this graph contributes the term $N^2\beta\gamma\,\mathrm{tr}(\Lambda\overline{\Lambda})$.

along the (real) line containing $0, 1, \infty$ coloring upper half plane white and lower half plane gray. This defines the partition of $C$ into white and grey triangles such that white triangles has common edges only with grey triangles. We then consider a dual graph in which edges are of three types (pre-images of the three edges shown in Fig. 1): the type of an edge depend on which segment—$f^{-1}([0,1]) \subset C$, $f^{-1}([1,\infty_+]) \subset C$, or $f^{-1}([\infty_-,0]) \subset C$—it intersects ($\infty_\pm$ indicate the directions of approaching the infinite point in $\mathbb{C}P^1$). Each face of the dual partition then contains a preimage of exactly one of the points $0, 1, \infty$, so they are of three sorts (bordered by solid, dotted, or dashed lines in the figure). We call such a graph a *Belyi fat graph*.

The type of ramification at infinity is determined by the set of solid-line bounded faces of a Belyi fat graph: the order of branching is $r$ for a $2r$-gon, so we introduce the generating function that distinguish between different types of branching at infinity. We let $n_1, n_2, n_3$ be the respective numbers of solid-, dotted-, and dashed-line cycles (faces) and let $m_r$ be the number of solid-line cycles of length $2r$ in a Belyi fat graph

We are interested in the following **counting problem**: we are going to calculate the generating function

$$\mathcal{F}\big[\{t_m\}, \beta, \gamma; N\big] = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} N^{2-2g}\beta^{n_2}\gamma^{n_3}\prod_{i=1}^{n_1} t_{r_i}, \tag{1.1}$$

where $N$, $\beta, \gamma$, and $t_r$ are formal independent parameters and the sum ranges all (connected) Belyi fat graphs. Often a factor $\alpha^{n_1}$ is also added; it can however be adsorbed into the times $t_r$ by scaling $t_r \to \alpha t_r$ for all $r$.

The structure of the paper is as follows. In Sec. 2, we show that generating function (1.1) is the free energy of a special matrix model. We demonstrate that this model is the two-logarithm matrix model of [14], and it therefore belongs to the class of generalized Kontsevich models (GKM) [20]. In Sec. 3, we present the solution of this model from paper [14] in which it was reduced, upon a special transformation of times, to a Hermitian one-matrix model with a general potential. In Sec. 4, we present the direct solution of the original generating function in terms of the Hermitian one-matrix model without appealing to the external field model thus again establishing the equivalence between the two models and describing the corresponding topological recursion. In Sec. 5, we construct the matrix model for *clean* Belyi morphisms (those having ramifications only of type $(2, 2, \ldots, 2)$ over 1) and show that the corresponding generating function is the original Kontsevich–Penner model of [12]. This model is also equivalent [13] to the Hermitian one-matrix model with a general potential. Finally, in Sec. 6, we combine the techniques of Secs. 2, 3, and 4 establishing that the generating function for the two-profile Belyi morphisms (with the given ramifications at two points, $\infty$ and 1) is again given by the GKM integral thus being a tau function of the KP hierarchy (that is, it satisfies the bilinear Hirota relations). We conclude with the discussion of our results.

Throughout the entire text we disregard all multipliers not depending on external fields; all equalities in the paper must therefore be understood modulo such factors.

## 2. The model

In our conventions the indices $i$, $i_1$, $i_2$, etc. take positive integer values between 1 and $\alpha N$, the indices $j$, $j_1$, etc. take positive integer values between 1 and $\beta N$, and the indices $k$, $k_1$, etc. take positive integer values between 1 and $\gamma N$. We introduce three complex-valued rectangular matrices $R_{k,i}$, $G_{i,j}$, and $B_{j,k}$ and one diagonal matrix (the external field) $\Lambda_{i_1,i_2} = \lambda_{i_1}\delta_{i_1,i_2}$. The action is given by the integral

$$\mathcal{F}[\{t_r\},\beta,\gamma;N] := \int DR\,D\overline{R}\,DB\,D\overline{B}\,DG\,D\overline{G}\,e^{N\mathrm{tr}(-B\overline{B}-R\overline{R}-G\overline{G}+R\Lambda GB+\overline{B}\,\overline{G}\,\overline{\Lambda}\,\overline{R})}. \tag{2.1}$$

The free energy $\mathcal{F}[\{t_r\},\beta,\gamma;N]$ is given by the sum over all connected bipartite three-valent fat graphs $\Gamma$ weighted by

$$\frac{1}{|\mathrm{Aut}\,\Gamma|}N^{2-2g}\beta^{n_2}\gamma^{n_3}\prod_r t_r^{m_r} \qquad \Big(\sum_r m_r = n_1\Big) \tag{2.2}$$

where $n_{1,2,3}$ are the respective numbers of solid-, dotted-, and dashed-line cycles in $\Gamma$,

$$t_r := \sum_{i=1}^{\alpha N} |\lambda_i|^{2r} \tag{2.3}$$

are the *times* of the model, and $m_r$ is the number of solid-line cycles of length $2r$ in $\Gamma$. Measures of integration are the standard Haar measures; for instance,

$$DR\,D\overline{R} := \prod_{k=1}^{\gamma N}\prod_{i=1}^{\alpha N} d\mathrm{Re}\,R_{k,i} d\mathrm{Im}\,R_{k,i}.$$

The logarithm of the integral (2.1) is therefore exactly generating function (1.1) for the Belyi graphs.

Integrating w.r.t. $B, \overline{B}$ we obtain the integral

$$\int DR\,D\overline{R}\,DG\,D\overline{G}\,e^{N\mathrm{tr}(-R\overline{R}-G\overline{G}+R\Lambda G\overline{G}\,\overline{\Lambda}\,\overline{R})} \tag{2.4}$$

in which we can perform the Gaussian integration w.r.t. $G, \overline{G}$ thus obtaining

$$\int DR\,D\overline{R}\,e^{-N\mathrm{tr}(R\overline{R})}\det\left[\delta_{i_1,i_2} - (\overline{\Lambda}\,\overline{R}R\Lambda)_{i_1,i_2}\right]^{-\beta N}. \tag{2.5}$$

After the change of variables $R \to R\Lambda$ this integral becomes

$$\prod_{i=1}^{\alpha N}|\lambda_i|^{-2\gamma N}\int DR\,D\overline{R}\,e^{-N\mathrm{tr}\left(\overline{R}R[\Lambda\overline{\Lambda}]^{-1}\right)}\det\left[\delta_{i_1,i_2} - (\overline{R}R)_{i_1,i_2}\right]^{-\beta N}. \tag{2.6}$$

For definiteness, let $\gamma \geq \alpha$. A general rectangular matrix $\overline{R}$ can then be reduced to the form $\overline{R} = U^\dagger \overline{M} V$, where $U \in U(\alpha N)$, $V \in U(\gamma N)/U((\gamma-\alpha)N)$, and

$$\overline{M} = \left(\begin{array}{ccc|cc} \overline{m}_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \overline{m}_{\alpha N} & 0 & 0 \end{array}\right).$$

In the vicinity of the unities of the unitary groups, we can write $U = e^{i\epsilon H}$ and $V = e^{i\epsilon Q}$ with the Hermitian $(\alpha N \times \alpha N)$-matrix $H$ and Hermitian $(\gamma N \times \gamma N)$-matrix $Q$ of the form

$$Q = \left( \begin{array}{c|c} \widetilde{H} & P \\ \hline P^\dagger & 0 \end{array} \right), \tag{2.7}$$

in which $\widetilde{H}$ is another Hermitian $(\alpha N \times \alpha N)$-matrix and $P$ is the general complex $(\alpha N \times (\gamma - \alpha)N)$-matrix. The Jacobian of the transformation

$$D\overline{R}\, DR = \mathrm{Jac}\, DU\, DV \prod_i dm_i d\overline{m}_i \tag{2.8}$$

can then be easily calculated (see Appendix A) to be

$$\mathrm{Jac} = \prod_{1\le i_1<i_2\le \alpha N} (|m_{i_2}|^2 - |m_{i_1}|^2)^2 \prod_{i=1}^{\alpha N} |m_i|^{2(\gamma-\alpha)N}, \tag{2.9}$$

Introducing the new variables $x_i = |m_i|^2$ ranging from zero to infinity, we reduce the integral in (2.6) to the $\alpha N$-fold integral w.r.t. $x_i$ and to the integration w.r.t. the unitary group:

$$\prod_{i=1}^{\alpha N} |\lambda_i|^{-2\gamma N} \int_0^\infty dx_1 \dots dx_{\alpha N} \Bigl[ \int DU e^{-N\sum_{i_1,i_2} x_{i_1} U_{i_1,i_2} |\lambda_{i_2}|^{-2} U^\dagger_{i_2,i_1}} \Bigr] \times$$

$$\times \bigl[\Delta(x)\bigr]^2 \prod_{i=1}^{\alpha N} \bigl[ x_i^{(\gamma-\alpha)N} (1-x_i)^{-\beta N} \bigr]. \tag{2.10}$$

The integral over $DU$ is given by the Itzykson–Zuber–Mehta formula (we write it having in mind that we subsequently integrate it over variables $x_i$ with a totally symmetric measure),

$$\int DU\, e^{-N\sum_{i_1,i_2} x_{i_1} U_{i_1,i_2} |\lambda_{i_2}|^{-2} U^\dagger_{i_2,i_1}} = \frac{e^{-N\sum_i x_i |\lambda_i|^{-2}}}{\Delta(x_i)\Delta(|\lambda_i|^{-2})},$$

so the final formula for the generating function reads

$$\frac{\prod_{i=1}^{\alpha N} |\lambda_i|^{-2\gamma N}}{\Delta(|\lambda|^{-2})} \int_0^\infty dx_1 \dots dx_{\alpha N} \Delta(x) e^{N\sum_i [-x_i|\lambda_i|^{-2} + (\gamma-\alpha)\log x_i - \beta \log(1-x_i)]}. \tag{2.11}$$

The integral (2.11) is equivalent to the matrix-model integral

$$\prod_{i=1}^{\alpha N} |\lambda_i|^{-2\gamma N} \int_{\alpha N\times\alpha N} DH_{\ge 0} e^{N\,\mathrm{tr}[-H\Lambda^{-2} + (\gamma-\alpha)\log H - \beta\log(1-H)]}, \tag{2.12}$$

where the integration goes over Hermitian $(\alpha N \times \alpha N)$-matrices with positive eigenvalues. We thus obtain the following statement.

**Lemma 2.1.** *The generating function for Grothendieck dessins d'enfants (Belyi fat graphs (1.1)) is the matrix-model integral (2.12).*

The integral (2.12) belongs to the class of *generalized Kontsevich models* (GKM) [20]; in terms of variables $\xi_i = 1/|\lambda_i|^2$ it can be calculated as the ratio of determinants of $(\alpha N \times \alpha N)$-matrices,

$$\left\| \frac{\partial^{i_1-1} f(\xi_{i_2})}{\partial \xi_{i_2}^{i_1-1}} \right\| / \Delta(\xi),$$

where

$$f(\xi) = \int_0^\infty dx e^{-Nx\xi} x^{(\gamma-\alpha)N} (1-x)^{-\beta N},$$

and as such is a tau-function of the Kadomtsev–Petviashvili (KP) hierarchy in times $t_n = \sum_i \xi_i^{-n} = \sum_i |\lambda_i|^{2n}$ (cf. (2.3)) i.e., we come to the following theorem proved by Zograf [23] by purely combinatorial means with the using of the cut-and-joint operator.

**Theorem 2.2.** *The generating function for Belyi fat graphs (1.1) is the tau-function of the KP hierarchy in times (2.3).*

Integral (2.12) was studied by one of the authors and Palamarchuk [14] in relation to exploring possible explicit solutions of matrix models with external fields. It was called the *two-logarithm model* there and it was proved that this integral admits Virasoro constraints that, upon a proper change of times, become the Virasoro constraints of the matrix model introduced in [12] (the term Kontsevich–Penner model was coined there), which, in turn, is equivalent [13] to a Hermitian one-matrix model with the potential related to the external-field variables $\xi_i$ via the Miwa transformation. As such, this integral must also satisfy the equations of the Toda chain hierarchy.

**Remark 2.3.** An important remark concerning integral (2.12) is that its asymptotic behavior as $N \to \infty$ is different depending on whether $\gamma - \alpha \simeq O(1)$ or $\gamma - \alpha \simeq O(1/N)$. In the first case, we have an infinite repulsive potential at the origin and an eigenvalue distribution is confined within an interval $[x'_-, x'_+]$ (see below) with $0 < x'_- < x'_+$. The $1/N$-expansion then is "insensitive" to the hard edge at the origin, and we can assume that we integrate over the whole real axis (the difference between the restricted and nonrestricted integrations is then exponentially small in $N$). If $\gamma = \alpha$ or $\gamma - \alpha \sim O(1/N)$, representation (2.12) still remains valid, but in this case the eigenvalue support is $[0, x'_+]$, so it reaches the hard edge $x = 0$ at the origin. We then again have a topological expansion (about $1/N$-expansion in matrix models with hard edges, see, e.g., review [8]) but with the differential $ydx$ finite at $x = 0$ ($y \sim 1/\sqrt{x}$ as $x \to 0$ and $y \sim \sqrt{x - x'_+}$ as $x \to x'_+$). The asymptotic expansions of integral (2.12) are therefore different in the corresponding regimes and do not admit an analytical transition as $\gamma \to \alpha$.

**Remark 2.4.** In Sec. 4, we present a simpler, straightforward way of proving that generating function (1.1) for general Belyi morphisms is indeed a Hermitian one-matrix model free energy. However, the external field technique of this and next sections will be instrumental when proving a general correspondence between the generating functions for clean (Sec. 5) and two-profile (Sec. 6) Belyi morphisms and free energies of the corresponding generalized Kontsevich models.

## 3. The two-logarithm matrix model

In this section, we present the results of [14] adapted to the notation of integral (2.12).

3.1. **Constraint equations for integral (2.12).** We first perform the variable changing

$$\begin{aligned} &\tilde N = \alpha N, \quad \tilde\Lambda = \Lambda^{-2}/(2\alpha), \quad \tilde H = 2H - 1 \\ &\tilde\alpha = \beta/\alpha, \quad \tilde\beta = 1 - \gamma/\alpha. \end{aligned} \tag{3.1}$$

in (2.12). Disregarding here and hereafter factors not depending on $\lambda$'s, the integral then takes the form

$$\prod_{i=1}^{\tilde N}\Big[|\tilde\lambda_i|^{\gamma N} e^{-\tilde N|\tilde\lambda_i|}\Big] \int_{\tilde N\times\tilde N} D\tilde H_{\geq 0} e^{-\tilde N \operatorname{tr}[\tilde H\tilde\Lambda + \tilde\alpha\log(1-\tilde H) + \tilde\beta\log(1+\tilde H)]} := \prod_{i=1}^{\tilde N}\Big[|\tilde\lambda_i|^{\gamma N} e^{-\tilde N|\tilde\lambda_i|}\Big]\mathcal{Z}[\tilde\lambda], \tag{3.2}$$

where we let $\mathcal{Z}[\tilde\lambda]$ denote the integral (2.12) without the normalization factor.

The Schwinger–Dyson equations for integral (3.2) follow from the identity (here all the indices range from 1 to $\alpha N$)

$$\left(\frac{1}{\tilde N^3}\frac{\partial}{\partial\tilde\Lambda_{jk}}\frac{\partial}{\partial\tilde\Lambda_{li}}-\frac{1}{\tilde N}\right)\int_{\tilde N\times\tilde N}D\tilde H\,\frac{\partial}{\partial\tilde H_{ij}}e^{-\tilde N\,\mathrm{tr}[\tilde H\tilde\Lambda+\tilde\alpha\log(1-\tilde H)+\tilde\beta\log(1+\tilde H)]}=0. \tag{3.3}$$

In terms of the eigenvalues $\tilde\lambda_i$ of the matrix $\tilde\Lambda$, the corresponding $\tilde N$ equations read
(3.4)

$$\left[-\frac{1}{\tilde N^2}\tilde\lambda_{i_1}\frac{\partial^2}{\partial\tilde\lambda_{i_1}^2}-\frac{1}{\tilde N^2}\sum_{i_2\neq i_1}\frac{\tilde\lambda_{i_2}}{\tilde\lambda_{i_2}-\tilde\lambda_{i_1}}\Big(\frac{\partial}{\partial\tilde\lambda_{i_2}}-\frac{\partial}{\partial\tilde\lambda_{i_1}}\Big)+\frac{\tilde\alpha+\tilde\beta-2}{\tilde N}\frac{\partial}{\partial\tilde\lambda_{i_1}}+\tilde\beta-\tilde\alpha+\tilde\lambda_{i_1}\right]\mathcal{Z}[\tilde\lambda]=0,$$

We can equivalently write the constraint equations (3.4) in terms of the *times*

$$t_n=\frac{1}{n}\sum_i\frac{1}{\tilde\lambda_i^n},\quad n\ge 1. \tag{3.5}$$

They then becomes the set of *Virasoro constraints*[1]

$$V_k\mathcal{Z}\big(\{t_n\}\big)=0,\quad k\ge 0, \tag{3.6}$$

where

$$\begin{aligned}V_k[t] \;:=\; & -\sum_{m=1}^{\infty}mt_m\frac{\partial}{\partial t_{m+k}}-\sum_{m=1}^{k}\frac{\partial}{\partial t_m}\frac{\partial}{\partial t_{k-m}}-\tilde N(\tilde\alpha-\tilde\beta+1)(1-\delta_{k,0}-\delta_{k,-1})\frac{\partial}{\partial t_k}\\ & +\big[2\tilde N(1-\delta_{k,-1})+\delta_{k,-1}t_1\big]\frac{\partial}{\partial t_{k+1}}+\tilde N^2\tilde\alpha(\tilde\beta-1)\delta_{k,0},\quad k=-1,0,1,\dots .\end{aligned} \tag{3.7}$$

(Here, for the future use, we have also introduced the operator $V_{-1}$.)

The operators $V_k$ enjoy the Virasoro algebra

$$[V_k,V_l]=(l-k)V_{k+l},\quad k,l\ge -1. \tag{3.8}$$

3.2. **Equivalence to the Hermitian one-matrix model.** In [14] it was shown that the two-logarithm model is equivalent to the Kontsevich–Penner model [12], which in turn was known [13], [20] to be equivalent to a Hermitian one-matrix model. In this paper, we skip the intermediate step and demonstrate the equivalence between (2.12) and a Hermitian one-matrix model defined as an integral

$$\mathcal{Z}_{1\mathrm{MM}}\big[\{\xi_m\},M\big]:=\int_{M\times M}DY\,e^{-V(Y)},\quad V(Y)=\sum_{m=1}^{\infty}\xi_m\,\mathrm{tr}\,Y^m. \tag{3.9}$$

It is well-known that this integral satisfies the set of Virasoro constraints uniformly written in the form
(3.10)

$$L_n\mathcal{Z}_{1\mathrm{MM}}\big[\{\xi_m\},M\big]=\Big\{\sum_{m=0}^{n}\frac{\partial^2}{\partial\xi_m\partial\xi_{n-m}}+\sum_{m=1}^{\infty}m\xi_m\frac{\partial}{\partial\xi_{n+m}}\Big\}\mathcal{Z}_{1\mathrm{MM}}\big[\{\xi_m\},M\big]=0,\ n\ge -1,$$

where we have used a convenient notation $\frac{\partial}{\partial\xi_0}\mathcal{Z}_{1\mathrm{MM}}\big[\{\xi_m\},M\big]=-M\mathcal{Z}_{1\mathrm{MM}}\big[\{\xi_m\},M\big]$.

In order to establish the correspondence it is necessary to shift the original variable $\tilde\lambda$,

$$\mu_i=\tilde\lambda_i-\rho,\quad \rho\in\mathbb{C}, \tag{3.11}$$

[1]The authors were informed by M. Kazarian that the same constraints can be derived by pure combinatorial means [M. Kazarian, P. Zograf, paper in preparation].

introducing an auxiliary parameter $\rho$. We also introduce the new times

$$\tau_n := \frac{1}{n}\sum_{i=1}^{\tilde N}\frac{1}{\mu_i^n}, \quad n \geq 1, \tag{3.12}$$

and the new normalizing factor

$$\mathcal{N}[\mu] := \prod_{i=1}^{\tilde N}\left[\mu_i^{\tilde N(\tilde\beta-1)}e^{\tilde N\mu_i}\right] \tag{3.13}$$

The following set of constraints was found in [14]:

**Lemma 3.1.** (see [14]) *The normalized integral $\mathcal{Z}[\tilde\lambda]/\mathcal{N}[\mu]$ where $\tilde\lambda_i = \mu_i + \rho$ satisfies the set of Virasoro constraints*

$$\mathcal{L}_k\big[\mathcal{Z}[\tilde\lambda]/\mathcal{N}[\mu]\big] = 0,\ \ k = -1, 0, 1, \ldots,$$

*in times (3.12) with*

$$\begin{aligned}\mathcal{L}_k \;=\; & -\sum_{m=1+\delta_{k,-1}}^{\infty} m(\tau_m - 2\tilde N\delta_{m,1})\frac{\partial}{\partial\tau_{m+k}} - \sum_{m=1}^{k-1}\frac{\partial^2}{\partial\tau_m\partial\tau_{k-m}} + 2\tilde N\alpha_{KP}(1-\delta_{k,0}-\delta_{k,1})\frac{\partial}{\partial\tau_k}\\ & -2\varphi\tilde N\sum_{m=1+\delta_{k,-1}}\frac{1}{(-\rho)^m}\frac{\partial}{\partial\tau_{k+m}} - (\tilde N\alpha_{KP})^2\delta_{k,0} + \tilde N\alpha_{KP}\Big(\tau_1 - 2\tilde N - \frac{2\varphi\tilde N}{\rho}\Big)\delta_{k,-1},\end{aligned} \tag{3.14}$$

*where $\alpha_{KP} = \tilde\beta - 1$ and $\varphi = -(\tilde\alpha + \tilde\beta - 1)/2$.*

**Remark 3.2.** In order to derive constraints (3.14) the following trick was used in [14]: constraint equations (3.4) after shift (3.11) were written in the form

$$\sum_{k=1}^{\infty}\mu_i^{-k}L_k\mathcal{Z}[\tilde\lambda] = 0,$$

where

$$\begin{aligned}L_k \;=\; & V_{k+1}[\tau] + \rho V_k[\tau] + \rho\tilde N(\tilde\alpha + \tilde\beta - 1)\Big((1-\delta_{k,0}-\delta_{k,-1})\frac{\partial}{\partial\tau_k} - (\tilde\beta-1)\tilde N\delta_{k,0}\Big)\\ & +\rho(\tilde\beta-1)\tilde N(\tau_1 - 2\tilde N)\delta_{k,-1}, \quad k \geq -1,\end{aligned}$$

were differential operators in (shifted) times $\tau_s$ and where we let $V_s[\tau]$ denote operators (3.7) upon the substitution $t \to \tau$. The "proper" Virasoro operators $\mathcal{L}_k$ (3.14) were finally obtained upon the upper-triangular transformation

$$\mathcal{L}_k = \sum_{s=0}^{\infty}\frac{(-1)^s}{\rho^{s+1}}L_{k+s}, \quad k \geq -1.$$

We see that in order to perform all these replacements we have to keep $\rho$ nonzero and finite.

**Lemma 3.3.** (see [14]) *Upon the substitution*

$$\xi_n = \tau_n + \frac{1}{n}\frac{2\varphi\tilde N}{(-\rho)^n} - 2\tilde N\delta_{n,1}, \quad M = \tilde N\alpha_{KP} \tag{3.15}$$

*Virasoro constraints (3.14) become Virasoro constraints (3.10) of the Hermitian one-matrix model. Because these conditions determine the corresponding integrals unambiguously, these two models are equivalent.*

In terms of the original variables, we have the following lemma.

**Lemma 3.4.** *The generating function* $\mathcal{F}[\{t_r\},\beta,\gamma;N]$ *(1.1) for the Belyi fat graphs is given by the exact formula*

$$
\begin{aligned}
e^{\mathcal{F}[\{t_r\},\beta,\gamma;N]} &= \prod_{i=1}^{\alpha N}\left[\left(\frac{1}{2\alpha}-\rho|\lambda_i|^2\right)^{-\gamma N} e^{\alpha N\left(\frac{1}{2\alpha|\lambda_i|^2}-\rho\right)}\right]\times \\
&\times\mathcal{Z}_{1MM}\Big[\xi_m=\tau_m+\frac{1}{m}\frac{(\gamma-\beta)N}{(-\rho)^m}-2\alpha N\delta_{n,1},\, M=-\gamma N\Big]
\end{aligned}
\tag{3.16}
$$

*with* $\tau_m = \frac{1}{m}\sum_{j=1}^{\alpha N}\frac{1}{\mu_j^m}$ *where* $\mu_i+\rho = 1/(2\alpha|\lambda_i|^2)$*. Here* $\mathcal{Z}_{1MM}\big[\{\xi_m\},M\big]$ *is matrix integral (3.9).*

In the next section we demonstrate that this statement enables us to write explicit formulas for terms of the genus expansion of $\mathcal{F}$ provided we know the answer for the free energy of matrix model (3.9) either in terms of momentums [2] or in terms of the topological recursion technique of [17], [10], [11], [1].

**Remark 3.5.** The shift of variables (3.11) is a convenient technical tool that was used in [14] for passing to the full half-Virasoro constraint algebra that includes also the operator $L_{-1}$. As we demonstrate in the next section, the final answers for genus expansion terms do not depend on this auxiliary parameter $\rho$.

3.3. **The genus expansion.** An extensive literature is devoted to solving the one-matrix model (3.9) in the topological (genus) expansion; its free energy $F$ admits a topological expansion $F=\sum_{h=0}^{\infty}M^{2-2h}F_h$, which can be interpreted as a semiclassical expansion of a (quasi)stationary statistical theory. As such, in the large-$M$ limit, we observe a stationary distribution of eigenvalues described by a *spectral curve* of the model. In the present paper, as in [14], we assume that this stationary distribution spans a single interval, and we therefore have a *one-cut* solution based on a spectral curve that is just a double cover of the complex plane with two branching points, $x_+$ and $x_-$ (a sphere). These two points are determined by the constraint equations for the so-called master loop equation [21]

$$
\oint_{C_D}\frac{dw}{2\pi i}\frac{V'(w)}{\sqrt{(w-x_+)(w-x_-)}}=0,\qquad \oint_{C_D}\frac{dw}{2\pi i}\frac{wV'(w)}{\sqrt{(w-x_+)(w-x_-)}}=2M,
\tag{3.17}
$$

where the integration contour encircles the eigenvalue domain (the interval $[x_-,x_+]$ in this case) and not other singularities (including possible singularities of $V'(w)$).

After the Miwa time transformation (3.15) we obtain for $V'(w)$ the expression

$$
V'(w)=-2\alpha N-\sum_{i=1}^{\alpha N}\frac{1}{w-\mu_i}-(\gamma-\beta)N\frac{1}{w+\rho}
\tag{3.18}
$$

and we assume that all $\mu_i$ and $-\rho$ are situated outside the integration contour. We can then take the integrals in (3.17) by residues at $\mu_i$, $-\rho$, and infinity. For the first equation we obtain

$$
-2\alpha N+\sum_{i=1}^{\alpha N}\frac{1}{\sqrt{(\mu_i-x_+)(\mu_i-x_-)}}+(\gamma-\beta)N\frac{1}{\sqrt{(p+x_+)(p+x_-)}}=0
$$

and shifting the branching points

$$
x_++\rho=x'_+,\quad x_-+\rho=x'_-
$$

and recalling that $\mu_i + \rho = \tilde{\lambda}_i$ we obtain the constraint equation solely in terms of $\tilde{\lambda}_i$:

$$-2\alpha N + \sum_{i=1}^{\alpha N} \frac{1}{\sqrt{(\tilde{\lambda}_i - x'_+)(\tilde{\lambda}_i - x'_-)}} + (\gamma - \beta)N \frac{1}{\sqrt{x'_+ x'_-}} = 0 \tag{3.19}$$

For the second constraint equation we obtain

$$-\alpha N(x'_+ + x'_- - 2\rho) + \sum_{i=1}^{\alpha N} \frac{\tilde{\lambda}_i - \rho}{\sqrt{(\tilde{\lambda}_i - x'_+)(\tilde{\lambda}_i - x'_-)}} - \alpha N - (\gamma - \beta)N + (\gamma - \beta)N \frac{-\rho}{\sqrt{x'_+ x'_-}} = -2\gamma N$$

and the term linear in $\rho$ is just the first constraint equation and thus vanishes. So, the second constraint equation becomes

$$(\gamma + \beta - \alpha)N - \alpha N(x'_+ + x'_-) + \sum_{i=1}^{\alpha N} \frac{\tilde{\lambda}_i}{\sqrt{(\tilde{\lambda}_i - x'_+)(\tilde{\lambda}_i - x'_-)}} = 0. \tag{3.20}$$

We see that, as expected, all the dependence on $\rho$ disappears from constraint equations (3.19) and (3.20).

**Remark 3.6.** Equations (3.19) and (3.20) exactly coincide with the respective first and second constraint equations in Eq. (2.14) of [14] upon the substitution

$$\begin{array}{lll} \lambda \to \tilde{\lambda}, & N \to \alpha N & \beta - \alpha \to 1 - \gamma/\alpha - \beta/\alpha, \\ c \to (\beta - \gamma)^2/4\alpha^2, & b/a \to -x'_+ - x'_-, & c/a \to x'_- x'_+. \end{array} \tag{3.21}$$

The answer for $\mathcal{F}_0$ (formula (2.16) in [14]) obtained on the base of these constraint equations therefore coincide (up to the normalization factor $\prod_{i=1}^{\alpha N}\Big[|\tilde{\lambda}_i|^{\gamma N} e^{-\alpha N|\tilde{\lambda}_i|}\Big]$) with the genus zero contribution to generating function (1.1).

3.3.1. *Genus-zero term.* It follows from Remark 3.6 that the genus-zero term $\mathcal{F}_0$ of our generating function (1.1) upon the substitutions (3.21) and (3.1) coincides with $F_0$ found in [14] with added normalization term $\sum_{i=1}^{\alpha N}\big[\gamma N \log \tilde{\lambda}_i - \alpha N \tilde{\lambda}_i\big]$. In terms of variables $x'_\pm$, $\tilde{\lambda}$ the corresponding expression reads

$$\begin{aligned} \mathcal{F}_0 \;=\; & \frac{1}{4}(\beta^2 N^2 + \gamma^2 N^2) \log\big[(x'_+ - x'_-)^2\big] \\ & + N^2(\alpha - \beta - \gamma)\left[|\beta - \gamma| \log\left(\frac{x'_+ + x'_- - 2\sqrt{x'_+ x'_-}}{x'_+ + x'_- + 2\sqrt{x'_+ x'_-}}\right) + \frac{x'_+ + x'_-}{2}\right] \\ & + N^2\Big[\frac{\alpha^2}{8}(x'_+ + x'_-)^2 + \alpha|\beta - \gamma|\sqrt{x'_+ x'_-} - \frac{(\beta - \gamma)^2}{4}\log[x'_+ x'_-]\Big] \\ & + N \sum_{i=1}^{\alpha N} \left\{ \frac{\beta + \gamma}{2} \log|\tilde{\lambda}_i| + g(\tilde{\lambda}_i) - \tilde{\lambda}_i + \frac{\alpha - \beta - \gamma}{2} \log\Big(\tilde{\lambda}_i - \frac{x'_+ + x'_-}{2} + g(\tilde{\lambda}_i)\Big) \right. \\ & \left. - \frac{|\beta - \gamma|}{4} \log \frac{g(\tilde{\lambda}_i) - \frac{\tilde{\lambda}_i (x'_+ + x'_-)}{2\sqrt{x'_+ x'_-}} + \sqrt{x'_+ x'_-}}{g(\tilde{\lambda}_i) + \frac{\tilde{\lambda}_i (x'_+ + x'_-)}{2\sqrt{x'_+ x'_-}} - \sqrt{x'_+ x'_-}} \right\} \\ & - \frac{1}{4} \sum_{i_1, i_2 = 1}^{\alpha N} \log\Big[g(\tilde{\lambda}_{i_1}) g(\tilde{\lambda}_{i_2}) + \tilde{\lambda}_{i_1} \tilde{\lambda}_{i_2} - \frac{\tilde{\lambda}_{i_1} + \tilde{\lambda}_{i_2}}{2}(x'_+ + x'_-) + x'_+ x'_-\Big] \end{aligned} \tag{3.22}$$

where we have introduced the notation $g(\tilde\lambda_i) := \sqrt{(\tilde\lambda_i - x'_+)(\tilde\lambda_i - x'_-)}$.

It is easy to see that in the domain of large $\tilde\lambda_i$, the expansion in (3.22) contains only negative powers of $\tilde\lambda$: the linear and the logarithmic in $\tilde\lambda_i$ terms vanish in this domain.

3.3.2. *Higher genus expressions.* All higher genus corrections to the Hermitian one-matrix model can be written in terms of *moments* [2] $M_r$, $J_r$ of the potential:

(3.23)
$$M_r = \oint_{C_D} \frac{dw}{2\pi i} \frac{V'(w)}{(w-x_+)^{r+1/2}(w-x_-)^{1/2}}, \qquad J_r = \oint_{C_D} \frac{dw}{2\pi i} \frac{V'(w)}{(w-x_+)^{1/2}(w-x_-)^{r+1/2}},\ r\ge 1.$$

Using representation (3.18), we obtain for the moments the following expressions

$$\begin{aligned} M_r &= \sum_{i=1}^{\alpha N} \frac{1}{(\tilde\lambda_i - x'_+)^{r+1/2}(\tilde\lambda_i - x'_-)^{1/2}} + (\gamma-\beta)N\frac{(-1)^r}{(x'_+)^{r+1/2}(x'_-)^{1/2}} \\ J_r &= \sum_{i=1}^{\alpha N} \frac{1}{(\tilde\lambda_i - x'_+)^{1/2}(\tilde\lambda_i - x'_-)^{r+1/2}} + (\gamma-\beta)N\frac{(-1)^r}{(x'_+)^{1/2}(x'_-)^{r+1/2}} \end{aligned} \qquad r\ge 1. \tag{3.24}$$

After substitution (3.24), the answer for $\mathcal{F}_h$ for generating function (1.1) is given by that of the standard Hermitian one-matrix model. We have thus proved the following lemma

**Lemma 3.7.** *In terms of moments (3.24), every term $\mathcal{F}_h$ corresponding to the genus $h>0$ has a polynomial form for higher $h$* [2],

$$\mathcal{F}_h = \sum_{r_s>1, q_s>1} \big\langle r_1\dots r_m; q_1\dots q_l | r\, q\, p\big\rangle_h \frac{M_{r_1}\cdots M_{r_m} J_{q_1}\cdots J_{q_l}}{M_1^r J_1^q |x'_+ - x'_-|^p}, \quad h>1, \tag{3.25}$$

*and* [3]

$$\mathcal{F}_1 = -\frac{1}{24}\log\big[M_1 J_1 |x'_+ - x'_-|^4\big]. \tag{3.26}$$

*Here $\big\langle r_1\dots r_m; q_1\dots q_l | r\, q\, p\big\rangle_h$ are finite (for a fixed $h$) sets of rational numbers given by the topological recursion technique for the standard Hermitian one-matrix model (see* [10]*). They are subject to restrictions: $m+l-r-q = 2-2h$, $\sum_{s=1}^m (r_1-1) + \sum_{s=1}^l (q_s-1) + p = 4h-4$, $p\ge h-1$.*

Using topological recursion we can effectively calculate the numbers $\big\langle r_1\dots r_m; q_1\dots q_l | r\, q\, p\big\rangle_h$. The quantity $|x'_+ - x'_-|$, which is often denoted by $d$, is the length of the interval of eigenvalue support. Formulas (3.25), (3.26), and (3.24) thus describe generating function (1.1) in all orders of the genus expansion.

## 4. Spectral curve and topological recursion

In this section, we directly derive the spectral curve without appealing to a matrix model with external fields. For this, we shrink all solid-line cycles assigning just the original times $t_r$ to the obtained $2r$-valent vertices of the field $B$, $\overline{B}$. The generating function (1.1) is then described by the matrix-model integral over rectangular $(\gamma N\times \beta N)$-matrices $B$:

$$\mathcal{Z}[t] = \int_{\gamma N\times\beta N} DB\, D\overline{B}\, e^{-N\,\mathrm{tr}[B\overline{B}] + N\sum_{r=1}^{\infty}\frac{1}{r}t_r\,\mathrm{tr}[(B\overline{B})^r]}, \tag{4.1}$$

which, using the Jacobian from Appendix A under assumption that $\beta > \gamma$, can be reduced to the $\gamma N$-fold integral over positive $x_k$:

$$\mathcal{Z}[t] = \int_0^\infty dx_1 \dots dx_{\gamma N} [\Delta(x)]^2 \prod_{k=1}^{\gamma N} x_k^{(\beta-\gamma)N} e^{-N \sum\limits_{r=1}^{\infty} \sum\limits_{k=1}^{\gamma N} \frac{1}{r}(\delta_{r,1}-t_r) x_k^r}. \tag{4.2}$$

This integral is again a Hermitian one-matrix model with a logarithmic term in the potential:

$$\mathcal{Z}[t] = \int\limits_{\gamma N \times \gamma N} DX_{\geq 0} e^{-N \operatorname{tr}\left[\sum\limits_{r=1}^{\infty} \frac{1}{r}(\delta_{r,1}-t_r)X^r - (\beta-\gamma)\log X\right]}, \tag{4.3}$$

We have thus obtained another representation of generating function (1.1).

**Lemma 4.1.** *Generating function (1.1) can be presented as a Hermitian one-matrix model integral (4.3) with a logarithmic term in the potential.*

Because we have reduced the original problem to a mere Hermitian one-matrix model integral, we can directly apply a standard topological recursion procedure [10] (see [7] where it was generalized to the case of rational functions $V'(x)$). We let

$$U'(x) := N \sum_{r=1}^{\infty} (\delta_{r,1} - t_r) x^{r-1} \tag{4.4}$$

denote the polynomial part of the potential with times $t_r$ with the shifted first time. The hyperelliptic spectral curve is a sphere with two branching points $x'_+$ and $x'_-$ whose positions are determined by the standard constraints (3.17) in which

$$V'(x) = U'(x) - \frac{N(\beta-\gamma)}{x}, \quad M = \gamma N. \tag{4.5}$$

Constraints (3.17) then become

$$\oint\limits_{C_D} \frac{dw}{2\pi i} \frac{U'(w)}{\sqrt{(w-x'_+)(w-x'_-)}} = \frac{N(\beta-\gamma)}{\sqrt{x'_+ x'_-}}, \quad \oint\limits_{C_D} \frac{dw}{2\pi i} \frac{wU'(w)}{\sqrt{(w-x'_+)(w-x'_-)}} = N(\beta+\gamma), \tag{4.6}$$

i.e., precisely constraints (3.19) and (3.20) after the inverse Miwa transformation.[2]

The $y$-variable of the topological recursion is given by the integral over the contour that encircles the eigenvalue support and the point $x$,

$$y(x) := \oint\limits_{C_{[x'_-,x'_+]}\cup\{x\}} \frac{dw}{2\pi i} \frac{V'(w)\sqrt{(x-x'_+)(x-x'_-)}}{(w-x)\sqrt{(w-x'_+)(w-x'_-)}}, \tag{4.7}$$

which can be evaluated by residues at infinity and at $w = 0$ (due to the presence of a pole term in $V'(w)$) The result reads

$$y(x) = \left(\operatorname{res}_\infty \left[\frac{U'(w)}{(w-x)\sqrt{(w-x'_+)(w-x'_-)}}\right] + \frac{N(\beta-\gamma)}{\sqrt{x'_+ x'_-}}\right) \sqrt{(x-x'_+)(x-x'_-)} \tag{4.8}$$

[2]The term $(\beta+\gamma)$ in the r.h.s. of the second equation is not a misprint.

The **genus expansion** for $h \geq 1$ has the same form as in Lemma (3.7) with the moments given by the standard integrals taken by residues at infinity and at $w = 0$:

$$
\begin{aligned}
M_r &= \operatorname{res}_{w=\infty}\Big[\frac{U'(w)}{(w-x'_+)^{r+1/2}(w-x'_-)^{1/2}}\Big] + (\gamma-\beta)N\frac{(-1)^r}{(x'_+)^{r+1/2}(x'_-)^{1/2}} \\
J_r &= \operatorname{res}_{w=\infty}\Big[\frac{U'(w)}{(w-x'_+)^{1/2}(w-x'_-)^{r+1/2}}\Big] + (\gamma-\beta)N\frac{(-1)^r}{(x'_+)^{1/2}(x'_-)^{r+1/2}}
\end{aligned}
\qquad r \geq 1. \tag{4.9}
$$

The term $\mathcal{F}_0$ has the general form [6] (for the number of eigenvalues equal $t_0N$)

$$
\mathcal{F}_0 = -\frac{1}{2}\int_{C_{[x'_-,x'_+]}} y(x)V(x) - \zeta t_0, \tag{4.10}
$$

where $\zeta$ is the Lagrange multiplier most conveniently obtained as the limit of the integral

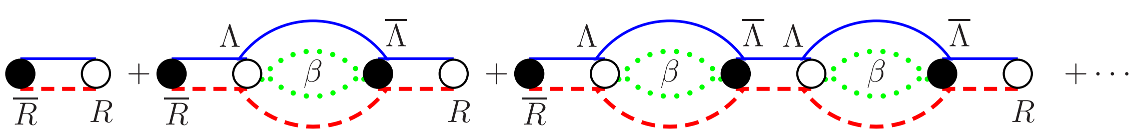
$$
\zeta = \lim_{\Lambda\to+\infty}\Big(\int_{x'_+}^{\Lambda} y(x)dx - V(\Lambda) - t_0\log\Lambda\Big). \tag{4.11}
$$

## 5. Generating functional for clean Belyi morphisms

5.1. **The model.** A *clean* Belyi morphism is a special class of Belyi pairs $(C, f)$ that have profile $(2, 2, \dots, 2, 1, 1, \dots, 1)$ over the branch point $1 \in \mathbb{C}P^1$. This means that all dotted cycles (in Fig. 1) have either lengths 2 (no ramification) or 4 (simple ramification). In [16] the authors demonstrated that the generating function for ramifications of sort $(2, 2, \dots, 2)$ satisfies the topological recursion relations with the spectral curve ($x = z + z^{-1}$; $y = z$).

In this section, we demonstrate that the matrix model corresponding to clean Belyi morphisms is just the Kontsevich–Penner model [12], which is in turn equivalent [13] to the Hermitian one-matrix model with a general potential.

We thus have to calculate generating function (1.1) in which the sum ranges over only clean Belyi morphisms. In terms of the diagrammatic technique of Sec. 2 this means that we count only dotted cycles of lengths 2 and 4. Counting cycles of length 2 reduces to a mere changing of the normalization of the $\langle \overline{R}\,R\rangle$-propagators:



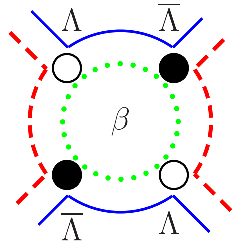

so that the propagator becomes

$$
\langle \overline{R}\,R\rangle \sim \frac{1}{N}\frac{\delta_{i_1,i_2}\delta_{k_1,k_2}}{1-\beta|\lambda_{i_1}|^2}
$$

and the corresponding quadratic form gets an external field addition:

$$
-N\operatorname{tr}[\overline{R}R(1-\beta|\Lambda|^2)]. \tag{5.1}
$$

The new interaction vertex arises from the dotted cycles of length four:




$$\sim \tfrac{1}{2} N\beta\, \mathrm{tr}[\overline{R}R\Lambda\overline{\Lambda}\,\overline{R}R\Lambda\overline{\Lambda}]$$

where the factor 1/2 takes into account the symmetry of the four-cycle.

We therefore have that the generating function $\mathcal{F}$ is the logarithm of the integral

$$\int DR\, D\overline{R}\, e^{N\,\mathrm{tr}[-\overline{R}R(1-\beta|\Lambda|^2)+\frac{1}{2}\beta\overline{R}R|\Lambda|^2\overline{R}R|\Lambda|^2]}, \tag{5.2}$$

where we integrate over rectangular complex $(\gamma N \times \alpha N)$-matrices $R$. We first rescale the integration variable $R \to R\Lambda$, which results in the integral

$$\prod_{i=1}^{\alpha N} |\lambda_i|^{-2\gamma N} \int DR\, D\overline{R}\, e^{N\,\mathrm{tr}[-\overline{R}R(|\Lambda|^{-2}-\beta)+\frac{1}{2}\beta\overline{R}R\overline{R}R]}. \tag{5.3}$$

Performing now the same chain of transformations as in Sec. 2, we obtain eventually that integral (5.3) is equivalent to the Hermitian one-matrix model integral

$$\prod_{i=1}^{\alpha N} |\lambda_i|^{-2\gamma N} \int_{\alpha N\times\alpha N} DH_{\geq 0} e^{N\,\mathrm{tr}[-H(\Lambda^{-2}-\beta)+(\gamma-\alpha)\log H+\frac{1}{2}\beta H^2]}. \tag{5.4}$$

**Lemma 5.1.** *The generating function for clean Belyi fat graphs ((1.1) with ramification profiles* $(2,\dots,2,1,\dots,1)$ *at the point* 1*) is the matrix-model integral (5.4). This matrix-model integral is the (original) Kontsevich–Penner matrix model* [12], [13].

**Remark 5.2.** If we demand the ramification profile at the point 1 to be just $(2,2,\dots,2)$ (no dotted two-cycles are allowed), then in order to obtain the corresponding generating function we must merely replace $\Lambda^{-2}-\beta$ by $\Lambda^{-2}$ in (5.4).

5.2. **Solving integral (5.4).** That the Kontsevich–Penner matrix model integral (5.4) is equivalent to the Hermitian one-matrix model integral (3.9) is well known. This equivalence was established using the Virasoro constraints in [13] or using explicit determinant relations in [20]. We recall here the logic of [20].

We begin with the standard eigenvalue representation for integral (3.9),

$$\int dy_1\dots dy_M\, [\Delta(y)]^2 e^{-\sum_{k=1}^{\infty}\sum_{i=1}^{M}\xi_k y_i^k} \tag{5.5}$$

in which we again perform the Miwa change of variables with the Gaussian shift,

$$\xi_k = \frac{1}{k}\sum_{j=1}^{N}\frac{1}{\mu_j^k} + \frac{1}{2}\delta_{k,2}. \tag{5.6}$$

Summing up the terms in the exponential into logarithms, we transform integral (5.5) to the form

$$\int dy_1\dots dy_M\, [\Delta(y)]^2 \prod_{i=1}^{M}\prod_{j=1}^{N}(\mu_j - y_i)\prod_{j=1}^{N}\mu_j^{-M} e^{-\frac{1}{2}\sum_{i=1}^{M}h_i^2}.$$

We now use that $\Delta(y)\prod_{i=1}^{M}\prod_{j=1}^{N}(\mu_j-y_i)=\Delta(y,\mu)/\Delta(\mu)$, where $\Delta(y,\mu)$ is the Vandermonde determinant of the set of variables $y_i$ and $\mu_j$, write each of the determinants $\Delta(y,\mu)$ and $\Delta(y)$ as determinants of the Hermitian polynomials $H_s(x)$, where $s$ ranges from 0 to $M+N-1$ and $x$ are either $y_i$ or $\mu_j$ in the first determinant and $s$ ranges from 0 to $M-1$ and $x$ are $y_i$ in the second determinant. Because the Hermitian polynomials are orthogonal with the measure $e^{-\frac{1}{2}x^2}$, we can integrate out all the $y$-variables; the remaining expression will be the determinant of the $(N\times N)$-matrix $\|H_{M+j_1-1}(\mu_{j_2})\|$, $j_1,j_2=1,\dots,N$, and the original integral (5.5) thus takes the form

$$\prod_{j=1}^{N}\mu_j^{-M}\frac{1}{\Delta(\mu)}\begin{vmatrix} H_M(\mu_1) & H_M(\mu_2) & \dots & H_M(\mu_N)\\ H_{M+1}(\mu_1) & H_{M+1}(\mu_2) & \dots & H_{M+1}(\mu_N)\\ \vdots & \vdots & \dots & \vdots \\ H_{M+N-1}(\mu_1) & H_{M+N-1}(\mu_2) & \dots & H_{M+N-1}(\mu_N)\end{vmatrix} \tag{5.7}$$

On the other hand, we obtain the same ratio of determinants multiplied by $e^{-\frac{1}{2}\sum_j\mu_j^2}$ if we consider the $N$-fold integral

$$\int dx_1\dots dx_N\,\frac{\Delta(x)}{\Delta(\mu)}\prod_{j=1}^{N}x_j^M e^{\sum_{j=1}^{N}(x_j\mu_j+\frac{1}{2}x_j^2)} \tag{5.8}$$

because $\int dx\,x^s e^{x\mu+\frac{1}{2}x^2}=e^{-\frac{1}{2}\mu^2}H_s(\mu)$. Expression (5.8) is nothing but the Kontsevich–Penner integral, so we obtain the relation between two matrix integrals of *different* sizes:

$$\int\limits_{N\times N} DX\,e^{\mathrm{tr}[X\mu+\frac{1}{2}X^2+M\log X]}=\prod_{j=1}^{M}\bigl[\mu_j^M e^{-\frac{1}{2}\mu_j^2}\bigr]\int\limits_{M\times M}DY\,e^{-\sum\limits_{k=1}^{\infty}\xi_k\,\mathrm{tr}\,Y^k},\ \xi_k=\tfrac{1}{k}\sum_{j=1}^{N}\tfrac{1}{\mu_j^k}+\tfrac{1}{2}\delta_{k,2}. \tag{5.9}$$

After a simple algebra, we come to the following lemma.

**Lemma 5.3.** *The generating function (1.1) for the* clean *Belyi morphisms with the ramification profile* $(2,2,\dots,2)$ *at the point* 1 *is given by the following Hermitian one-matrix model integral for* $\gamma-\alpha\simeq O(1)$*:*

$$\mathcal{Z}[t;\gamma,\beta]=\prod_{i=1}^{\alpha N}|\lambda_i|^{-2\gamma N}\int\limits_{M\times M}DY\,e^{-\sum\limits_{k=1}^{\infty}\frac{t_k}{k}(-1)^k\,\mathrm{tr}\,Y^k-\frac{N}{2\beta}\mathrm{tr}\,Y^2},\ t_k=\sum_{i=1}^{\alpha N}\lambda_i^{2k},\ M=(\gamma-\alpha)N. \tag{5.10}$$

*Because this integral is also equivalent to Kontsevich–Penner matrix model (5.4) (with the external field term* $\Lambda^{-2}$ *instead of* $\Lambda^{-2}-\beta$*), it also belongs to the GKM class thus being a tau function of the KP hierarchy.*

**Remark 5.4.** Note again that the above correspondence is valid only in the $1/N$ asymptotic expansion and only when $\gamma-\alpha\simeq O(1)$. If $\gamma-\alpha\lesssim O(1/N)$ the above correspondence fails because in this case we must take into account that we integrate in formula (5.4) over positive definite matrices, contrary to formula (5.9) in which no restriction on integration domain is assumed. So, again, the case $\gamma=\alpha$ is special and must be treated separately.

## 6. A general case of two-profile Belyi morphisms

Combining the techniques of Secs. 2 and 4 we now address the most general case of Belyi morphisms with given profiles at *two* branching points: infinity and 1. We take these profiles into account in two different ways: at infinity we, as in Sec. 4, introduce the times $t_m$ responsible for the profile whereas the times at 1 will be taken into account by introducing, as in Sec. 2, the external field $\Lambda$ with

$$\mathfrak{t}_s = \mathrm{tr}\Big[(\Lambda\overline{\Lambda})^s\Big] = \sum_{k=1}^{\gamma N} |\lambda_k|^{2s}. \tag{6.1}$$

We then have the following statement

**Lemma 6.1.** *The generating function*

$$\mathcal{F}[\{t_1,t_2,\dots\},\{\mathfrak{t}_1,\mathfrak{t}_2,\dots\},\beta;N] = \sum_{\Gamma} \frac{1}{|Aut\,\Gamma|} N^{2-2g}\beta^{n_2} \prod_{i=1}^{n_1} t_{r_i} \prod_{k=1}^{n_3} \mathfrak{t}_{s_k} \tag{6.2}$$

*of Belyi morphisms in which we have two sets of ramification profiles: $\{t_{r_1},\dots,t_{r_{n_1}}\}$ at infinity and $\{\mathfrak{t}_{s_1},\dots,\mathfrak{t}_{s_{n_3}}\}$ at 1 is given by the integral over complex rectangular $(\beta N\times\gamma N)$-matrices $B,\overline{B}$:*

$$\mathcal{Z}[t,\mathfrak{t}] := e^{\mathcal{F}[\{t\},\{\mathfrak{t}\},\beta;N]} = \int_{\gamma N\times\beta N} DB\,D\overline{B}\,e^{-N\,\mathrm{tr}[B\overline{B}]+N\sum_{m=1}^{\infty}\frac{1}{m}t_m\,\mathrm{tr}[(B\overline{B}\,\overline{\Lambda}\Lambda)^m]}, \tag{6.3}$$

*where the times $\mathfrak{t}_s$ are given by (6.1).*

Performing the same operation as in (4.1)–(4.3), we obtain that integral (6.3) is equal to the integral over Hermitian positive definite $(\gamma N\times\gamma N)$-matrix $X$ with the external matrix field $\tilde{\Lambda} = |\Lambda|^{-2}$:

$$\mathcal{Z}[t,\tau] = \prod_{k=1}^{\gamma N} |\lambda_k|^{-2\beta N} \int\limits_{\gamma N\times\gamma N} DX_{\geq 0} e^{N\,\mathrm{tr}\Big[-X|\Lambda|^{-2}+\sum\limits_{m=1}^{\infty}\frac{t_m}{m}X^m+(\beta-\gamma)\log X\Big]}, \tag{6.4}$$

Integral (6.4) is again a GKM integral [20]; after integration over eigenvalues $x_k$ of the matrix $X$ it takes the form of the ratio of two determinants,

$$\mathcal{Z}[t,\tau] = \prod_{k=1}^{\gamma N} |\lambda_k|^{-2\beta N} \frac{\Big\| \frac{\partial^{k_1-1}}{\partial\tilde{\lambda}_{k_2}^{k_1-1}} f(\tilde{\lambda}_{k_2}) \Big\|_{k_1,k_2=1}^{\gamma N}}{\Delta(\tilde{\lambda})}, \tag{6.5}$$

where

$$f(\tilde{\lambda}) = \int_0^{\infty} x^{N(\beta-\gamma)} e^{-Nx\tilde{\lambda}+N\sum\limits_{m=1}^{\infty}\frac{t_m}{m}x^m}. \tag{6.6}$$

Because any GKM integral (in the proper normalization) is a $\tau$-function of the KP hierarchy, we immediately obtain the following theorem.

**Theorem 6.2.** *The exponential $e^{\mathcal{F}[\{t\},\{\mathfrak{t}\},\gamma;N]}$ of generating function (6.2) modulo the normalization factor $\prod_{k=1}^{\gamma N}|\lambda_k|^{-2\beta N}$ is a $\tau$-function of the KP hierarchy (that is, it satisfies the bilinear Hirota relations) in times $\mathfrak{t}_s$ given by (6.1).*

## 7. Conclusion

We have proved that generating functions for numbers of three different types of Belyi morphisms are free energies of special matrix models all of which are in the GKM class thus being tau functions of the KP hierarchy. Besides this, it is interesting to establish other relations between, say, generating function (1.1) for clean Belyi morphisms and the free energy of the Kontsevich–Penner matrix model, which is known (see [9],[22],[15]) to be related to the numbers of integer points in moduli spaces $\mathcal{M}_{g,n}$ of curves of genus $g$ with $n$ holes with fixed (integer) perimeters; the very same model is also related [9] by a canonical transformation to two copies of the Kontsevich matrix model expressed in times related to the discretization of the moduli spaces $\mathcal{M}_{g,n}$. It is tempting to find possible relations between these discretizations, cut-and-join operators of [23], and Hodge integrals of [19].

Of course, the possibility of using GKM techniques when studying enumeration problems for Belyi morphisms deserves more detailed studies; we consider this note a first step in exploring this perspective field of knowledge.

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## Appendix A Deriving the Jacobian of transformation (2.8)

The invariant measure $DU\,DV$ in the vicinity of the unity becomes $DH\,D\tilde H\,DP\,D\overline{P}$. For $d\overline{R}_{i,k}$ we then obtain

$$d\overline{R}_{i,k}=\Bigl\{d\overline{m}_i\delta_{i,k}+idH_{i,k}\overline{m}_k+im_id\tilde H_{i,k},\ k\le\alpha N\ \Bigm|\ \overline{m}_idP_{i,k-\alpha N},\ k>\alpha N\Bigr\}. \tag{A.1}$$

The elements $dm_i$ appear only for $i=k$ with the unit factor, so we have to calculate only "non-diagonal" differentials $DR\,D\overline{R}$. For $i<k\le\alpha N$ we have:

$$\begin{aligned} d\overline{R}_{i,k}&=idH_{i,k}\overline{m}_k+i\overline{m}_id\tilde H_{i,k}, & d\overline{R}_{k,i}&=idH^*_{i,k}\overline{m}_i+i\overline{m}_kd\tilde H^*_{i,k},\\ dR_{k,i}&=-idH_{i,k}m_i-im_kd\tilde H_{i,k}, & dR_{i,k}&=-idH^*_{i,k}m_k-im_id\tilde H^*_{i,k}. \end{aligned} \tag{A.2}$$

Combining the columns in these relations, we obtain

$$\begin{aligned} d\overline{R}_{i,k}\wedge dR_{k,i}&=dH_{i,k}\wedge d\tilde H_{i,k}[m_k\overline{m}_k-m_i\overline{m}_i],\\ d\overline{R}_{k,i}\wedge dR_{i,k}&=dH^*_{i,k}\wedge d\tilde H^*_{i,k}[m_i\overline{m}_i-m_k\overline{m}_k], \end{aligned}\qquad 1\le i<k\le\alpha N, \tag{A.3}$$

and we obtain that

$$\bigwedge_{i,k=1}^{\alpha N}d\overline{R}_{i,k}\wedge dR_{k,i}=DH\wedge D\tilde H\wedge\prod_{i=1}^{\alpha N}dm_i\wedge d\overline{m}_i\prod_{1\le i<k\le\alpha N}\bigl[|m_i|^2-|m_k|^2\bigr]^2. \tag{A.4}$$

For the remaining part we merely obtain from (A.1) that

$$\bigwedge_{\substack{i=1,\dots,\alpha N\\ k=\alpha N+1,\dots,\gamma N}}d\overline{R}_{i,k}\wedge dR_{k,i}=DP\wedge D\overline{P}\prod_{i=1}^{\alpha N}|m_i|^{2(\gamma-\alpha)N}, \tag{A.5}$$

so we finally obtain formula (2.9) for the Jacobian of transformation (2.8).

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