A POLYNOMIAL INVARIANT OF GRAPHS ON ORIENTABLE SURFACES

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1. *Introduction*

Our aim in this paper is to construct a polynomial invariant of cyclic graphs, that is, graphs with cyclic orders at the vertices, or, equivalently, of 2-cell embeddings of graphs into closed orientable surfaces. We shall call this invariant the *cyclic graph polynomial*, and denote it by the letter C. The cyclic graph polynomial is a three-variable polynomial which generalizes the Tutte polynomial in an essential way. In the next section we define cyclic graphs from two different viewpoints, and give some background on cyclic graphs and the Tutte polynomial. In § 3 we discuss one-vertex cyclic graphs, thought of as chord diagrams, introducing an algebraic notion of the rank of a chord diagram needed to define the cyclic graph polynomial. In § 4 we define the cyclic graph polynomial in terms of recurrence relations and a 'boundary condition' on one-vertex cyclic graphs, and state our main result – that these relations have a (unique) solution. This is proved in § 5. In § 6 we state and prove a universal property of the cyclic graph polynomial. In § 7 we give an alternative description of the rank of a chord diagram, as the genus of the surface naturally associated to the diagram. Finally, in § 8 we give some further properties of the cyclic graph polynomial, showing that it has a spanning tree expansion, and, more importantly, that it depends on the orders (or embedding) in an essential way.

2. *Background*

2.1. *Cyclic graphs*

The objects we consider in this paper arise in many different contexts, and have thus been given many different names. Following Dennis Sullivan we shall call them *cyclic graphs*. We shall think of a cyclic graph G^π as an abstract graph G, in which loops and multiple edges are allowed, together with a cyclic order on the edges at each vertex v of G (so a loop at v appears twice in this order). Two cyclic graphs are isomorphic as cyclic graphs if there is a graph isomorphism between the underlying graphs which respects the cyclic orders. These objects are called 'graphs with rotation systems' in [**18**, **19**]; in [**18**] it is shown that a certain class of directed cyclic graphs classifies Morse–Smale flows up to topological equivalence. A similar result for foliations is proved in [**19**]. Cyclic graphs are

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also known as ‘fatgraphs’, for example in [**17**], where they are used to analyse the moduli spaces of Riemann surfaces with distinguished points. In [**9**] cyclic graphs, referred to as ‘fat graphs’, are used to study moduli spaces of flat connections on Riemann surfaces. Note, however, that the term ‘fatgraphs’, or ‘fat graphs’, is also used for distinct but related objects, in which each edge is ‘thickened’ to form a band.

Cyclic graphs arise naturally from graphs embedded into oriented surfaces. In fact, as has been known for a long time, they correspond exactly to 2-*cell embeddings*, where each component of the complement of the embedded graph is a disc; see [**8**, **13**, **14**]. In abstract terms, cyclic graphs without isolated vertices are sometimes known as *combinatorial premaps*, a *combinatorial map* being a connected cyclic graph. Combinatorial premaps give perhaps the most concise formal definition of cyclic graphs: a combinatorial premap is a triple (S, θ, π) where S is a finite set, θ a fixed-point free involution on S, and π any permutation of S. One can think of S as the set of *half-edges*, or *darts*, or *flags* of the graph, θ as identifying which pairs of half-edges are joined to form an edge, and the cycles of π as corresponding to the vertices, with each cycle giving the cyclic order on the half-edges meeting at the corresponding vertex. The survey paper [**24**] gives the definition and basic properties of combinatorial maps.

Trivalent cyclic graphs, known as Chinese characters or Feynman diagrams, arise in rather different contexts. Examples include perturbative Chern–Simons theory and its application to the construction of 3-manifold invariants; see [**11**, **12**, **16**], for example. Trivalent cyclic graphs also arise in the study of Vassiliev invariants (see [**2**] for example), as do cyclic graphs with only one vertex; these, as described in § 3, are equivalent to chord diagrams.

2.2. *The Tutte polynomial*

The *Tutte polynomial*, or *Whitney–Tutte dichromate* is an important graph invariant introduced by Tutte in [**22**], based on the work of Whitney [**27**]. Given a graph G and an edge e of G we shall say that e is a *loop* if its endvertices are the same, a *bridge* if its removal increases the number of components of G, and that e is *ordinary* if it is neither a loop nor a bridge. We write $G - e$ for the graph formed from G by deleting the edge e, and, if e is not a loop, G/e for the graph formed from G by *contracting* e, that is, deleting e and identifying its endvertices to form a single vertex. We may define the Tutte polynomial $T(G; X, Y)$ in terms of contraction and deletion as follows:

$$T(G; X, Y) = Y^n$$

if G is a graph with n loops and no other edges, and

$$T(G; X, Y) = \begin{cases} T(G/e; X, Y) + T(G - e; X, Y) & \text{if } e \text{ is an ordinary edge of } G, \\ XT(G/e; X, Y) & \text{if } e \text{ is a bridge in } G. \end{cases}$$

It is not *a priori* clear that these relations have a solution – this was shown by Tutte, who defined the polynomial in a very different way, in terms of a spanning tree expansion. Note that the Tutte polynomial is genuinely a polynomial, rather than an array of coefficients written as a polynomial, as shown by the multiplicative structure: if G_1 and G_2 are graphs which are either completely disjoint, or share no edges and exactly one vertex, then we have $T(G_1 \cup G_2) = T(G_1)T(G_2)$.

The Tutte polynomial is important because it carries a huge amount of information about the graph. For example, evaluating the Tutte polynomial at particular points one obtains the number of spanning trees of G, the number of acyclic orientations of G, the number of k-colourings of G, and the number of nowhere-zero k-flows on G. For a much longer list see [**26**]. In fact, the Tutte polynomial is the *universal* contraction-deletion invariant of graphs: if a graph invariant ϕ satisfies

$$\phi(G) = \begin{cases} \sigma\phi(G/e) + \tau\phi(G-e) & \text{if } e \text{ is an ordinary edge of } G, \\ X\phi(G/e) & \text{if } e \text{ is a bridge in } G, \end{cases}$$

then ϕ can be calculated from the Tutte polynomial, together with the number of components, number of vertices, and number of edges of G; see [**4**, **20**].

In other contexts, natural generalizations of the Tutte polynomial play important roles. For example, the Tutte polynomial for signed graphs described in [**15**] specializes to the Kauffman bracket for link diagrams, and thus the Jones polynomial, while the Tutte polynomials for weighted graphs defined in [**10**, **21**] play an important role in statistical physics. A single invariant of graphs with arbitrary labels, or colours, on the edges is introduced in [**4**]. This has a universal property like that of the Tutte polynomial – from it can be calculated any invariant satisfying contraction deletion relations where the coefficients may depend upon the colour of the edge. This includes all the examples mentioned above.

Our aim in this paper is to generalize the Tutte polynomial to graphs with a very different kind of extra structure, namely to cyclic graphs. For coloured graphs, it is clear where to start looking for a generalization of the Tutte polynomial – one takes the defining contraction-deletion relations, or the spanning tree expansion, and replaces the coefficients involved with coefficients depending on the colour of the edge. The problem is then to show that for the coefficients one chooses the result is well defined, or, taking the more general approach of [**4**], to find the set of all possible coefficients for which the result is well defined. In the case of cyclic graphs the situation is very different, as it is not at all clear how we should attempt to generalize the Tutte polynomial. What turns out to be the key concept, the algebraic rank of a chord diagram defined in § 3, has nothing whatsoever to do with any definition of the Tutte polynomial.

2.3. *Contraction-deletion for cyclic graphs*

If e is an edge of a cyclic graph G^π, let $G^\pi - e$ be the cyclic graph obtained by deleting the edge e from the underlying graph and from whichever cyclic order(s) it occurs in. The definition for contraction is slightly more complicated, but just as natural: if $e = xy$ is not a loop, G^π/e has G/e as underlying graph, and inherits all cyclic orders from G^π except at the vertex (xy) formed by identifying x and y. Here the cyclic order is obtained by uniting those at x and y using the position of the edge e, as shown in Figure 1.

As in the graph case, contraction and deletion behave well with respect to each other – whenever the cyclic graphs involved are defined we have $G^\pi/e/f \cong G^\pi/f/e$, $G^\pi/e - f \cong G^\pi - f/e$ and, of course, $G^\pi - e - f \cong G^\pi - f - e$. These important operations arise naturally in many contexts. For one example specific to cyclic graphs see [**9**].

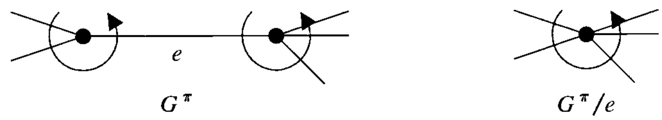


FIGURE 1. *The contraction operation for cyclic graphs.*

The notion of disjoint union for cyclic graphs is exactly the same as for graphs. The one-point join needs a little more explanation. If $G_1^{\pi_1}$ and $G_2^{\pi_2}$ are cyclic graphs sharing no edges and exactly one vertex v, then G^π is the *one-point join* of $G_1^{\pi_1}$ and $G_2^{\pi_2}$ if G^π is the union of $G_1^{\pi_1}$ and $G_2^{\pi_2}$ as graphs, all its cyclic orders respect those of $G_1^{\pi_1}$ and $G_2^{\pi_2}$, and in the order at v, all edges of $G_1^{\pi_1}$, say, precede all those of $G_2^{\pi_2}$. If this is the case we write $G^\pi = G_1^{\pi_1} \cdot G_2^{\pi_2}$. This is illustrated in Figure 2.

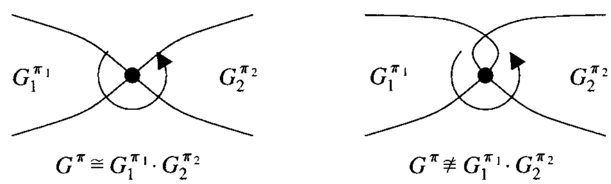


FIGURE 2. *The one-point join operation for cyclic graphs.*

Our aim in this paper is to construct the universal cyclic graph invariant C satisfying the contraction-deletion relations

$$C(G^\pi) = \begin{cases} C(G^\pi/e) + C(G^\pi - e) & \text{if } e \text{ is an ordinary edge of } G^\pi, \\ XC(G^\pi/e) & \text{if } e \text{ is a bridge in } G^\pi. \end{cases} \quad (1)$$

This turns out to be a three-variable polynomial $C(G^\pi; X, Y, Z)$, the *cyclic graph polynomial*. Like the Tutte polynomial, the cyclic graph polynomial is multiplicative on disjoint unions and one-point joins. However, as we shall see later, the cyclic graph polynomial is an essential generalization of the Tutte polynomial, and depends very strongly on the cyclic orders, that is, on the embedding of the underlying graph. In the next section we discuss one-vertex cyclic graphs, that is, chord diagrams, and do the groundwork necessary to define the cyclic graph polynomial and to show that it exists.

3. *Chord diagrams*

We shall construct the cyclic graph polynomial using contraction-deletion relations to reduce an arbitrary cyclic graph to cyclic graphs consisting of a single vertex and some number of loops. While ordinary graphs with one vertex and n loops are all isomorphic, this is not the case for cyclic graphs. Labelling the loops with $1, 2, \ldots, n$, say, we find that the cyclic order at the vertex gives rise to a *cyclic pairing*, that is, a cyclic order on the multi-set $\{1, 1, 2, 2, \ldots, n, n\}$, and two cyclic graphs are isomorphic if and only if the corresponding cyclic pairings can be interchanged by permuting the labels.

Cyclic pairings also arise in other contexts. One example is the study of Vassiliev invariants (see [**3**], for example), where they have been called *n-configurations* [**25**]. Let K be an oriented singular knot with n singular points. When K is traversed once, each singular point is visited twice, so the cyclic order in which the singular points are visited generates a cyclic pairing. Any invariant of order n depends by definition only on the cyclic pairing generated by K.

A very different context in which cyclic pairings arise is that of DNA sequencing by hybridization [**1**]. The task is to reconstruct a sequence from the multi-set of its short subsequences. In a certain limit, the only ambiguities likely to arise come from subsequences occurring exactly twice. The pattern in which these occur is a pairing; for the details of why it is only the corresponding cyclic pairing which matters see [**1**].

However they arise, cyclic pairings are conveniently represented by *chord diagrams*, or just *diagrams*. A chord diagram of *degree n* consists of $2n$ distinct points on the unit circle, together with n chords pairing them off. Two chord diagrams are isomorphic if one can be transformed into the other by a map from the unit circle to itself which preserves cyclic order. Note that if two one-vertex cyclic graphs $G_1^{\pi_1}$, $G_2^{\pi_2}$ are represented by diagrams D_1, D_2 then $G_1^{\pi_1}$ is isomorphic to $G_2^{\pi_2}$ if and only if D_1 is isomorphic to D_2. In what follows we shall not distinguish a one-vertex cyclic graph from the corresponding chord diagram.

In order to construct the cyclic graph polynomial C it turns out that we have to consider functions ϕ on chord diagrams satisfying a certain four-term relation. This arises from considering cyclic graphs G^{π} with exactly two vertices having at least two edges e and f between these vertices, as well as any number of loops. Applying the contraction-deletion relations (1) starting with e or with f will give two different expressions for $C(G^{\pi})$ in terms of the values of C on one-vertex cyclic graphs, that is, on chord diagrams. The four-term relation we consider arises from equating these expressions; the details are in § 5.

Let D_1, D_1', D_2 and D_2' be the diagrams shown in Figure 3, where the part of the diagram indicated by dotted lines is arbitrary, but the same in all four cases. In the figure, the chord f, as well as any chords not shown, may or may not cross the chord e.

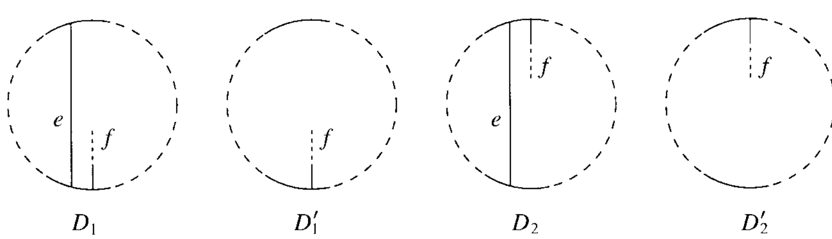


FIGURE 3. *The chord diagrams involved in the four-term relation* (2).

We consider functions ϕ satisfying the relation

$$\phi(D_1) - \phi(D_1') = \phi(D_2) - \phi(D_2'). \tag{2}$$

There is a similar four-term relation that arises in the study of Vassiliev invariants (see [**2**], for example); this relation is implied by (2). However, (2) does not imply the Vassiliev condition that ϕ should vanish on a diagram containing a chord intersecting no other chords. Thus the space of solutions to (2) neither contains nor is contained in the space of weight systems giving rise to Vassiliev invariants.

In studying solutions to (2) we will work upwards through diagrams with increasing numbers of chords. Thus we shall be particularly interested in solutions ϕ that vanish on all diagrams with at most $n-1$ chords. Such ϕ then satisfy $\phi(D_1)=\phi(D_2)$ when the diagrams D_1 and D_2 shown in Figure 3 have degree n. We shall thus say that two diagrams are related by an *R-operation*, or a *single step rotation about the chord* e, if they are related as D_1 and D_2 in Figure 3, that is, if one can be obtained from the other by fixing a chord e and that part of the diagram on one side of e, and rotating the rest of the diagram by one step. We shall say that diagrams D_1 and D_2 are *R-equivalent*, and write $D_1 \sim D_2$, if they are related by a sequence of R-operations. In the rest of the paper we shall show that constructing contraction-deletion invariants of arbitrary cyclic graphs can be reduced to the study of functions on one-vertex cyclic graphs, that is, chord diagrams, that are invariants of R-equivalence.

Part of the information in a chord diagram can be conveniently represented as a graph, the *interlace graph* or *intersection graph* of the pairing. The vertices are the chords of the diagram, and two chords are adjacent if they intersect. In the context of Vassiliev invariants, these graphs have been studied in depth by Chmutov, Duzhin and Lando [**5**, **6**, **7**]. For the importance of interlace graphs in a very different context see [**1**], for example. By the *interlace matrix* $M(D)$ of a diagram D we shall mean the adjacency matrix of the interlace graph, so if the chords are labelled with $1, 2, \ldots, n$, then $M(D)_{ij}$ is 1 if $i \neq j$ and the chords i and j intersect, and 0 otherwise. We shall consider $M(D)$ as a matrix over the field $\mathbb{F}_2$ with two elements, so $M(D)$ is both symmetric and antisymmetric, and has zeros on the diagonal. In particular, the rank of $M(D)$ is even. The key notion we shall need is that of the *rank* $r(D)$ of a diagram D, given by

$$r(D) = \tfrac{1}{2}\operatorname{rank}(M(D)).$$

Note that $r(D)$ has no connection with the usual graph rank.

Suppose that D_1 and D_2 are related by a single step rotation about a chord e, and that f is the chord one of whose endpoints is moved by the rotation. Then whether f intersects a chord g is changed by the rotation if and only if $g \neq e, f$ and e intersects g. Thus $M(D_1)$ can be transformed into $M(D_2)$ by adding (over $\mathbb{F}_2$) the row corresponding to e to that corresponding to f, and then adding the column corresponding to e to that corresponding to f. Thus $M(D_1)$ and $M(D_2)$ have the same rank over $\mathbb{F}_2$. This gives the main property of the rank $r(D)$ that we shall need, namely that

$$r(D_1) = r(D_2) \quad \text{if } D_1 \sim D_2. \tag{3}$$

Another invariant of R-equivalence is of course $n(D)$, the degree or number of chords of D. Rather surprisingly, it will turn out that these invariants together classify chord diagrams up to R-equivalence.

For two diagrams D_1 and D_2 there is a standard (ambiguous) notion of their sum: we write $D = D_1 + D_2$ if D can be obtained from D_1 and D_2 as shown in Figure 4, noting that there are in general many such D, as the points at which the diagrams are attached may be chosen freely.

When the diagrams represent singular knots then the sum of the diagrams corresponds to the connected sum of the knots, which is ambiguous in the same way. Here, where the diagrams represent one-vertex cyclic graphs, the sum of the diagrams corresponds to the one-point join of the cyclic graphs.

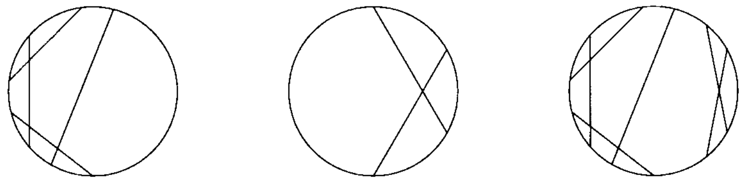


FIGURE 4. *The sum operation for chord diagrams.*

We shall need the following simple fact about the sum of two diagrams: since the matrix $M(D)$ is the direct sum of $M(D_1)$ and $M(D_2)$, we have

$$r(D) = r(D_1) + r(D_2) \tag{4}$$

whenever D is the sum of D_1 and D_2.

To define the cyclic graph polynomial we need one more natural definition: a *subdiagram* of a diagram D is a diagram obtained by deleting some subset of the chords of D. Thus if D has n chords, it has exactly 2^n subdiagrams, some of which will be isomorphic. We shall write $D - e$ for the subdiagram of D formed by deleting a chord e.

4. *Construction of the cyclic graph polynomial*

In this section we describe the construction of the cyclic graph polynomial C; in the next section we shall prove that it is well defined.

For a cyclic graph G^π with exactly one vertex set

$$C(G^\pi) = \sum_{D' \subset D} Y^{n(D')} Z^{r(D')}, \tag{5}$$

where the sum is over all $2^{n(D)}$ subdiagrams of the chord diagram D corresponding to G^π. For a connected cyclic graph G^π with more than one vertex consider the relations

$$C(G^\pi) = C(G^\pi/e) + C(G^\pi - e) \tag{6}$$

for each ordinary edge e of G^π, and

$$C(G^\pi) = XC(G^\pi/e) \tag{7}$$

for each bridge e in G^π. We shall use the relations to define C inductively on connected cyclic graphs, although at first sight it is not clear this is possible, as the relations may be inconsistent. Our main result is that there is indeed an isomorphism-invariant function of cyclic graphs satisfying (5), (6) and (7). We write $\mathcal{G}$ for the set of isomorphism classes of cyclic graphs, and $\mathcal{G}^*$ for the set of isomorphism classes of connected cyclic graphs.

THEOREM 1. *There is a unique function* $C\colon \mathcal{G}^* \to \mathbb{Z}[X, Y, Z]$ *satisfying conditions* (5)–(7).

Note that uniqueness is immediate given the existence of C: we can use (5)–(7) inductively to calculate C on cyclic graphs with increasing numbers of edges.

Having defined C on connected cyclic graphs, we extend it to all cyclic graphs by setting

$$C(G_1^{\pi_1} \cup G_2^{\pi_2}) = C(G_1^{\pi_1})C(G_2^{\pi_2}) \tag{8}$$

for vertex-disjoint cyclic graphs $G_1^{\pi_1}$ and $G_2^{\pi_2}$. Theorem 1 then has the following immediate corollary.

COROLLARY 2. *The invariant* $C\colon \mathscr{G} \to \mathbb{Z}[X, Y, Z]$ *defined above satisfies* (6) *and* (7) *for an arbitrary cyclic graph* G^{π}. *Also,*

$$C(G_1^{\pi_1} \cdot G_2^{\pi_2}) = C(G_1^{\pi_1})C(G_2^{\pi_2}) \tag{9}$$

whenever $G_1^{\pi_1} \cdot G_2^{\pi_2}$ *is the one-point join of two cyclic graphs* $G_1^{\pi_1}$ *and* $G_2^{\pi_2}$.

Proof. The first part can be seen immediately from Theorem 1 by applying relations (6) and (7) to a component of G^{π}. If D_1 and D_2 are diagrams and $D = D_1 + D_2$, then as D_1' and D_2' run over all subdiagrams of D_1 and D_2, the sum $D_1' + D_2'$ runs over all subdiagrams of D. As both $n(\ )$ and, from (4), $r(\ )$ are additive on diagrams, this proves (9) for one-vertex cyclic graphs. The same equation for all cyclic graphs follows by applying relations (6) and (7). □

5. *Existence*

We shall prove Theorem 1 by verifying that the result obtained by applying (5)–(7) to calculate $C(G^{\pi})$ depends only on G^{π}, and not on the order in which we apply relations (6) and (7) to the edges of G^{π}. The proof will come in two parts – most of the work will be needed to reduce the various possible cases to one simple special case. In this case the condition needed turns out to be exactly our four-term relation (2), so we start by showing that the function C defined by (5) satisfies this relation.

It turns out to be convenient to consider separately the coefficients of different powers of Y and Z arising in C. For a chord diagram D, let $C_{ij}(D)$ be the number of subdiagrams of D which have i chords and rank j. Recalling that we do not distinguish D and the associated one-vertex cyclic graph G^{π}, we have, from (5),

$$C(D) = \sum_{i,j} C_{ij}(D) Y^i Z^j.$$

LEMMA 3. *Let* D_1, D_1', D_2 *and* D_2' *be chord diagrams related as in Figure* 3. *Then we have*

$$C_{ij}(D_1) - C_{ij}(D_1') = C_{ij}(D_2) - C_{ij}(D_2'), \tag{10}$$

for every i and j. Also, the same equation holds with C_{ij} *replaced by* C *throughout.*

Proof. Let e be the chord about which we rotate to convert D_1 to D_2, so $D_1' = D_1 - e$ and $D_2' = D_2 - e$. Then the left-hand side of (10) counts the number of subdiagrams of D_1 containing e with degree i and rank j, while the right-hand side counts the same quantity for D_2. Suppose that E_1 is the subdiagram of D_1 formed by deleting a certain set S of chords, and that $e \notin S$, so e is a chord of E_1. Let E_2 be formed by deleting the same set S of chords from D_2. Then E_1 and E_2 are either identical, or are related by a single step rotation about e. Thus we have $n(E_1) = n(E_2)$ and, from (3), $r(E_1) = r(E_2)$, so (10) follows. The same equation for C is an immediate consequence. □

We are now ready to prove Theorem 1. The strategy of the proof is the same as (one approach) for the Tutte polynomial, or the coloured Tutte polynomial (see [**4**]). However, the key part of the proof, where the chord diagrams come into play, is very different from the corresponding parts of these other proofs. In fact, with the benefit of hindsight, the existence of the Tutte polynomial itself can be proved in this way with essentially no work – what is the heart of the proof here, showing that the results obtained by expanding using (6) for each of two parallel edges are the same, is a triviality for the Tutte polynomial, as the cyclic graphs obtained differ only in the cyclic orders.

Proof of Theorem 1. Throughout we consider only connected cyclic graphs, even when this is not explicitly stated. As mentioned above, relations (6) and (7) give a recipe for calculating $C(G^\pi)$ from its values on cyclic graphs with fewer edges, unless G^π has only loops, when (5) applies. However, there may be many choices as to how to apply the relations. More formally, let G^π be a cyclic graph and $<$ a total order on the edges of G^π. We define $C_<(G^\pi)$ to be the result of applying relations (6) and (7) to the edges of G^π in the opposite order to that given by $<$. Thus if e is the last edge of G^π which is not a loop, then $C_<(G^\pi) = XC_<(G^\pi/e)$ if e is a bridge, and $C_<(G^\pi) = C_<(G^\pi/e) + C_<(G^\pi - e)$ if e is ordinary. If G^π consists only of loops then $C_<(G^\pi)$ is given by (5).

Writing $e(G^\pi) = e(G)$ for the number of edges of G^π, we shall take as induction hypothesis the following statement (H_r):

$$\text{if } e(G^\pi) \leqslant r, \text{ then } C_<(G^\pi) = C_{<'}(G^\pi) \qquad (\mathrm{H}_r)$$

for any two orders $<$ and $<'$ on the edges of G^π. Assuming (H_r), and given G^π with $e(G^\pi) \leqslant r$, we shall write $C(G^\pi)$ for the common value of the $C_<(G^\pi)$. Note that if (H_r) holds then the function $C(G^\pi)$ defined on cyclic graphs G^π with $e(G^\pi) \leqslant r$ satisfies (5)–(7) by construction. Expanding down to one-vertex cyclic graphs shows that this function also satisfies (9).

The statements (H_0) and (H_1) are vacuously true, so suppose from now on that $r \geqslant 2$, and that (H_{r-1}) holds. Given a connected cyclic graph G^π with $e(G^\pi) = r$, and an edge e of G^π that is not a loop, let

$$C_e(G^\pi) = \begin{cases} C(G^\pi/e) + C(G^\pi - e) & \text{if } e \text{ is ordinary,} \\ XC(G^\pi/e) & \text{if } e \text{ is a bridge.} \end{cases}$$

To prove (H_r) it suffices to show that

$$C_e(G^\pi) = C_f(G^\pi) \qquad (11)$$

whenever $e(G^\pi) = r$ and e and f are distinct edges of G^π that are not loops. If e and f are bridges, this is immediate: we have $C_e(G^\pi) = XC(G^\pi/e) = X^2C(G^\pi/e/f)$, as (7) holds for cyclic graphs with fewer than r edges. Similarly, $C_f(G^\pi) = X^2C(G^\pi/f/e)$, but $G^\pi/e/f$ and $G^\pi/f/e$ are isomorphic, so (11) holds. If e is a bridge, say, and f is ordinary, then the argument is similar; note that f is ordinary in G^π/e, and e is a bridge in G^π/f and $G^\pi - f$.

We may thus suppose that e and f are both ordinary. We now consider four cases, according to whether e and f are parallel (have the same endpoints), and whether deleting both disconnects G^π.

Case 1: *e, f not parallel, $G^\pi - e - f$ connected.* In this case e is ordinary in G^π/f and $G^\pi - f$, while f is ordinary in G^π/e and $G^\pi - e$, so (11) is again immediate.

Case 2: *e, f not parallel, $G^\pi - e - f$ disconnected.* In this case e is ordinary in G^π/f, but is a bridge in $G^\pi - f$, and similarly with e and f swapped. Thus

$$\begin{aligned} C_e(G^\pi) &= C(G^\pi/e) + C(G^\pi - e) \\ &= C(G^\pi/e/f) + C(G^\pi/e - f) + XC(G^\pi - e/f), \end{aligned}$$

while

$$C_f(G^\pi) = C(G^\pi/f/e) + C(G^\pi/f - e) + XC(G^\pi - f/e).$$

Now $G^\pi/e/f \cong G^\pi/f/e$. Although $G^\pi/e - f$ and $G^\pi - e/f$ are not in general isomorphic, they are both joins of $G_1^{\pi_1}$ and $G_2^{\pi_2}$, the components of $G^\pi - e - f$. Since (H_{r-1}) holds, and the cyclic graphs involved have fewer than r edges, we can apply (9), showing that

$$C(G^\pi/e - f) = C(G_1^{\pi_1})C(G_2^{\pi_2}) = C(G^\pi - e/f).$$

Thus $C_e(G^\pi) = C(G^\pi/e/f) + (X+1)C(G_1^{\pi_1})C(G_2^{\pi_2}) = C_f(G^\pi)$, so (11) holds in this case.

Case 3: *e, f parallel, $G^\pi - e - f$ connected.* This case is the heart of the proof. In this case f is ordinary in $G^\pi - e$, but is a loop in G^π/e. We thus have

$$C_e(G^\pi) = C(G^\pi/e) + C(G^\pi - e/f) + C(G^\pi - e - f),$$

$$C_f(G^\pi) = C(G^\pi/f) + C(G^\pi - f/e) + C(G^\pi - f - e),$$

and (11) is equivalent to

$$C(G^\pi/e) - C(G^\pi/e - f) = C(G^\pi/f) - C(G^\pi/f - e). \tag{12}$$

(i) Suppose first that G^π has a bridge h. Then h is a bridge in all four cyclic graphs appearing in (12). Thus (12) reduces to

$$XC(G^\pi/e/h) - XC(G^\pi/e - f/h) = XC(G^\pi/f/h) - XC(G^\pi/f - e/h). \tag{13}$$

Since contractions and deletions commute, this is X times condition (12) for the smaller cyclic graph G^π/h. As (12) is *equivalent* to (11), which holds for cyclic graphs with fewer than r edges, (13) and thus (12) hold in this case.

(ii) Suppose now that G^π has an ordinary edge h not parallel to e and f. Then h is ordinary in G^π/e and $G^\pi - e$. As C satisfies (6) for cyclic graphs with fewer than r edges, we have

$$\begin{aligned} C_e(G^\pi) &= C(G^\pi/e) + C(G^\pi - e) \\ &= C(G^\pi/e/h) + C(G^\pi/e - h) + C(G^\pi - e/h) + C(G^\pi - e - h) \\ &= C(G^\pi/h/e) + C(G^\pi/h - e) + C(G^\pi - h/e) + C(G^\pi - h - e) \\ &= C(G^\pi/h) + C(G^\pi - h). \end{aligned}$$

The same argument also gives $C_f(G) = C(G^\pi/h) + C(G^\pi - h)$, so (11) holds in this case.

(iii) By (i) and (ii), in proving (12) we may assume that every edge of G^π is either a loop, or is parallel to e and f. Thus G^π has only two vertices, the endvertices x and y of e and f. Let us write α for that part of the cyclic order at x lying between e and f, and β for the part between f and e. Similarly, let γ be the order at y between e and f, and δ that between f and e. This is illustrated in Figure 5, where the order at each vertex is anticlockwise around the vertex. For reasons of space we have not labelled the vertices; x is the vertex on the left, y the one on the right. The dotted edges indicate one possibility for the edges of $G^\pi - e - f$.

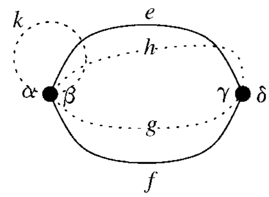


FIGURE 5. *A two-vertex cyclic graph with* (*at least*) *two ordinary edges, e and f.*

For such a cyclic graph G^π the four cyclic graphs involved in (12) are as shown in Figure 6.

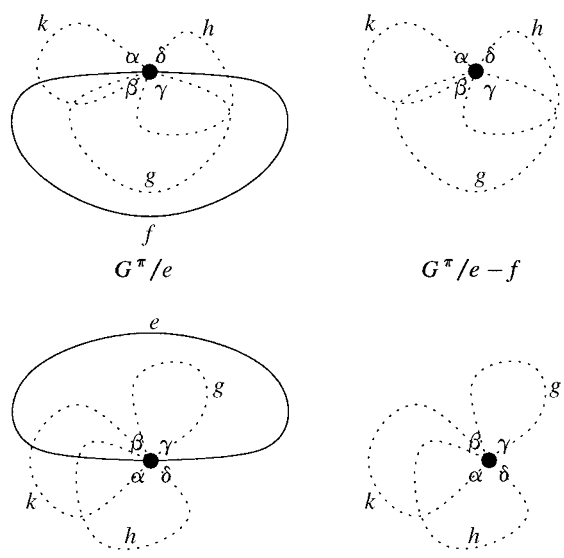


FIGURE 6. *The four cyclic graphs appearing in* (12).

In terms of chord diagrams, (12) thus becomes

$$C(D_1) - C(D_1') = C(D_2) - C(D_2'), \tag{14}$$

where the diagrams D_1, D_1', D_2 and D_2' are as shown in Figure 7.

Now (14) asserts that $C(D) - C(D - e)$ is unaffected by rotating the parts of D on either side of a chord e of D. Any such rotation can be made up of single step rotations, so (14) and hence (12) follow from Lemma 3, completing the proof in this case.

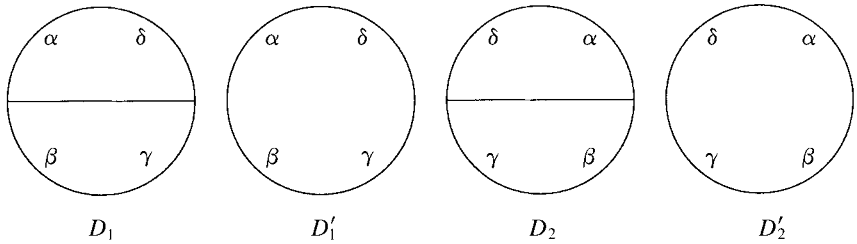


FIGURE 7. *The four chord diagrams appearing in* (14).

Case 4: *e, f parallel,* $G^{\pi}-e-f$ *disconnected.* Now f is a loop in G^{π}/e and a bridge in $G^{\pi}-e$, so

$$C_e(G^{\pi}) = C(G^{\pi}/e) + XC(G^{\pi}-e/f),$$

$$C_f(G^{\pi}) = C(G^{\pi}/f) + XC(G^{\pi}-f/e),$$

and (11) can be written as

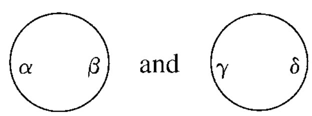
$$C(G^{\pi}/e) - XC(G^{\pi}/e-f) = C(G^{\pi}/f) - XC(G^{\pi}/f-e). \tag{15}$$

This appears to contradict (12), but we shall see that this is not the case, as (15) only applies to rather special cyclic graphs G^{π}.

Arguing as before, we can reduce the proof of (15) to the case where G^{π} has only two vertices. Defining α, β, γ, δ and the diagrams D_1, D_1', D_2, D_2' as above, we see that (15) is exactly the condition

$$C(D_1) - XC(D_1') = C(D_2) - XC(D_2'). \tag{16}$$

Since $G^{\pi}-e-f$ is disconnected, no chord joins either of α or β to either of γ or δ. Thus D_1' and D_2' are both sums of the diagrams



Arguing as in the proof of Corollary 2, we see that $C(D_1') = C(D_2')$, so (16) follows from (14) which we have already proved. Thus (11) holds in this case also.

As the cases checked cover all possibilities, (11) holds for all connected r-edge cyclic graphs. This proves (H_r), and hence, by induction on r, the theorem. □

Having established the existence and uniqueness of the cyclic graph polynomial, we next describe its universal property.

6. *Universality*

In this section we show that any invariant of connected cyclic graphs satisfying (6) and (7) can be calculated from the cyclic graph polynomial. This of course implies that any cyclic graph invariant satisfying (6), (7) and (8) can be obtained from the cyclic graph polynomial. The key step is to show that $n(D)$ and $r(D)$ together form a complete set of invariants for the relation of R-equivalence on chord diagrams.

Recall that the sum of two diagrams D_1 and D_2 is obtained by choosing a point

P_i (not the endpoint of a chord) on the boundary of each D_i, joining the bounding circles at these points and then deforming the result until it is again a circle. By choosing the P_i differently this sum can in general be formed in many different ways. We start by showing that all these possible sums are R-equivalent.

LEMMA 4. *If diagrams D and D' are both sums of diagrams D_1 and D_2, then we have $D \sim D'$.*

Proof. As we can vary the points P_1 and P_2 used to define the sum one at a time, it is sufficient to consider moving P_1 past an endpoint x of a chord in D_1, that is, to show the equivalence of the two diagrams in Figure 8.

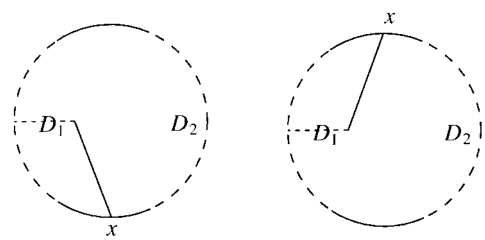
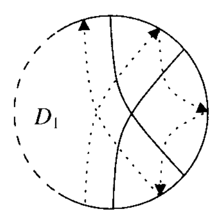


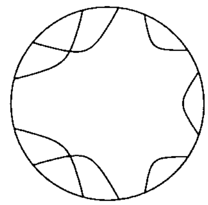
FIGURE 8. *Moving D_2 past an endpoint of a chord of D_1.*

We do this using anticlockwise rotations about chords of D_2, each moving only the point x. This is illustrated for an example below, the dotted line indicating the path followed by the point x:



In fact the illustration more or less gives our proof. For each position of the point x with respect to the chords of D_2 other than the final position, there is a rotation as described above: we rotate about the first chord of D_2 encountered when moving anticlockwise around the circle from x. Also, this operation is invertible, except from the initial position, by rotating clockwise about the first chord encountered when moving clockwise around the circle from x. There are only finitely many positions for x, so performing a sequence of anticlockwise rotations as above starting from the initial position, we either repeat or reach the desired final position of x. To see that we cannot repeat, suppose that the position of x after i steps is the same as after j steps, with $i < j$. Using invertibility, we find that the positions after $i - 1$ and $j - 1$ steps are the same, and so are those after 0 and $j - i$ steps. In other words, after $j - i > 0$ steps we again reach the initial position of x. However, there is no rotation as above leading to this position. This contradiction proves that there is a sequence of rotations transforming one diagram in Figure 8 into the other, completing the proof of the lemma. □

For $i \geqslant 0$ and $0 \leqslant j \leqslant \frac{1}{2}i$, let $D_{i,j}$ be the diagram consisting of j pairs of chords intersecting only each other, and $i-2j$ chords with no intersections, arranged around the circle as illustrated below for $D_{7,2}$:



$D_{7,2}$

LEMMA 5. *Any diagram D is R-equivalent to some $D_{i,j}$.*

Proof. We use induction on $n(D)$; the case $n(D)=0$ is trivial. Suppose then that $n(D) \geqslant 1$, that the lemma holds for diagrams with fewer than $n(D)$ chords, and that e is a chord of D. Then $D-e \sim D_{i,j}$ for some i, j. Consider the sequence S of rotations transforming $D-e$ into $D_{i,j}$. We can construct a corresponding sequence S' starting with D such that S' becomes S when the chord e is deleted from every diagram in the sequence. (If the edge e is adjacent to the chord that we rotate about at some step in S, we may have to rotate two steps in S' to achieve the effect of a single step rotation in S.) The sequence S' ends in some D' with $e \in D'$, such that $D \sim D'$, and $D'-e = D_{i,j}$. The structure of $D_{i,j}$ is such that the chord e added can only intersect two components of $D_{i,j}$. It is thus easy to check that the component C of D' containing e is always equivalent to some $D_{i',j'}$. Using Lemma 4 we can move the remaining components of $D' \setminus C = D_{i'',j''}$, say, 'out of the way'. Thus from $C \sim D_{i',j'}$ we can deduce that $D' \sim D_{i'+i'',j'+j''}$. As $D \sim D'$, this completes the proof of the lemma. □

It is now easy to prove the universal property of the cyclic graph polynomial C. As before, we split C according to the powers of Y and Z, writing $C = \sum_{i,j} C_{ij} Y^i Z^j$, where each C_{ij} is a map from $\mathcal{G}^*$ to $\mathbb{Z}[X]$. This definition of C_{ij} agrees on chord diagrams with that given in the previous section. Also, by equating coefficients of $Y^i Z^j$, we see from Theorem 1 that each C_{ij} satisfies (6) and (7).

Suppose that R is a commutative ring, and that $x \in R$. For each i, j, as the map C_{ij} takes values in $\mathbb{Z}[X]$, we can compose it with the natural ring homomorphism from $\mathbb{Z}[X]$ to R mapping X to x, obtaining a map $C_{ij}(x)$ from $\mathcal{G}^*$ to R. Note that infinite sums of the functions $C_{ij}(x)$ make sense, as only finitely many are non-zero on any given cyclic graph.

THEOREM 6. *Let R be a commutative ring and $x \in R$. Then ϕ: $\mathcal{G}^* \to R$ satisfies*

$$\phi(G^\pi) = \begin{cases} \phi(G^\pi/e) + \phi(G^\pi - e) & \text{if } e \text{ is ordinary,} \\ x\phi(G^\pi/e) & \text{if } e \text{ is a bridge} \end{cases} \tag{17}$$

if and only if there are coefficients $\lambda_{ij} \in R$, with $0 \leqslant j \leqslant \frac{1}{2}i$, such that

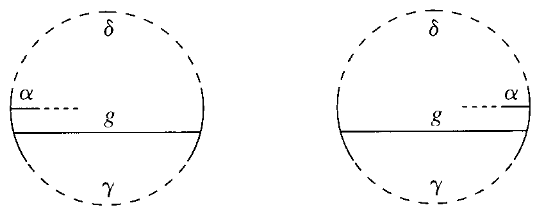
$$\phi = \sum_{i,j} \lambda_{ij} C_{ij}(x). \tag{18}$$

Furthermore, when ϕ *does satisfy* (17), *the* λ_{ij} *are uniquely determined by* (18).

Proof. Theorem 1 implies that each $C_{ij}(x)$ satisfies (17). Thus if ϕ has the form (18), then ϕ satisfies (17).

Suppose then that ϕ satisfies (17). We shall define the λ_{ij} inductively, starting by setting λ_{00} equal to the value of ϕ on the cyclic graph with one vertex and no loops.

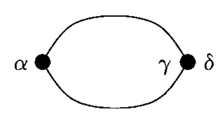
Suppose that $n \geqslant 1$, and that we have chosen λ_{ij}, with $i < n$, such that $\phi(G^{\pi}) = \sum_{i<n} \lambda_{ij} C_{ij}(G^{\pi}; x)$ for all one-vertex cyclic graphs G^{π} with fewer than n loops. Let $\phi' = \phi - \sum_{i<n} \lambda_{ij} C_{ij}(x)$, so ϕ' satisfies (17) and vanishes on one-vertex cyclic graphs with fewer than n loops. Suppose that D_1 and D_2 are diagrams with n chords related by a single step rotation, as shown in Figure 9.



D_1 D_2

FIGURE 9. *Two diagrams related by a single step rotation about the chord g.*

Let α be the point moved by the rotation, β be empty, and γ and δ be the parts of the diagrams shown in Figure 9. Consider the following two-vertex cyclic graph:



Applying the relation (17) for ϕ' to this cyclic graph (a special case of that shown in Figure 5), we can deduce that

$$\phi'(D_1) - \phi'(D_1 - g) = \phi'(D_2) - \phi'(D_2 - g).$$

As $D_1 - g$ and $D_2 - g$ have fewer than n chords, this gives $\phi'(D_1) = \phi'(D_2)$. Thus, on diagrams D with n chords, $\phi'(D)$ depends only on the R-equivalence class of D, and hence, by Lemma 5, only on $r(D)$. As $C_{nj}(D; x)$ is 1 if $r(D) = j$ and 0 otherwise, we have $\phi' = \sum_j \lambda_{nj} C_{nj}(D; x)$ on n-chord diagrams if and only if $\lambda_{nj} = \phi'(D_{n,j})$.

Proceeding by induction on n, we see that there are unique λ_{ij} such that (18) holds for all one-vertex cyclic graphs G^{π}. The same equation for all connected cyclic graphs follows from (17). □

7. *The genus of a chord diagram*

As noted in the introduction, connected cyclic graphs correspond to graphs embedded into oriented surfaces in such a way that each face (component of the complement of the embedded graph) is a topological disk. In particular, to each cyclic graph G^{π} there is associated a closed orientable surface $\sigma(G^{\pi})$, which can be specified by giving its genus. It thus makes sense to define the *genus* $g(G^{\pi})$ of a connected cyclic graph G^{π} as the genus of $\sigma(G^{\pi})$. For a chord diagram D we define the *genus* $g(D)$ to be the genus of the corresponding one-vertex cyclic graph.

Suppose now that D_1 and D_2 are two cyclic graphs related by a single step rotation, that is, related as D_1 and D_2 in Figure 3. A straightforward local check shows that $g(D_1) = g(D_2)$, so on chord diagrams the genus g is an invariant of R-equivalence. Now $g(D_{i,j}) = j = r(D_{i,j})$. As the rank r is also an invariant of R-equivalence, it follows from Lemma 5 that $g(D) = r(D)$ for all chord diagrams D. Thus an alternative way of writing our definition of the cyclic graph polynomial on one-vertex cyclic graphs G^{π} is

$$C(G^{\pi}) = \sum_{D' \subset D} Y^{n(D')} Z^{g(D')},$$

where D is the chord diagram corresponding to G^{π}. This definition is perhaps more natural than (5) when thinking of graphs embedded into surfaces.

However, while we could have introduced the cyclic graph polynomial considering only $g(D)$ and not $r(D)$, the use of $r(D)$ has the advantage that it is defined as a function of the interlace graph of the diagram D. This notion of rank can be extended to graphs that are not interlace graphs of chord diagrams, which may be useful in some other context. More importantly, considering $r(D)$ shows immediately that $g(D)$ depends only on the interlace graph of D, which does not by any means contain all the information about D. Also, the operation of considering subdiagrams is rather unnatural when thought of in terms of graphs embedded into surfaces, but the effect on the interlace graph, or on the interlace matrix, is very natural.

8. *Further properties*

We start by showing that, like the Tutte polynomial, the cyclic graph polynomial has a spanning tree expansion, although one very different from that for the Tutte polynomial.

Let G^{π} be a connected cyclic graph with a total order $<$ on its edges, and let T be a spanning tree of G^{π}. We say that an edge e of T is *internally active* (with respect to T, G^{π} and $<$) if e is the first edge of G^{π} (in the order $<$) between the two components of $T - e$. We say that an edge e of G^{π} which is not in T is *externally active* (with respect to T, G^{π} and $<$) if e is the first edge on the unique cycle of $T \cup e$. These definitions were introduced by Tutte [**23**].

Given a set S of edges of $G^{\pi} \setminus T$, we write $D_T(S)$ for the *chord diagram induced by* S (with respect to T), defined as follows: starting from the cyclic graph G^{π}, first contract all edges of T to obtain a one-vertex cyclic graph. Then delete all edges other than those in S. One can also think of $D_T(S)$ as the chord diagram with chords corresponding to the edges of S where the outer circle is given by traversing the spanning tree T in the unique way specified by the cyclic orders at the vertices.

We are now ready to give the spanning tree expansion of $C(G^{\pi})$: we define the *weight* of T with respect to G^{π} and $<$ as

$$w(T, G^{\pi}, <) = X^{i} \sum_{S} Y^{|S|} Z^{r(D_T(S))},$$

where i is the number of internally active edges of T, and the sum is over all subsets of the set of externally active edges. Summing over spanning trees we set

$$w(G^\pi, <) = \sum_T w(T, G^\pi, <).$$

THEOREM 7. *For any connected cyclic graph G^π and any order $<$ on the edges of G^π, we have $w(G^\pi, <) = C(G^\pi)$.*

Proof. We use induction on the number of edges of G^π which are not loops. If G^π consists only of loops, then G^π has a unique spanning tree (with no edges), and every edge of G^π is externally active. It is easy to see that the definition of $w(G^\pi, <)$ then coincides with that of $C(G^\pi)$.

Suppose then that G^π does not consist entirely of loops, and let e be the last edge of G^π (in the order $<$) which is not a loop. If e is a bridge, then e is contained in every spanning tree of G^π, and is always internally active. Thus $w(G^\pi, <) = Xw(G^\pi/e, <)$. If e is ordinary, then as it is the last edge which is not a loop, it is never active, either internally or externally. Also, the spanning trees of G^π containing e correspond to those of G^π/e, while the spanning trees not containing e correspond to those of $G^\pi - e$, where the correspondence preserves the activities of the edges. Thus $w(G^\pi, <) = w(G^\pi/e, <) + w(G^\pi - e, <)$. The result now follows from the induction hypothesis and the recurrence relations for C. □

We now turn to the relationship between the cyclic graph polynomial and the Tutte polynomial.

Setting $Z = 1$ removes the dependence of $C(G^\pi)$ on the cyclic orders at the vertices. More precisely, writing G for the graph underlying G^π, one can easily check that $C(G^\pi; X, Y, 1) = T(G; X, Y + 1)$. Thus the cyclic graph polynomial C inherits the full power of the Tutte polynomial as a graph invariant. More importantly, C genuinely does strengthen the Tutte polynomial, depending very strongly on the embedding of the graph G into an orientable surface. Computer calculations show that C distinguishes all three non-isomorphic 2-cell embeddings of K_4, all fourteen of the cube on eight vertices, and forty-eight of the fifty of K_5. Note that it makes sense to talk of the value of C on an embedding into an orientable (as opposed to oriented) surface: such an embedding gives rise to two cyclic graphs, one for each orientation, related by reversing all the cyclic orders. As a chord diagram and its mirror image have the same rank, C takes the same value on these two cyclic graphs.

Finally we give another small illustration of the order dependence by mentioning one specific evaluation of the cyclic graph polynomial which behaves in an interesting way. On cyclic graphs in which each vertex has degree 3, one can show that $C(0, -4, \frac{1}{4})$ satisfies the IHX and AS (antisymmetry) relations arising in perturbative Chern–Simons theory; see [**11**] for example. To prove this one can use the contraction-deletion relations to reduce to the case when all edges not directly involved in the relation are loops. This case reduces to a finite check by using the R-equivalence of any diagram to some $D_{i,j}$. Of course this is not a good way of looking for many solutions to these relations on trivalent cyclic graphs – we have imposed a very strong additional condition, that the solution should extend in a certain way to arbitrary cyclic graphs. Nevertheless, it is interesting to see that this one particular function does extend to all cyclic graphs, and it illustrates the essential dependence of the cyclic graph polynomial on the embedding.

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