RELATIONS BETWEEN CUMULANTS IN NONCOMMUTATIVE PROBABILITY

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Abstract. We express classical, free, Boolean and monotone cumulants in terms of each other, using combinatorics of heaps, pyramids, Tutte polynomials and permutations. We completely determine the coefficients of these formulas with the exception of the formula for classical cumulants in terms of monotone cumulants whose coefficients are only partially computed.

1. Introduction

Cumulants provide a combinatorial description of independence of random variables. While Fourier analysis is the tool of choice for most problems in classical probability, cumulants are an indispensable ingredient for many investigations in noncommutative probability. An intriguing aspect of noncommutative probability is the existence of several kinds of independence [Voi85, SW97, Mur01, Leh04] with corresponding cumulants introduced in [Voi85, Spe94, SW97, HS11b, Leh04] sharing many common features. In a certain sense (which can be made precise, see [Spe97, Mur02]) these are the only "natural" notions of independence and the combinatorics of cumulants in particular show very close analogies between the different theories. Roughly speaking, one can pass from classical to free/boolean/monotone independence by replacing the lattice of all set partitions by noncrossing/interval/monotone partitions respectively.

On the other hand, the generating functions of cumulants correspond to various transforms of probability measures and one major application is the calculation of walk generating functions (or Green's functions) of certain graph products, see [Woe00] for details of the following concepts. The cartesian product of graphs corresponds to classical convolution as observed by Polya [Pól21], the free product of graphs corresponds to Voiculescu's free convolution [Voi86, Woe86, CS86] and the star product of graphs [Woe00, Section 9.7] corresponds to Boolean convolution [Oba04]. The comb product entered the graph theory literature rather recently [KP04] (a special case having been considered earlier in physics, see [WH86]) in order to construct an example of a recurrent random walk with so-called *finite collision property*. It was observed in [ABGO04] that this graph product corresponds to monotone convolution.

The starting point of this paper is the following relations between (univariate) classical $((\kappa_n)_{n\geq1})$, free $((r_n)_{n\geq1})$ and Boolean $((b_n)_{n\geq1})$ cumulants, shown by the third author [Leh02] some time ago:

$$b_n = \sum_{\pi\in \mathrm{NC}_{\mathrm{irr}}(n)} r_\pi, \tag{1.1}$$

$$r_n = \sum_{\pi\in \mathcal{P}_{\mathrm{conn}}(n)} \kappa_\pi, \tag{1.2}$$

$$b_n = \sum_{\pi\in \mathcal{P}_{\mathrm{irr}}(n)} \kappa_\pi. \tag{1.3}$$

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where $\mathrm{NC}_{\mathrm{irr}}(n)$, $\mathcal{P}_{\mathrm{conn}}(n)$, $\mathcal{P}_{\mathrm{irr}}(n)$ are, respectively, the sets of irreducible noncrossing partitions, connected partitions and irreducible partitions[1]. Relation (1.2) was used in [BBLS11] to attack the problem of free infinite divisibility of the normal law.

We denote by K_n, H_n, B_n, R_n the multivariate classical, monotone, Boolean and free cumulants respectively. The univariate cumulants κ_n, h_n, b_n, r_n are obtained by evaluating the multivariate cumulants at n copies of a single variable.

Relation (1.1) was extended by Belinschi and Nica in [BN08] to the case of multivariate cumulants B_n, R_n. In addition, they obtained the inverse formula:

$$R_n = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} B_\pi. \tag{1.4}$$

It is interesting to notice the similarity to formula (4) in [LM11]. We will give a different proof of (1.4) in Section 4 which clarifies this coincidence.

The extensions of (1.2) and (1.3) to the multivariate case can be shown by using the same proofs as in [Leh02] for the univariate case, see below for details. An interesting inverse formula for (1.3) was proved recently by M. Josuat-Vergès [JV13], expressing classical cumulants in terms of free cumulants:

$$\kappa_n = \sum_{\pi \in \mathcal{P}_{\mathrm{conn}}(n)} (-1)^{|\pi|-1} T_{G(\pi)}(1,0) r_\pi, \tag{1.5}$$

where $G(\pi)$ is the crossing graph of π and $T_{G(\pi)}$ its Tutte polynomial. The proof of (1.5) in [JV13] is also valid for the multivariate case.

The purpose of the present article is to complete the picture for the relations between classical, Boolean, free and monotone cumulants, extending some identities to the multivariate case. More precisely, we are able to prove the following cumulant identities.

Theorem 1.1. *The following identities hold for multivariate cumulants:*

$$B_n = \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(n)} \frac{1}{|\pi|!} H_\pi = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{\tau(\pi)!} H_\pi, \tag{1.6}$$

$$R_n = \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{|\pi|!} H_\pi = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{(-1)^{|\pi|-1}}{\tau(\pi)!} H_\pi, \tag{1.7}$$

where $\mathcal{M}_{\mathrm{irr}}(n)$ is the set of irreducible monotone partitions.

Theorem 1.2. *The following identities hold for univariate cumulants:*

$$h_n = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \alpha_\pi r_\pi, \tag{1.8}$$

$$h_n = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} \alpha_\pi b_\pi, \tag{1.9}$$

$$h_n = \sum_{\pi \in \mathcal{P}_{\mathrm{irr}}(n)} \alpha_{\bar{\pi}} \kappa_\pi, \tag{1.10}$$

where $\bar{\sigma} \in \mathrm{NC}(n)$ denotes the noncrossing closure of $\sigma \in \mathcal{P}(n)$ and α_π is the linear part of the number of nonincreasing labellings of the nesting forest of π (which in the case of irreducible partitions consists of precisely one tree). This quantity will be defined rigorously in Section 5.

Remark 1.3. Calculations indicate that a multivariate analogue of Theorem 1.2 also holds, but at present we do not know how to prove it.

The proof of the Boolean-to-classical cumulant formula follows the techniques of the proof (1.5) used in [JV13].

[1]Partitions and notations are defined in Section 2.

Theorem 1.4.

$$K_n = \sum_{\pi \in \mathcal{P}_{\mathrm{irr}}(n)} (-1)^{|\pi|-1} T_{\tilde{G}(\pi)}(1,0) B_\pi, \tag{1.11}$$

where $\tilde{G}(\pi)$ is the anti-interval graph of π and $T_{\tilde{G}(\pi)}$ is its Tutte polynomial (see Section 6).

Alternatively, the values of the Tutte polynomials in (1.5) and (1.11) can be interpreted as certain pyramids in the sense of Cartier-Foata, see Section 6 for details.

Yet another interpretation expresses classical cumulants in terms of Boolean cumulants via permutation statistics.

Theorem 1.5. *Denote by $\mathcal{C}_n$ the set of cyclic permutations of order n and let* $\mathrm{cruns}(\sigma)$ *be the set partition defined by the cycle runs of σ (see Section 7). Then*

$$K_n = \sum_{\sigma \in \mathcal{C}_n} (-1)^{\#\,\mathrm{cruns}(\sigma)-1} B_{\mathrm{cruns}(\sigma)}. \tag{1.12}$$

There is a bijection between $\mathcal{C}_n$ and $\{\sigma \in \mathfrak{S}_n \mid \sigma(1) = 1\}$: given $\pi = (1, \pi(1), \ldots, \pi^{n-1}(1)) \in \mathcal{C}_n$, we define a permutation $\sigma(1) = 1, \sigma(k) = \pi^{k-1}(1)$, $2 \leq k \leq n$. Then we may rewrite Theorem 1.5 into

Corollary 1.6. *Let $\mathfrak{S}_n$ be the set of permutations of order n, let* $\mathrm{runs}(\sigma)$ *be the set partition associated to the runs of $\sigma \in \mathfrak{S}_n$ and let $d(\sigma)$ be the number of descents of $\sigma \in \mathfrak{S}_n$ (see Section 7). Then*

$$K_n = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=1}} (-1)^{d(\sigma)} B_{\mathrm{runs}(\sigma)}. \tag{1.13}$$

Understanding the coefficients of the remaining, monotone-to-classical cumulant formula

$$K_n = \sum_{\pi \in \mathcal{P}(n)} \beta(\pi) H_\pi,$$

seems to require a more detailed treatment. We compute β for some particular cases and list some of its properties. In particular, we show that the coefficients β only depend on an anti-interval digraph, and moreover we show that

Theorem 1.7. (1) *If π is reducible, then $\beta(\pi) = 0$.*
(2) *If π is irreducible and has no nestings, then $\beta(\pi)$ coincides with the coefficient* $(-1)^{|\pi|-1} T_{G(\pi)}(1,0)$ *from formula* (1.5)*.*
(3) *If $\pi \in \mathrm{NC}_{\mathrm{irr}}$ and has depth 1 or 2, then $\beta(\pi) = \frac{(-1)^{|\pi|-1}}{|\pi|}$.*

Thus various combinatorial objects contribute to the proofs and the paper is organized accordingly. Types of partitions and notation are defined in Section 2. We collect some combinatorial properties of monotone partitions in Section 3. Theorems 1.1, 1.2, 1.4, 1.5, 1.7 are proved in Sections 4, 5, 6, 7, 8, respectively.

2. Definitions and preliminary results

Concepts on partitions and ordered partitions are summarized below, first let us recall some well known facts from the theory of *posets* (partially ordered sets). For details on the latter the standard reference is [Sta12].

Proposition 2.1 (Principle of Möbius inversion)**.** *On any poset $(P, \leq)$ there is a unique* Möbius function $\mu : P \times P \to \mathbf{Z}$ *such that for any pair of functions $f, g : P \to \mathbf{C}$ (in fact any abelian group in place of $\mathbf{C}$) the identity*

$$f(x) = \sum_{y \leq x} g(y) \tag{2.1}$$

holds for every $x \in P$ if and only if

$$g(x) = \sum_{y \leq x} f(y)\, \mu(y, x). \tag{2.2}$$

In particular, if, given f, two functions g_1 and g_2 satisfy (2.1)*, then g_1 and g_2 coincide.*

Definition 2.2. Let P be a poset. A map $c : P \to P$ is called *closure operator* if:

(1) it is *increasing*, i.e., $x \leq c(x)$ for every $x \in P$;
(2) it is *order preserving*, i.e., if $x \leq y$ then $c(x) \leq c(y)$;
(3) it is *idempotent*, i.e., $c \circ c = c$.

In the present paper we will be concerned with posets (mostly lattices) of set partitions exclusively and make use of the noncrossing closure and interval closure defined next.

Definition 2.3. (1) A *partition* of a set is a decomposition into disjoint subsets, called *blocks*. The set of partitions of the set $[n] := \{1, \dots, n\}$ is denoted by $\mathcal{P}(n)$. It is a lattice under refinement order with maximal element $\{[n]\}$ denoted by $\hat{1}_n$ and minimal element $\{\{1\}, \dots, \{n\}\}$ is denoted by $\hat{0}_n$.
We write $\mathcal{P} = \bigcup_{n \geq 1} \mathcal{P}(n)$ and similar notations will be used such as NC.

(2) Any partition defines an equivalence relation on $[n]$ and vice versa. Given $\pi \in \mathcal{P}(n)$, $i \sim_\pi j$ holds if and only if there is a block $V \in \pi$ such that $i, j \in V$.
(3) A partition $\pi \in \mathcal{P}(n)$ is *noncrossing* if there is no quadruple of elements $1 \leq i < j < k < l \leq n$ such that $i \sim_\pi k$, $j \sim_\pi l$ and $i \not\sim_\pi j$. The noncrossing partitions of order n form a sub-lattice which we denote by $\mathrm{NC}(n)$.
(4) For two blocks V, W of a partition, we say V is an *inner block* of W or equivalently V nests inside W or W is an *outer block* of V if there are $i, j \in W$ such that $i < k < j$ for each $k \in V$.
(5) The *depth* of a block V of a noncrossing partition is the number of blocks (including V itself) which graphically cover the block V. The *depth* of a noncrossing partition is the maximal depth among all the blocks. For example $\hat{1}_n$ has depth 1.
(6) A block V of a partition is called an *interval block* if V is of the form $V = \{k, k+1, \dots, k+l\}$ for $k \geq 1$ and $0 \leq l \leq n-k$. We denote by $IB(n)$ the set of all interval blocks of $[n]$.
(7) An *interval partition* is a partition π for which every block is an interval. The set of interval partitions of $[n]$ is denoted by $\mathcal{I}(n)$ and is a sub-lattice of $\mathcal{P}(n)$. Sometimes these are called *linear partitions* and in fact they are in obvious bijection with *compositions* of a number n, i.e., sequences of integers $(k_1, k_2, \dots, k_r)$ such that $k_i > 0$ and $k_1 + k_2 + \cdots + k_r = n$.
(8) The *noncrossing closure* $\bar{\pi}$ of a partition π is the smallest noncrossing partition which dominates π.
(9) A partition π is *connected* if its noncrossing closure is equal to the maximal partition $\hat{1}_n$, or, equivalently, the diagram of π is a connected graph. The set of connected partitions is denoted by $\mathcal{P}_{\mathrm{conn}}(n)$.
(10) The *connected components* of a partition π are the connected sub-partitions of π, i.e., the partitions induced on the blocks of the noncrossing closure $\bar{\pi}$.
(11) The *interval closure* $\hat{\pi}$ of a partition π is the smallest interval partition which dominates π.
(12) A partition $\pi \in \mathcal{P}(n)$ is *irreducible* if its interval closure is equal to the maximal partition $\hat{1}_n$. For a noncrossing partition of $[n]$ this is equivalent to the property that $1 \sim_\pi n$. Every partition π can be "factored" into irreducible factors which we denote by $\pi = \pi_1 \cup \cdots \cup \pi_r$. The factors π_j are sub-partitions induced on the blocks of the interval closure $\hat{\pi}$.
The sets of irreducible partitions and irreducible noncrossing partitions are respectively denoted by $\mathcal{P}_{\mathrm{irr}}(n)$ and $\mathrm{NC}_{\mathrm{irr}}(n)$.

Different types of partitions are shown in the following figure.

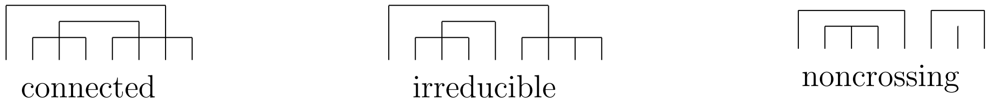


Figure 1. Typical partitions

Definition 2.4. (1) An *ordered partition* is a pair (π, λ) of a set partition π and a linear order λ on its blocks. An ordered partition can be regarded as a sequence of blocks: $(\pi, \lambda) = (V_1, \dots, V_k)$ by understanding that $V_i <_\lambda V_j$ iff $i < j$.
(2) A *monotone partition* is an ordered partition (π, λ) with $\pi \in \mathrm{NC}(n)$ such that, for $V, W \in \pi$, $V >_\lambda W$ whenever V is an inner block of W.
(3) An ordered partition (π, λ) is *irreducible* if π is irreducible. Let $\mathcal{M}_{\mathrm{irr}}(n)$ denote the set of irreducible monotone partitions.

Positivity of random variables or states is irrelevant in this paper, our treatment is purely algebraic. One reason for this is the introduction of a formal random variable $\widetilde{\mathbf{X}}$ in (4.12) whose positivity is not guaranteed. Let $(\mathcal{A}, \varphi)$ be a pair of a unital algebra over $\mathbb{C}$ and a unital linear functional on $\mathcal{A}$, i.e. $\varphi(1_{\mathcal{A}}) = 1$.

Let A_n (resp., a_n) be one of the cumulant functionals B_n, H_n, R_n (resp., b_n, h_n, r_n). Given a partition $\pi \in \mathcal{P}(n)$ and $X, X_i \in \mathcal{A}$, we define the associated multivariate and univariate *partitioned cumulant functionals*

$$A_\pi(X_1, \dots, X_n) := \prod_{V \in \pi} A_{|V|}(X_V), \qquad a_\pi(X) := A_\pi(X, \dots, X) = \prod_{V \in \pi} a_{|V|}(X),$$

where we use the notation

$$A_{|V|}(X_V) := A_m(X_{v_1}, \dots, X_{v_m})$$

for a block $V = \{v_1, \dots, v_m\}$, $v_1 < \dots < v_m$. The linear functional φ gives rise to the multilinear functional

$$(X_1, \dots, X_n) \mapsto \varphi(X_1 \cdots X_n)$$

on $\mathcal{A}^n$ for each n and φ_π is defined analogously.

The following formulas implicitly define the classical, free, Boolean and monotone cumulants.

Theorem 2.5.

$$\varphi_\pi(X_1, \cdots, X_n) = \sum_{\substack{\sigma \in \mathcal{P}(n) \\ \sigma \leq \pi}} K_\sigma(X_1, \dots, X_n), \qquad \text{[Sch47, Rot64]} \tag{2.3}$$

$$\varphi_\pi(X_1, \cdots, X_n) = \sum_{\substack{\sigma \in \mathrm{NC}(n) \\ \sigma \leq \pi}} R_\sigma(X_1, \dots, X_n), \qquad \text{[Spe94]} \tag{2.4}$$

$$\varphi_\pi(X_1, \cdots, X_n) = \sum_{\substack{\sigma \in \mathcal{I}(n) \\ \sigma \leq \pi}} B_\sigma(X_1, \dots, X_n), \qquad \text{[SW97]} \tag{2.5}$$

$$\varphi(X_1 \cdots X_n) = \sum_{(\sigma, \lambda) \in \mathcal{M}(n)} \frac{1}{|\sigma|!} H_\sigma(X_1, \dots, X_n). \qquad \text{[HS11a]} \tag{2.6}$$

The multiplicative extension of the monotone case (2.6) is not very useful because the summand would depend on both σ, π (but if π is an interval partition, the summand does not depend on σ; see the proof of Theorem 1.1 in Section 4).

Let $\mu_{\mathcal{P}}, \mu_{\mathrm{NC}}, \mu_{\mathcal{I}}$ be the Möbius functions on the posets $\mathcal{P}, \mathrm{NC}, \mathcal{I}$ respectively. The values are

$$\mu_{\mathcal{P}}(\hat{0}_n, \hat{1}_n) = (-1)^{n-1}(n-1)!, \qquad \text{[Sch47, Rot64]} \tag{2.7}$$

$$\mu_{\mathrm{NC}}(\hat{0}_n, \hat{1}_n) = (-1)^{n-1} C_{n-1}, \qquad \text{[Kre72]} \tag{2.8}$$

$$\mu_{\mathcal{I}}(\hat{0}_n, \hat{1}_n) = (-1)^{n-1}, \tag{2.9}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number. The values $\mu(\pi,\sigma)$ for general intervals $[\pi,\sigma]$ are products of these due to the fact that in all lattices considered here any such interval is isomorphic to a direct product of full lattices of different orders, see [DRS72, Spe94].

In fact it is easy to see that the lattice of interval partitions of order n is antiisomorphic to the lattice of subsets of a set with $n-1$ elements and formula (2.9) is equivalent to the inclusion-exclusion principle.

From the Möbius principle we may express the classical, free and Boolean cumulants as

$$K_\pi(X_1,\cdots,X_n) = \sum_{\substack{\sigma\in\mathcal{P}(n)\\ \sigma\leq\pi}} \varphi_\sigma(X_1,\ldots,X_n)\,\mu_{\mathcal{P}}(\sigma,\pi), \tag{2.10}$$

$$R_\pi(X_1,\cdots,X_n) = \sum_{\substack{\sigma\in\mathrm{NC}(n)\\ \sigma\leq\pi}} \varphi_\sigma(X_1,\ldots,X_n)\,\mu_{\mathrm{NC}}(\sigma,\pi), \tag{2.11}$$

$$B_\pi(X_1,\cdots,X_n) = \sum_{\substack{\sigma\in\mathcal{I}(n)\\ \sigma\leq\pi}} \varphi_\sigma(X_1,\ldots,X_n)\,\mu_{\mathcal{I}}(\sigma,\pi). \tag{2.12}$$

Alternatively, univariate cumulants can be defined via generating functions as follows. Let $(m_n)_{n\geq1}$ be a sequence with $m_0=1$ and $F(z)=\sum_{n=0}^\infty \frac{m_n}{n!}z^n$ its exponential generating function and $M(z)=\sum_{n=0}^\infty m_n z^n$ the ordinary generating function.

(1) The exponential generating function of the classical cumulants satisfies the identity

$$\sum_{n=1}^\infty \frac{\kappa_n}{n!}\,z^n = \log F(z).$$

(2) The ordinary generating function of the free cumulants

$$R(z) = \sum_{n=1}^\infty r_n z^n$$

is called *R-transform* and satisfies the equivalent identities

$$1+R(zM(z)) = M(z), \tag{2.13}$$

$$M(z/(1+R(z))) = 1+R(z). \tag{2.14}$$

(3) The ordinary generating function of the Boolean cumulants

$$B(z) = \sum_{n=1}^\infty b_n z^n$$

satisfies the identity

$$M(z) = \frac{1}{1-B(z)}. \tag{2.15}$$

Our proofs make use of both the set partition machinery and multivariate versions of the following generating function relations. Consider the following identities:

$$1+R\left(\frac{z}{1-B(z)}\right) = \frac{1}{1-B(z)}, \qquad 1-B\left(\frac{z}{1+R(z)}\right) = \frac{1}{1+R(z)}. \tag{2.16}$$

The left hand identity is obtained by substituting (2.15) into (2.13); and the right hand identity follows from (2.14) by taking reciprocals on both sides and substituting $1/M(z) = 1-B(z)$.

Define a map $\tilde{}$ on the set of generating functions by setting $\tilde{R} = -B$, $\tilde{B} = -R$. This map swaps the identities (2.16) and explains the occurrence of the factor $(-1)^{|\pi|-1}$ in formulas (1.4),(1.7) and (1.9). It turns out that $\tilde{H} = -H$ under this transformation, where $H(z)$ is the generating function of monotone cumulants. The details and the multivariate generalization of this observation are worked out in Section 4.

3. Counting monotone partitions

We want to count the number of monotone labellings of a noncrossing partition π, i.e., the number of possible orders λ on the blocks of π such that (π, λ) becomes a monotone partition. For this purpose it is convenient to map the nesting structure of noncrossing partitions to trees.

Definition 3.1. The *nesting forest* $\tau(\pi)$ of a noncrossing partition π with k blocks is the forest of planar rooted trees on k vertices built recursively as follows.

(1) If π is an irreducible partition, then $\tau(\pi)$ is the planar rooted tree, whose vertices are the blocks of π, the root being the unique outer block, and branches $\tau(\pi_i)$ where π_i are the irreducible components of π without the outer block.

(2) If π has irreducible components $\pi_1, \pi_2, \dots, \pi_k$, then $\tau(\pi)$ is the forest consisting of the rooted trees $\tau(\pi_1), \tau(\pi_2), \dots, \tau(\pi_k)$.

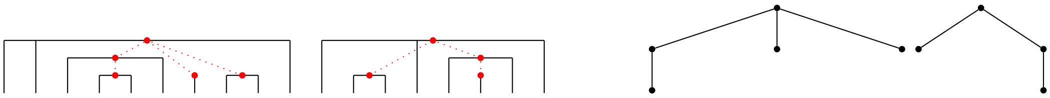


Figure 2. A noncrossing partition and its nesting forest

Every monotone labelling of the noncrossing partition π corresponds to an increasing labelling of its nesting forest $\tau(\pi)$. For the enumeration of the latter the so-called *tree factorial* is useful.

Definition 3.2. The *tree factorial* $t!$ of a finite rooted tree t is recursively defined as follows. Let t be a rooted tree with $n > 0$ vertices. If t consists of a single vertex, set $t! = 1$. Otherwise t can be decomposed into its root vertex and branches t_1, t_2, $\dots$, t_r and we define recursively the number

$$t! = n \cdot t_1!\, t_2! \cdots t_r!.$$

The tree factorial of a forest is the product of the factorials of the constituting trees.

Proposition 3.3. *(a) The number $m(\pi)$ of monotone labellings of a noncrossing partition π depends only on its nesting forest $\tau(\pi)$ and is given by*

$$m(\pi) = \frac{|\pi|!}{\tau(\pi)!}.$$

(b) The function $w(\pi) = \frac{m(\pi)}{|\pi|!} = \frac{1}{\tau(\pi)!}$ is multiplicative, i.e., if π has irreducible components $\pi_1, \pi_2, \dots, \pi_k$, then

$$w(\pi) = w(\pi_1)\, w(\pi_2) \cdots w(\pi_k).$$

Proof. (a) Here is an inductive proof of the well known fact that the number of increasing labellings of a tree t is equal to $|t|!/t!$.

If the tree has only one vertex the claim is obviously true. So assume that there are at least 2 vertices. The root must get the smallest label, so there is no choice. Then we have to distribute the remaining labels among the branches t_1, t_2, $\dots$,t_r. There are

$$\binom{|t|-1}{|t_1|\,, |t_2|\,, \dots, |t_r|}$$

possibilities to do so and by induction on each branch t_i there are $\frac{|t_i|!}{t_i!}$ monotone labellings. Putting these together we obtain

$$\binom{|t|-1}{|t_1|\,, |t_2|\,, \dots, |t_r|} \frac{|t_1|!}{t_1!} \frac{|t_2|!}{t_2!} \cdots \frac{|t_r|!}{t_r!} = \frac{1}{|t|} \frac{|t|!}{t_1! t_2! \cdots t_r!} = \frac{|t|!}{t!}.$$

(b) is immediate from the definition of the tree factorial.

□

From this we can rewrite the formula expressing moments in terms of monotone cumulants (2.6) into

$$\varphi(X_1 \cdots X_n) = \sum_{\sigma \in \mathrm{NC}(n)} \frac{1}{\tau(\sigma)!} H_\sigma(X_1, \dots, X_n). \tag{3.1}$$

We can count the number of monotone partitions as follows. Let $IB(n,k)$ be a subset of $IB(n)$ defined by $\{V \in IB(n); |V| = k\}$. We notice that $|IB(n,k)| = n-k+1$.

Proposition 3.4. $|\mathcal{M}(n)| = \frac{(n+1)!}{2}$.

Proof. There is a bijection $\Phi : \mathcal{M}(n) \to \left(\bigcup_{k=1}^{n-1} \mathcal{M}(k) \times IB(n, n-k)\right) \cup \{\hat{1}_n\}$ defined by

$$\Phi : (V_1, \dots, V_{|\pi|}) \mapsto ((V_1, \dots, V_{|\pi|-1}), V_{|\pi|}).$$

Now let $a_n := |\mathcal{M}(n)|$. In view of the above bijection Φ, we have $a_n = 1 + \sum_{k=1}^{n-1}(k+1)a_k$. By calculating $a_n - a_{n-1}$, one gets the relation $a_n = (n+1)a_{n-1}$ and so the result follows. □

4. Symmetries of generating functions and proof of Theorem 1.1

Proof of Theorem (1.6). The proof follows the same line as the proof of (1.1) in [Leh02] by a simple application of the principle of Möbius inversion (Prop. 2.1).

We know that the Boolean cumulants are uniquely determined by the property that

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\substack{\rho \in \mathcal{I}(n) \\ \rho \leq \pi}} B_\rho(X_1, X_2, \dots, X_n)$$

for every $\pi \in \mathcal{I}(n)$. We define

$$\hat{B}_n = \sum_{\pi \in \mathcal{M}_{\mathrm{irr}}(n)} \frac{1}{|\pi|!} H_\pi = \sum_{\pi \in \mathrm{NC}_{\mathrm{irr}}(n)} \frac{1}{\tau(\pi)!} H_\pi$$

and show that

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\substack{\rho \in \mathcal{I}(n) \\ \rho \leq \pi}} \hat{B}_\rho(X_1, X_2, \dots, X_n),$$

which then implies $\hat{B}_\pi = B_\pi$ for all $\pi \in \mathcal{I}(n)$ by the Möbius principle.

First note that multiplicativity of the nesting tree factorial (Proposition 3.3(b)) implies that

$$\hat{B}_\rho = \sum_{\substack{\pi \in \mathcal{M}(n) \\ \hat{\pi} = \rho}} \frac{1}{|\pi|!} H_\pi = \sum_{\substack{\pi \in \mathrm{NC}(n) \\ \hat{\pi} = \rho}} \frac{1}{\tau(\pi)!} H_\pi.$$

any interval partition ρ. Moreover given an interval partition $\pi = \{V_1, V_2, \dots, V_p\}$ we have

$$\begin{aligned}
&\varphi_\pi(X_1, X_2, \dots, X_n) \\
&= \sum_{\sigma_1 \in \mathcal{M}(V_1)} \sum_{\sigma_2 \in \mathcal{M}(V_2)} \cdots \sum_{\sigma_p \in \mathcal{M}(V_p)} \frac{1}{|\sigma_1|!} H_{\sigma_1}(X_{V_1}) \frac{1}{|\sigma_2|!} H_{\sigma_2}(X_{V_2}) \cdots \frac{1}{|\sigma_p|!} H_{\sigma_p}(X_{V_p}) \\
&= \sum_{\sigma_1 \in \mathrm{NC}(V_1)} \sum_{\sigma_2 \in \mathrm{NC}(V_2)} \cdots \sum_{\sigma_p \in \mathrm{NC}(V_p)} \frac{1}{\tau(\sigma_1)!} H_{\sigma_1}(X_{V_1}) \frac{1}{\tau(\sigma_2)!} H_{\sigma_2}(X_{V_2}) \cdots \frac{1}{\tau(\sigma_p)!} H_{\sigma_p}(X_{V_p}) \\
&= \sum_{\substack{\sigma \in \mathrm{NC}(n) \\ \sigma \leq \pi}} \frac{1}{\tau(\sigma)!} H_\sigma(X_1, X_2, \dots, X_n) \\
&= \sum_{\substack{\rho \in \mathcal{I}(n) \\ \rho \leq \pi}} \sum_{\substack{\sigma \in \mathrm{NC}(n) \\ \hat{\sigma} = \rho}} \frac{1}{\tau(\sigma)!} H_\sigma(X_1, X_2, \dots, X_n) \\
&= \sum_{\substack{\rho \in \mathcal{I}(n) \\ \rho \leq \pi}} \hat{B}_\rho(X_1, X_2, \dots, X_n),
\end{aligned}$$

where the multiplicativity of the nesting tree factorial $\tau(\sigma)!$ was used again for the third equality. □

We are going to show (1.7) and (1.4) by using generating functions. The latter was shown in [BN08], but our proof is different. Let $\mathbb{C}[\![z_1, \dots, z_r]\!]$ be the ring of formal power series on r free indeterminates $z_1, \dots, z_r$. Let $\mathbf{z}$ denote the vector $(z_1, \dots, z_r)$. For a vector of noncommutative elements $\mathbf{X} = (X_1, \dots, X_r)$, we introduce generating functions:

$$M_{\mathbf{X}}(\mathbf{z}) := 1 + \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{r} \varphi(X_{i_1} \cdots X_{i_n}) z_{i_1} \cdots z_{i_n}, \tag{4.1}$$

$$B_{\mathbf{X}}(\mathbf{z}) := \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{r} B_n(X_{i_1}, \dots, X_{i_n}) z_{i_1} \cdots z_{i_n}, \tag{4.2}$$

$$R_{\mathbf{X}}(\mathbf{z}) := \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{r} R_n(X_{i_1}, \dots, X_{i_n}) z_{i_1} \cdots z_{i_n}, \tag{4.3}$$

$$H_{\mathbf{X}}(\mathbf{z}) := \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{r} H_n(X_{i_1}, \dots, X_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{4.4}$$

We also introduce vectors of generating functions:

$$\mathfrak{M}_{\mathbf{X}}(\mathbf{z}) := \mathbf{z} M_{\mathbf{X}}(\mathbf{z}) = (z_1 M_{\mathbf{X}}(\mathbf{z}), \dots, z_r M_{\mathbf{X}}(\mathbf{z})), \tag{4.5}$$

$$\mathfrak{H}_{\mathbf{X}}(\mathbf{z}) := \mathbf{z} H_{\mathbf{X}}(\mathbf{z}) = (z_1 H_{\mathbf{X}}(\mathbf{z}), \dots, z_r H_{\mathbf{X}}(\mathbf{z})). \tag{4.6}$$

Lemma 4.1. *The following identities hold.*

$$B_{\mathbf{X}}(\mathbf{z}) M_{\mathbf{X}}(\mathbf{z}) = M_{\mathbf{X}}(\mathbf{z}) - 1, \tag{4.7}$$

$$R_{\mathbf{X}}(\mathfrak{M}_{\mathbf{X}}(\mathbf{z})) = M_{\mathbf{X}}(\mathbf{z}) - 1. \tag{4.8}$$

Proof. The first identity is a straightforward consequence of the Boolean moment-cumulant formula. One has

$$\begin{aligned}
&B_{\mathbf{X}}(\mathbf{z}) M_{\mathbf{X}}(\mathbf{z}) \\
&= \left(\sum_{p=1}^{\infty} \sum_{i_1, \dots, i_p = 1}^{r} B_p(X_{i_1}, \dots, X_{i_p}) z_{i_1} \cdots z_{i_p} \right) \left(1 + \sum_{q=1}^{\infty} \sum_{j_1, \dots, j_q = 1}^{r} \varphi(X_{j_1} \cdots X_{j_q}) z_{j_1} \cdots z_{j_q} \right).
\end{aligned}$$

On the right hand side, the coefficient of the term $z_{k_1}\cdots z_{k_n}$ is equal to

$$B_n(X_{k_1},\dots,X_{k_n})+\sum_{p=1}^{n-1}B_p(X_{k_1},\dots,X_{k_p})\,\varphi(X_{k_{p+1}}\cdots X_{k_n}),$$

which, by virtue of the Boolean moment-cumulant formula, equals

$$\begin{aligned}B_n(X_{k_1},\dots,X_{k_n})+\sum_{p=1}^{n-1}\sum_{\pi\in\mathcal{I}(n-p)}B_p(X_{k_1},\dots,X_{k_p})B_\pi(X_{k_{p+1}},\dots,X_{k_n})&\\ =B_n(X_{k_1},\dots,X_{k_n})+\sum_{\sigma\in\mathcal{I}(n),\sigma\neq\hat{1}_n}B_\sigma(X_{k_1},\dots,X_{k_n})&\\ =\varphi(X_{k_1}\cdots X_{k_n}).&\end{aligned}$$

Formula (4.8) is the fundamental identity defining the multivariate R-transform, see [NS06, Lecture 16]. □

For $t\in\mathbb{R}$ and $\mathbf{X}=(X_1,\dots,X_r)$, let $\mathbf{X}(t)=(X_1(t),\dots,X_r(t))$ be a vector whose joint distribution is characterized by

$$H_n(X_{i_1}(t),\dots,X_{i_n}(t))=tH_n(X_{i_1},\dots,X_{i_n}),\qquad i_1,\dots,i_n\in[r],\ n\geq 1, \tag{4.9}$$

or equivalently

$$\varphi(X_{i_1}(t)\cdots X_{i_n}(t))=\sum_{\pi\in\mathcal{M}(n)}\frac{t^{|\pi|}}{|\pi|!}H_\pi(X_{i_1},\dots,X_{i_n}).$$

Let us denote

$$\begin{aligned}M_{\mathbf{X}}(t,\mathbf{z})&:=M_{\mathbf{X}(t)}(\mathbf{z}),\\ \mathfrak{M}_{\mathbf{X}}(t,\mathbf{z})&:=\mathbf{z}M_{\mathbf{X}}(t,\mathbf{z})=(z_1M_{\mathbf{X}}(t,\mathbf{z}),\dots,z_rM_{\mathbf{X}}(t,\mathbf{z})).\end{aligned}$$

Then the following differential equation holds as formal power series [HS11a]:

$$\frac{d}{dt}\mathfrak{M}_{\mathbf{X}}(t,\mathbf{z})=\mathfrak{H}_{\mathbf{X}}(\mathfrak{M}_{\mathbf{X}}(t,\mathbf{z})),\qquad \mathfrak{M}_{\mathbf{X}}(0,\mathbf{z})=\mathbf{z}. \tag{4.10}$$

Moreover, $(\mathfrak{M}_{\mathbf{X}}(t,\cdot))_{t\in\mathbb{R}}$ becomes a flow on $\mathbb{C}[\![z_1,\dots,z_r]\!]$:

$$\mathfrak{M}_{\mathbf{X}}(t+s,\mathbf{z})=\mathfrak{M}_{\mathbf{X}}(t,\mathfrak{M}_{\mathbf{X}}(s,\mathbf{z})),\qquad t,s\in\mathbb{R}, \tag{4.11}$$

which is proved by standard techniques from ordinary differential equations using the uniqueness of the solution in $\mathbb{C}[\![z_1,\dots,z_r]\!]$.

Definition 4.2. For a vector $\mathbf{X}=(X_1,\dots,X_r)\in\mathcal{A}^r$, let $\widetilde{\mathbf{X}}=(\widetilde{X}_1,\dots,\widetilde{X}_r)$ be a vector satisfying the relation

$$R_n(\widetilde{X}_{i_1},\dots,\widetilde{X}_{i_n})=-B_n(X_{i_1},\dots,X_{i_n}) \tag{4.12}$$

for any tuple $(i_1,\dots,i_n)\in[r]^n$.

Lemma 4.3. *The following relations hold for any $\mathbf{X}$ and any tuple $(i_1,\dots,i_n)$:*

(1) $B_n(\widetilde{X}_{i_1},\dots,\widetilde{X}_{i_n})=-R_n(X_{i_1},\dots,X_{i_n})$, *or equivalently* $B_{\widetilde{\mathbf{X}}}(\mathbf{z})=-R_{\mathbf{X}}(\mathbf{z})$;
(2) $H_n(\widetilde{X}_{i_1},\dots,\widetilde{X}_{i_n})=-H_n(X_{i_1},\dots,X_{i_n})$, *or equivalently* $H_{\widetilde{\mathbf{X}}}(\mathbf{z})=-H_{\mathbf{X}}(\mathbf{z})$.

Proof. (1) We will show that

$$\mathfrak{M}_{\mathbf{X}}\circ\mathfrak{M}_{\widetilde{\mathbf{X}}}=\mathrm{Id}. \tag{4.13}$$

The definition (4.12) of $\widetilde{\mathbf{X}}$ reads $R_{\widetilde{\mathbf{X}}}(\mathbf{z})=-B_{\mathbf{X}}(\mathbf{z})$, so that

$$-R_{\widetilde{\mathbf{X}}}(\mathbf{z})M_{\mathbf{X}}(\mathbf{z})=M_{\mathbf{X}}(\mathbf{z})-1 \tag{4.14}$$

from (4.7). Replace $\mathbf{z}$ by $\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z})$ and then (4.14) becomes

$$-\left(M_{\widetilde{\mathbf{X}}}(\mathbf{z})-1\right)M_{\mathbf{X}}(\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z}))=M_{\mathbf{X}}(\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z}))-1,$$

where the relation (4.8) was used for $\widetilde{\mathbf{X}}$. Hence, $z_iM_{\widetilde{\mathbf{X}}}(\mathbf{z})M_{\mathbf{X}}(\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z}))=z_i$ for each i, implying the claim (4.13). In particular, $M_{\mathbf{X}}(\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z}))=\frac{1}{M_{\widetilde{\mathbf{X}}}(\mathbf{z})}$. Replacing $\mathbf{z}$ by $\mathfrak{M}_{\widetilde{\mathbf{X}}}(\mathbf{z})$ in (4.8), one obtains $R_{\mathbf{X}}(\mathbf{z})=\frac{1}{M_{\widetilde{\mathbf{X}}}(\mathbf{z})}-1$, which coincides with $-B_{\widetilde{\mathbf{X}}}(\mathbf{z})$ thanks to (4.7) for $\widetilde{\mathbf{X}}$.

(2) The flow property (4.11) for $t=1,s=-1$ reads $\mathfrak{M}_{\mathbf{X}}\circ\mathfrak{M}_{\mathbf{X}(-1)}=\mathrm{Id}$, which together with (4.13) implies that $\mathbf{X}(-1)=\widetilde{\mathbf{X}}$ in distribution regarding φ. From (4.9) we get

$$H_n(\widetilde{X}_{i_1},\dots,\widetilde{X}_{i_n})=H_n(X_{i_1}(-1),\dots,X_{i_n}(-1))=-H_n(X_{i_1},\dots,X_{i_n}).$$

□

Proof of Theorem 1.1(1.7). From (1.6) and Lemma 4.3, we obtain

$$\begin{aligned}-R_n(X_1,\dots,X_n)=B_n(\widetilde{X}_1,\dots,\widetilde{X}_n)&=\sum_{\pi\in\mathcal{M}_{\mathrm{irr}}(n)}\frac{1}{|\pi|!}H_\pi(\widetilde{X}_1,\dots,\widetilde{X}_n)\\&=\sum_{\pi\in\mathcal{M}_{\mathrm{irr}}(n)}\frac{(-1)^{|\pi|}}{|\pi|!}H_\pi(X_1,\dots,X_n).\end{aligned}$$

□

Proof of (1.4). From the multi-variate version generalization of (1.1) and Lemma 4.3, we have the following:

$$\begin{aligned}-R_n(X_1,\dots,X_n)=B_n(\widetilde{X}_1,\dots,\widetilde{X}_n)&=\sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)}R_\pi(\widetilde{X}_1,\dots,\widetilde{X}_n)\\&=\sum_{\pi\in\mathrm{NC}_{\mathrm{irr}}(n)}(-1)^{|\pi|}B_\pi(X_1,\dots,X_n).\end{aligned}$$

□

Remark 4.4. $\mathrm{NC}_{\mathrm{irr}}(n)$ is a lattice and is isomorphic to $\mathrm{NC}(n-1)$; however we can not apply the Möbius inversion directly to the free-to-boolean formula (1.1) to get the boolean-to-free formula (1.4) since it does not respect multiplicativity.

5. Colored trees and proof of Theorem 1.2

The concept of colored partitions was introduced by Lenczewski [Len12].

Definition 5.1. An N-*colored partition* of $[n]$ is a pair (π,f), where $\pi=\{V_1,\dots,V_k\}$ is a partition of $[n]$ and f is a map from the set $\{V_1,\dots,V_k\}$ to $[N]$. The set of noncrossing N-colored partitions of $[n]$ is denoted by $\mathrm{NC}(n,N)$. When $i=f(V)$, we say that $V\in\pi$ is colored by i.

Remark 5.2. An ordered partition of $[n]$ is a $|\pi|$-colored partition (π,f) of $[n]$ such that every block has a different color.

We will express monotone cumulants in terms of free cumulants. For this purpose, we are going to associate a polynomial to a rooted tree.

Definition 5.3. An N-labelling of a graph is a map f from its vertices to $[N]$. A labelling f of a rooted tree is called *nondecreasing* if for every vertex v and every child vertex u of v we have $f(v)\le f(u)$. An N-labelling of a forest is called nondecreasing if the labels on each of its trees are nondecreasing.

Proposition 5.4. *Let t be a rooted tree and let $P_t(N)$ be the number of nondecreasing N-labellings of t. Then $P_t(N)$ is a polynomial in N of degree $\le|t|$ with zero constant term. An analogous statement holds for forests.*

Proof. It is easy to see that the polynomial $P_t(N)$ of a nesting forest consisting of rooted trees $t_1, \dots t_k$ is just $\prod_i P_{t_i}(N)$, therefore it suffices to show that $P_t(N)$ is a polynomial without constant term for every rooted tree t. We proceed by induction. If t has only one vertex, then $P_t(N) = N$ satisfies the claimed property. Otherwise t decomposes into the root vertex and branches $t'_1, \dots, t'_m$. If the root gets label k, all other vertices must get labels not smaller than k and the number of such nondecreasing labellings is

$$Q(N,k) = \prod_{i=1}^{m} P_{t'_i}(N-k+1)$$

which by assumption is a polynomial in N and k of degree $\leq |t| - 1$ without constant term. The number of N-labellings can be enumerated recursively by conditioning on the label of the root and we obtain

$$P_t(N) = \sum_{k=1}^{N} Q(N,k).$$

We can now apply Faulhaber's summation formula

$$\sum_{k=1}^{N} k^n = \frac{B_{n+1}(N+1) - B_{n+1}(0)}{n+1} \tag{5.1}$$

to each term to express $P_t(N)$ in terms of Bernoulli polynomials $B_n(x)$ defined by their generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

It follows that $P_t(N)$ is a polynomial of degree $\leq |t|$ without constant term. $\square$

Definition 5.5. Let α_t be the coefficient of the linear term of P_t, i.e.,

$$\alpha_t = P'_t(0).$$

For $\pi \in \mathrm{NC}(n)$, we define $P_\pi(N) := P_{\tau(\pi)}(N)$ and $\alpha_\pi := \alpha_{\tau(\pi)}$.

Example 5.6.
(1) If t is not connected then $\alpha_t = 0$.
(2) Let t be a tree consisting of the vertices $\{1, 2, \dots, n+1\}$ and n edges, each connecting 1 and k for $2 \leq k \leq n+1$. The vertex 1 is the root of t. Then $P_t(N)$ is equal to Faulhaber's formula (5.1) and $\alpha_t = P'_t(0) = \frac{1}{n+1} B'_{n+1}(1) = B_n(1)$ because of the identity $B'_k(x) = kB_{k-1}(x)$. So $\alpha_t = B_n(1)$ is the nth Bernoulli number.
(3) A tree t has n vertices $\{1, 2, \dots, n\}$ and $n-1$ edges, each connecting k and $k+1$ for $1 \leq k \leq n-1$, and 1 is the root of t. Then $P_t(x) = \binom{x+n-1}{n}$ and $\alpha_t = \frac{1}{n}$.

Proof of Theorem 1.2. Let J be a subset of $[N] \times [N]$ including the diagonal set $\{(j,j) \mid j \in [N]\}$. Lenczewski introduced the concept of strong matricial freeness for an array $(X_{ij})_{(i,j)\in J}$ of elements in a noncommutative probability space $(\mathcal{A}, (\varphi_{ij})_{(i,j)\in J})$, where $(\varphi_{ij})_{(i,j)\in J}$ is an array of unital linear functionals. Following [Len12], we assume that $\varphi_{ii} = \varphi$, the same linear functional, and φ_{ij} do not depend on i if $i \neq j$. Let $(X_{ij})_{(i,j)\in J}$ be an array of elements with low-identical distributions, that is, the moments $\varphi(X_{ij}^n)$ do not depend on j.

Given $(\pi, f) \in \mathrm{NC}(n, N)$, we associate a product of free cumulants $r_{(\pi,f)} := r_{(V_1,f)} \cdots r_{(V_{|\pi|},f)}$ as follows. Take a block V_k of π with color i. If its outer blocks are all colored by i, we define $r_{(V_k,f)} := r_{|V_k|}(X_{ii})$. If V_k has an outer block with another color j, then $r_{(V_k,f)} := r_{|V_k|}(X_{ij})$ where j is the color of the outer block nearest to V_k whose color j is different from i. If a pair (i,j) is not an element of J, then we understand $r_k(X_{ij}) = 0$, $k = 1, 2, 3, \dots$.

We need the following results (see Lemma 7.1 and Theorem 3.1 of [Len12] and also Proposition 4.1 of [Len10]):

(i)

$$\varphi\left(\left(\sum_{(i,j)\in J} X_{ij}\right)^n\right) = \sum_{(\pi,f)\in \mathrm{NC}(n,N)} r_{(\pi,f)}. \tag{5.2}$$

(ii) Let $(X_{ij})_{(i,j)\in J}$ be strongly matricially free. If $J = \{(i,j) \mid 1 \le i \le j \le N\}$ and the distributions of X_{ij} do not depend on j, then $X_i := \sum_{j=1}^i X_{ij}$, $i \in [N]$, are monotonically independent.

In statement (ii), note that X_i has the same distribution as X_{ii} with respect to φ as mentioned in the proof of Proposition 4.1 of [Len10]. In statement (ii) we assume in addition that the distributions of X_{ij} do not depend on (i,j). Then formula (5.2) in statement (i) becomes

$$\varphi((X_1 + \cdots + X_N)^n) = \sum_{(\pi,f)\in \mathrm{NC}(n,N)} r_{(\pi,f)}$$

for monotonically i.i.d. random variables X_i. Each summand $r_{(\pi,f)}$ is either $\prod_{V\in\pi} r_{|V|}(X_1)$ or 0. The summand $r_{(\pi,f)}$ does not vanish if and only if, for each block V with color i, its outer blocks have colors not greater than i. The number of such colorings is just equal to $P_\pi(N)$. By definition, the nth monotone cumulant of X_1 coincides with the coefficient of N in $\varphi((X_1 + \cdots + X_N)^n)$ (which, in particular, is zero unless $\pi \in \mathrm{NC}_{\mathrm{irr}}(n)$), so (1.8) follows.

The identity (1.9) is proved similarly to Theorem 1.1(1.7). From (1.8) and Lemma 4.3, it follows that

$$-h_n(X) = h_n(\widetilde{X}) = \sum_{\pi\in \mathrm{NC}(n)} \alpha_\pi r_\pi(\widetilde{X}) = \sum_{\pi\in \mathrm{NC}(n)} (-1)^{|\pi|}\alpha_\pi b_\pi(X).$$

Identity (1.10) follows from (1.8) together with the easy fact (following from (1.2)) that more generally

$$r_\sigma = \sum_{\substack{\pi\in\mathcal{P}(n)\\ \bar{\pi}=\sigma}} \kappa_\pi, \qquad \sigma \in \mathrm{NC}(n).$$

Indeed, we observe that $\pi \in \mathcal{P}_{\mathrm{irr}}(n)$ if and only if $\bar{\pi} \in \mathrm{NC}_{\mathrm{irr}}(n)$. Hence

$$\sum_{\pi\in\mathcal{P}_{\mathrm{irr}}(n)} \alpha_{\bar{\pi}}\kappa_\pi = \sum_{\sigma\in \mathrm{NC}_{\mathrm{irr}}(n)} \alpha_\sigma \sum_{\substack{\pi\in\mathcal{P}(n)\\ \bar{\pi}=\sigma}} \kappa_\pi = \sum_{\sigma\in \mathrm{NC}_{\mathrm{irr}}(n)} \alpha_\sigma r_\sigma = h_n.$$

□

Remark 5.7. The moment-cumulant formula (5.2) is known only for the univariate case, and so we can prove Theorem 1.2 only for univariate cumulants.

6. Tutte polynomials and proof of Theorem 1.4

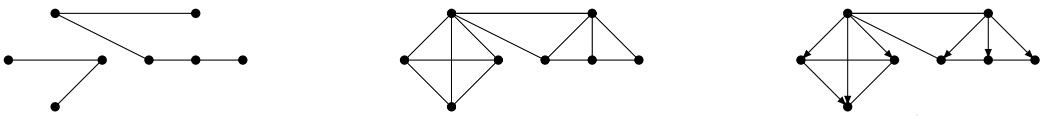
For an arbitrary finite set $\mathcal{S}$ we denote by $\mathcal{P}(\mathcal{S})$ its set of partitions. Any bijection between $\mathcal{S}$ and $\{1,\dots|\mathcal{S}|\}$ induces a poset isomorphism $\mathcal{P}(\mathcal{S})$ to $\mathcal{P}(|\mathcal{S}|)$. If $\mathcal{S}$ is totally ordered we consider the bijection which preserves this order and define $\mathrm{NC}(\mathcal{S}), \mathcal{I}(\mathcal{S})$ via this isomorphism.

Definition 6.1. Let $\pi \in \mathcal{P}(n)$.

(1) We define the crossing graph $G(\pi) := (V,E)$ of π, where the set of vertices $V = \{V_1,\dots,V_{|\pi|}\}$ is indexed by the blocks[2] of π and an edge joins the vertices V_i, V_j if and only if they cross, i.e., $W = (V_i,V_j) \in (\mathcal{P}(V_i\cup V_j)\setminus \mathrm{NC}(V_i\cup V_j))$.

(2) Similarly, the vertices of the anti-interval graph $\tilde{G}(\pi) := (V,E)$ of π are just the blocks of π. An edge joining (V_i,V_j) is drawn if and only if $W = (V_i,V_j) \in (\mathcal{P}(V_i\cup V_j)\setminus\mathcal{I}(V_i\cup V_j))$. (For a noncrossing partition this is the nesting forest from Definition 3.1, augmented by the edges from all vertices to all their descendents).

[2] It should not cause confusion that we regard V_i simultaneously as a vertex of $G(\pi)$ and as a block of π



$G(\pi)$ $\tilde{G}(\pi)$ $\overrightarrow{G}(\pi)$

$\pi = \{\{1,10\},\{2,6\},\{3,5\},\{4,7\},\{8,16\},\{9,12\},\{11,14\},\{13,15\}\} =$ 

FIGURE 3. A partition and its associated graphs (see Definition 8.2 for the third graph).

(3) For a finite graph $G=(V,E)$ and $e\in E$, we let $G\backslash e=(V,E\backslash e)$, and $G/e=(V/e,E\backslash e)$ be the graph obtained from removing e and identifying the endpoints of e. The *Tutte polynomial* $T_G(x,y)$ of G can be defined recursively by setting $T_G(x,y)=1$ if $E=\emptyset$ and:

$$T_G(x,y)=\begin{cases} xT_{G/e}(x,y) & \text{if } e \text{ is a bridge,}\\ yT_{G\backslash e}(x,y) & \text{if } e \text{ is a loop,}\\ T_{G/e}(x,y)+T_{G\backslash e}(x,y) & \text{otherwise.}\end{cases}$$

Remark 6.2. Let $(A_i)_{i\in I}$ be a family of sets. Recall that the *intersection graph* is the graph with vertex set $\{A_i : i\in I\}$ where there is an edge $i\sim j$ if and only if $A_i\cap A_j\neq 0$, see [Pri98] for more information. An *interval graph* is the intersection graph of a family of intervals on the real line. Coincidentally, the anti-interval graph defined above is exactly the interval graph generated by the convex hulls of the blocks of π.

Remark 6.3. Let $G=(V,E)$ be any finite graph. For $\pi\in\mathcal{P}(V)$ we define $i(E,\pi)$ to be the number of edges in E which connect vertices with both endpoints belonging to the same block of π. It was shown in [JV13] that for any graph G and $q\neq 1$ we have:

$$\frac{1}{(q-1)^{\#V-1}}\sum_{\pi\in\mathcal{P}(V)} q^{i(E,\pi)}\mu_{\mathcal{P}}(\pi,1_V)=\begin{cases} T_G(1,q) & \text{if } G \text{ is connected,}\\ 0 & \text{otherwise}\end{cases} \tag{6.1}$$

with the convention that $q^0=1$ for $q=0$.

We obtain the Boolean-to-classical cumulant formula by following the lines of the proof of (1.5) in [JV13].

Proof of Theorem 1.4. Let $X_1,\dots,X_n\in\mathcal{A}$. Using subsequently the classical and the Boolean moment-cumulant formulas (2.10), (2.5), we obtain

$$\begin{aligned} K_n(X_1,\dots,X_n) &= \sum_{\pi\in\mathcal{P}(n)}\varphi_\pi(X_1,\dots,X_n)\mu_{\mathcal{P}}(\pi,\hat{1}_n)\\ &= \sum_{\substack{\sigma,\pi\in\mathcal{P}(n)\\ \sigma\preceq\pi}} B_\sigma(X_1,\dots,X_n)\mu_{\mathcal{P}}(\pi,\hat{1}_n)\\ &= \sum_{\sigma\in\mathcal{P}(n)} B_\sigma(X_1,\dots,X_n)\sum_{\pi\succeq\sigma}\mu_{\mathcal{P}}(\pi,\hat{1}_n), \end{aligned}$$

where, for $\pi\geq\sigma$, we write $\pi\succeq\sigma$ if the restriction $\sigma|_W\in\mathcal{P}(W)$ to any block W of π is an interval partition.

Let us fix $\sigma\in\mathcal{P}(n)$ and consider its anti-interval graph $\tilde{G}(\sigma)=(V,E)$. There is a one-to-one correspondence $\pi\mapsto\tilde{\pi}$ between partitions $\{\pi:\pi\geq\sigma\}\subset\mathcal{P}(n)$ and $\mathcal{P}(V)$: $\pi\geq\sigma$ is obtained by gluing blocks of σ, and $\tilde{\pi}$ describes which blocks of σ are glued together. In view of the formula (6.1) for $q=0$, we observe that $i(E,\tilde{\pi})=0$ exactly for those $\pi\geq\sigma$ such that

$\pi \succeq \sigma$. Furthermore, $|\pi| = |\tilde{\pi}|$ and hence $\mu_{\mathcal{P}}(\tilde{\pi}, 1_V) = (-1)^{|\tilde{\pi}|-1}(|\tilde{\pi}|-1)! = (-1)^{|\pi|-1}(|\pi|-1)! = \mu_{\mathcal{P}}(\pi, \hat{1}_n)$. Therefore

$$\sum_{\pi \succeq \sigma} \mu_{\mathcal{P}}(\pi, \hat{1}_n) = \sum_{\pi \geq \sigma} 0^{i(E,\tilde{\pi})} \mu_{\mathcal{P}}(\pi, \hat{1}_n) = \sum_{\tilde{\pi} \in \mathcal{P}(V)} 0^{i(E,\tilde{\pi})} \mu_{\mathcal{P}}(\tilde{\pi}, 1_V)$$

and thus, formula (6.1) yields

$$\sum_{\pi \succeq \sigma} \mu_{\mathcal{P}}(\pi, \hat{1}_n) = \sum_{\tilde{\pi} \in \mathcal{P}(V)} 0^{i(E,\tilde{\pi})} \mu_{\mathcal{P}}(\tilde{\pi}, 1_V) = \begin{cases} (-1)^{|\sigma|-1} T_{\tilde{G}(\sigma)}(1,0) & \text{if } \tilde{G}(\sigma) \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see that the number of blocks of the interval closure of σ is equal to the number of connected components of its anti-interval graph. Therefore $\tilde{G}(\sigma)$ is connected iff $\sigma \in \mathcal{P}_{\mathrm{irr}}(n)$ and the formula follows. □

Several evaluations of the Tutte polynomial have combinatorial interpretations. For our purposes it will be interesting to note the fact that $T_G(1, 0)$ equals the number of rooted acyclic orientations with unique specified source; this number does not depend on the choice of the source [GZ83]. Recall that an *acyclic orientation* of a graph is an orientation without oriented cycles. An acyclic orientation has source v if there is a directed path from v to every other vertex. Alternatively, this can be interpreted as the Hasse diagram of an ordering of the vertices of G with prescribed unique minimal element.

Returning from graphs to partitions this has the following pictorial interpretation.

Definition 6.4. (1) Let π be a connected set partition. A *crossing heap* on π is a poset structure on the blocks of π such that any pair of crossing blocks is comparable. A *pyramid* is a crossing heap such that the first block is the only maximal element.

(2) Let π be an irreducible set partitions. An *interval heap* on π is a poset structure on the blocks of π such that any pair of blocks whose convex hulls have nonempty intersection are comparable. A *pyramid* is an interval heap such that the first block is the only maximal element.

The following proposition is immediate from the definition, see Fig. 4 and Fig. 5.

Proposition 6.5. *(1) The crossing heaps of a connected partition are in bijection with the acyclic orientations of its crossing graph. Pyramids correspond to those rooted acyclic orientations which are rooted in the first block, thus $T_{G(\pi)}(1, 0)$ equals the number of crossing heaps on π which are pyramids.*

(2) The interval heaps of an irreducible partition are in bijection with the acyclic orientations of its anti-interval graph. Pyramids correspond to those rooted acyclic orientations which are rooted in the first block, thus $T_{\tilde{G}(\pi)}(1, 0)$ equals the number of interval heaps on π which are pyramids.

From these facts one can make a connection to the Cartier-Foata-Viennot theory of heaps [CF69] and in fact some proofs of [JV13] make use of this machinery. The heap interpretation of formulas (1.5) and (1.11) read as follows.

Corollary 6.6. *The classical cumulants can be expressed in terms of the free and Boolean cumulants as follows.*

$$K_n = \sum (-1)^{|\pi|-1} R_\pi \tag{6.2}$$

where the sum runs over all pyramidal crossing heaps.

$$K_n = \sum (-1)^{|\pi|-1} B_\pi \tag{6.3}$$

where the sum runs over all pyramidal interval heaps.

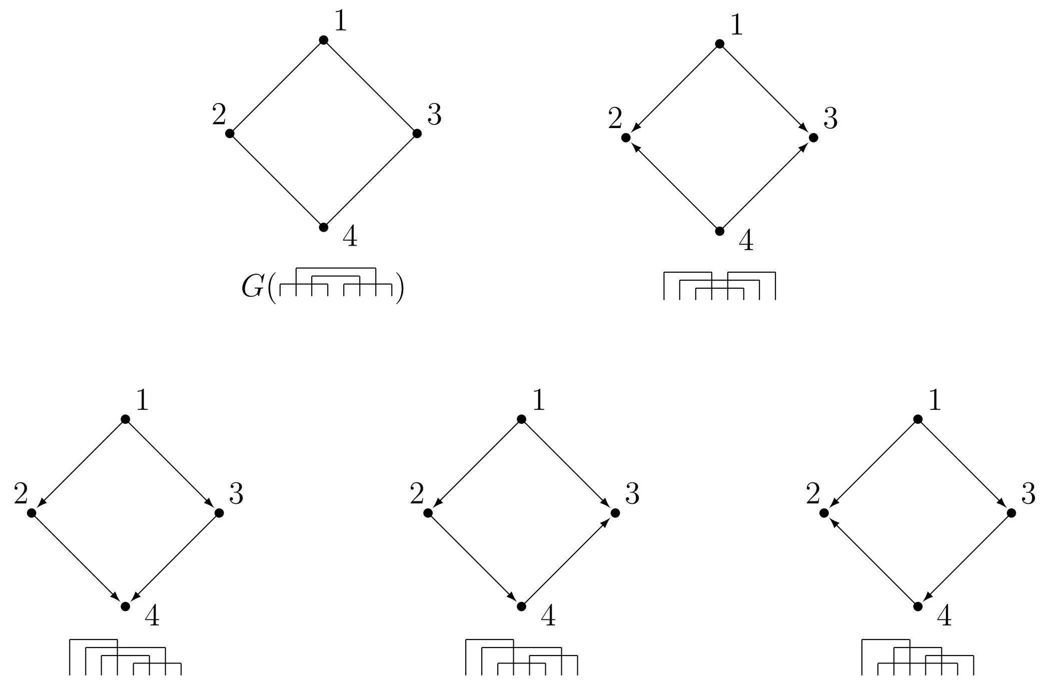


Figure 4. The crossing graph, some acyclic orientations and crossing heaps of the partition $\pi =$; blocks are numbered in the canonical order. Note that the upper right orientation is not a pyramid.

7. Permutation statistics and proof of Theorem 1.5

There are no cancellations in formula (1.11) and one might wonder whether there is another combinatorial interpretation. The following formulas suggest that a statistic on permutations might be involved.

Corollary 7.1.

$$\sum_{\pi \in \mathcal{P}_{\mathrm{irr}}(n)} T_{\tilde{G}(\pi)}(1,0) = (n-1)!.$$

Proof. The sum is obtained by evaluating the negative classical cumulants of a (formal) distribution with Boolean cumulants $b_n = -1$ for all n. The "moments" have generating function

$$M(z) = \frac{1}{1 + \sum_{n=1}^{\infty} z^n} = 1 - z$$

and in this case $F(z) = M(z)$. Thus the negative classical cumulants have exponential generating series

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

□

We collect key concepts for statistics of permutations.

(1) Recall that a *run* in a permutation $\sigma \in \mathfrak{S}_n$ is a maximal increasing segment of the sequence $(\sigma(1), \sigma(2), \dots, \sigma(n))$. Any permutation decomposes uniquely into runs. With the exception of the last run, at the end of each run there is a descent, therefore the number of runs is equal to the number of descents incremented by one. Given a permutation σ we denote by $\mathrm{runs}(\sigma)$ the set partition associated to the set of runs.

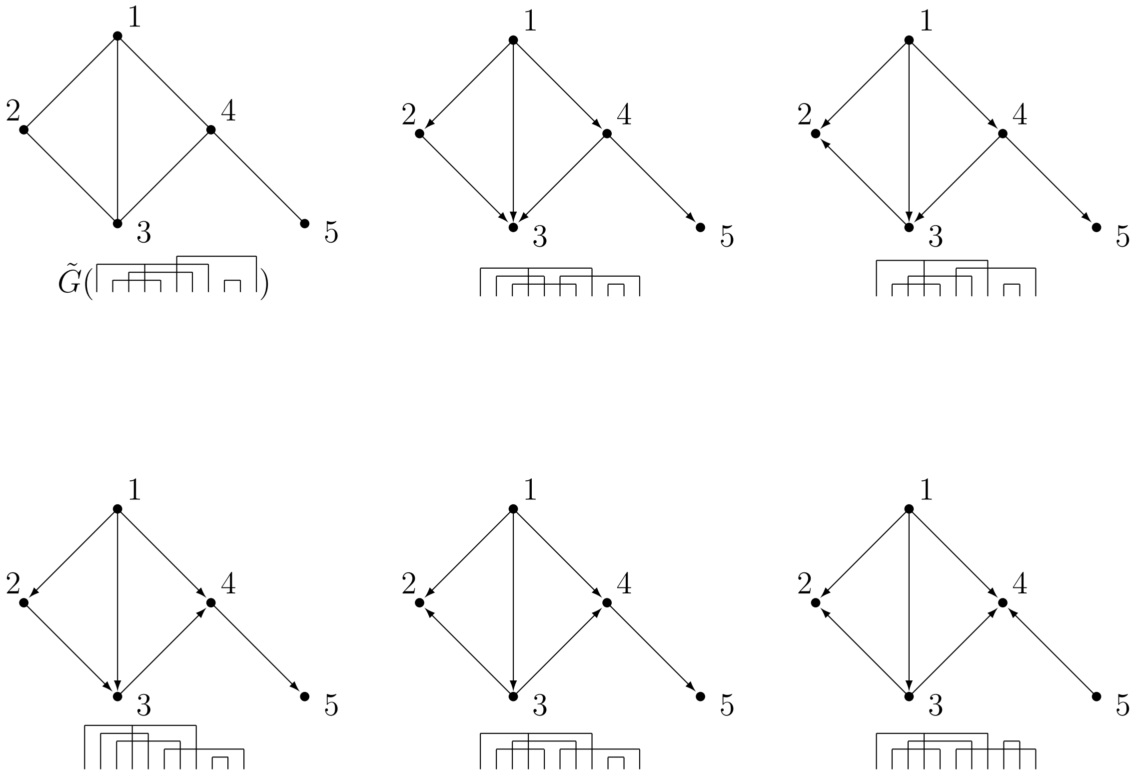


FIGURE 5. The anti-interval graph, some acyclic orientations and interval heaps of the partition $\pi =$. All heaps except the last are pyramidal.

(2) Every permutation π has a unique factorization into cycles $\sigma = \gamma_1\gamma_2\cdots\gamma_k$, where the cycles are sorted in increasing order with respect to their minimal elements. Moreover, every cycle is written starting with its minimal element. A *cycle run* in a permutation σ is a maximal contiguous increasing subsequence of one of its cycles; in other words, a maximal sequence $i_1 < i_2 < \cdots < i_r$ such that $\sigma(i_k) = i_{k+1}$.

It is easy to see that distinct cycle runs of a given permutation are disjoint and therefore give rise a set partition of order n. We denote this set partition by $\operatorname{cruns}(\sigma)$.

We denote by $\mathrm{c}(\sigma) \in \mathcal{P}(n)$ the *cycle partition* of σ, i.e., the set partition whose blocks are given by the cycles of σ.

The key identity for showing Theorem 1.5 is contained in the following lemma.

Lemma 7.2.

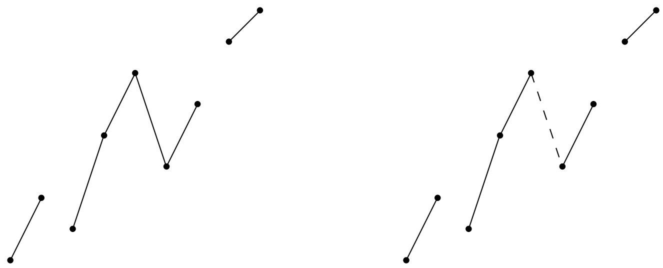
$$\varphi(X_1X_2\cdots X_n) = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\#\operatorname{cruns}(\sigma)-\#\mathrm{c}(\sigma)} B_{\operatorname{cruns}(\sigma)}(X_1, X_2, \ldots, X_n). \tag{7.1}$$

Proof. The starting point is formula (2.5) for $\pi = \hat{1}_n$, expressing moments in terms of Boolean cumulants. The right hand side of formula (2.5) for $\pi = \hat{1}_n$ arise from the terms

$$(-1)^{\#\operatorname{cruns}(\sigma)-\#\mathrm{c}(\sigma)} B_{\operatorname{cruns}(\sigma)}(X_1, X_2, \ldots, X_n)$$

where σ factorizes into "interval cycles", i.e., cycles of the form

$$(k, k+1, k+2, \ldots, l).$$



$(1,3)(2,5,7,4,6)(8,9)$ $\qquad$ $(1,3)(2,5,7)(4,6)(8,9)$

FIGURE 6. The involution Φ

Note that in this case we have $\#\,\mathrm{c}(\sigma) = \#\,\mathrm{cruns}(\sigma)$ and so the sign is $+1$. Hence the proof follows the common strategy of finding an inversion which shows cancellation of all permutations which do not have a factorization into interval cycles.

Assume that a permutation σ is not of interval type, and let $\sigma = \gamma_1\gamma_2\cdots\gamma_m$ be its cycle decomposition in standard order and $\gamma_i = (r_{i1}, r_{i2}, \ldots, r_{ik_i})$ be the run decompositions of the cycles. Since σ is not of interval type, there are descents. Our involution Φ operates on the last descent. We say that σ is of type A if the last descent occurs within one cycle and type B if it occurs between two consecutive cycles. We set up an involution Φ as follows.

If σ is of type A and the last descent occurs within one cycle, then it occurs necessarily between the last two runs of this cycle. We split the cycle at this descent to obtain $\Phi(\sigma)$ which is of type B.

If σ is of type B, the descent occurs between two cycles and the second cycle necessarily only consists of one run. We join the two cycles to obtain $\Phi(\sigma)$ which is of type A.

Figure 6 shows an example of the action of the involution Φ. The permutation on the left (type A) is mapped to the permutation on the right (type B) and vice versa.

In both cases the position of the last descent remains the same and therefore Φ maps type A to type B bijectively and $\Phi\circ\Phi = \mathrm{Id}$.

On the other hand, the total number of runs is unchanged, while the total number of cycles is changed by ±1. It follows that the contributions of type A and type B permutations in the sum (7.1) cancel. $\square$

First proof of Theorem 1.5. Denote by $\tilde{K}_n$ the right hand side of (1.12):

$$\tilde{K}_n = \sum_{\sigma\in\mathcal{C}_n} (-1)^{\#\,\mathrm{cruns}(\sigma)-1} B_{\mathrm{cruns}(\sigma)}.$$

The full cycles are exactly the permutations σ such that $\mathrm{c}(\sigma) = \hat{1}_n$ and thus

$$\tilde{K}_n = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{c}(\sigma)=\hat{1}_n}} (-1)^{\#\,\mathrm{cruns}(\sigma)-1} B_{\mathrm{cruns}(\sigma)}.$$

Taking the product of them, it is then easy to see that $\tilde{K}_\pi = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{c}(\sigma)=\pi}} (-1)^{\#\,\mathrm{cruns}(\sigma)-\#\,\mathrm{c}(\sigma)} B_{\mathrm{cruns}(\sigma)}$, and so from Lemma 7.2

$$\varphi(X_1\cdots X_n) = \sum_{\pi\in\mathcal{P}(n)} \tilde{K}_\pi(X_1,\ldots,X_n).$$

Since the same formula holds with $\tilde{K}_\pi$ replaced by K_π, then multiplicativity and the Möbius principle imply that $\tilde{K}_n = K_n$. $\square$

Another proof of Theorem 1.5 can be obtained from the following identities.

Lemma 7.3. *For any irreducible partition $\pi\in\mathcal{P}(n)$ the evaluation of the Tutte polynomial $T_{\tilde{G}(\pi)}(1,0)$ is equal to the following numbers.*

(1) The number of pyramidal interval heaps on π.
(2) The number of cyclic permutations $\sigma \in \mathcal{C}_n$ such that and $\operatorname{cruns}(\sigma) = \pi$.
(3) The number of permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = 1$ and $\operatorname{runs}(\sigma) = \pi$.

In particular, $\operatorname{cruns}(\sigma)$ *is an irreducible permutation for every $\sigma \in \mathcal{C}_n$.*

Proof. The first identity has already been seen in Proposition 6.5 and the equality of the other two numbers follows from the obvious bijection between cyclic permutations and permutations fixing 1. It remains to provide a bijection between runs of cyclic permutations and pyramidal interval heaps.

Given a cyclic permutation σ, we assign to it an ordered interval partition $\Psi(\sigma)$ consisting of the cycle runs of σ read from left to right. This order induces a heap structure by placing subsequent blocks below their predecessors. We claim that $\Psi(\sigma)$ is a pyramid.

Let $r_1, r_2, \dots, r_m$ be the cycle runs of σ, in the order of their appearance when σ is written as $(1, \sigma(1), \sigma^2(1), \dots, \sigma^{n-1}(1))$. Clearly r_1 is not a singleton and for every run r_k with $k \geq 2$ the following statements are true.

(a) The maximal element in the run r_{l-1} exceeds the starting element of r_l $(2 \leq l \leq k)$.
(b) There is a run r_i with $i \leq k-1$ such that as an element of the heap, r_i covers r_k, i.e., $\max r_i > \min r_k$ and $\min r_i < \max r_k$. In other words, the convex hulls of r_i and r_k intersect.

Since $r_1, \dots, r_{l-1}, r_l, \dots$ is the decomposition of σ into cycle runs, we must have $\max r_{l-1} > \min r_l$ and this implies (a).

To show (b), we perform the following steps.

(1) If $\min r_{k-1} < \max r_k$, then r_{k-1} itself covers r_k from (a) and we are done.
(2) If $\min r_{k-1} > \max r_k$, then $\max r_{k-2} > \min r_{k-1} > \max r_k$. If $\min r_{k-2} < \max r_k$, then r_{k-2} covers r_k and we are done. If not, we go to r_{k-3}. We repeat this until we reach some run r_i $(i < k-1)$ such that $\min r_i < \max r_k < \max r_i$. This must happen at some point because ultimately we reach $r_1 = (1, \dots)$ and then clearly $1 = \min r_1 < \max r_k$.

The inverse map can be defined recursively as follows: Given an interval heap, take the leftmost minimal element and write it to the left of the previously written cycle run. There are two possibilities. Either the previous run was not covered by the current one, then it came from a block to the left, or it was covered by the current one. In either case the maximum of the current block is larger than the minimum of the previous one and thus the current block starts a new cycle run. For an example of this process see Figure 7. This map clearly reverses the map Ψ defined above. ☐

With this lemma, Theorem 1.5 follows from Corollary 6.6, equation (6.3) or from Theorem 1.4.

Recall that the *Eulerian polynomials* are defined by

$$E_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{d(\sigma)} = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k \tag{7.2}$$

where $\left\langle {n \atop k} \right\rangle$ is the number of permutations with k descents. A good reference for these and the following facts is the book [GKP94]. The Eulerian polynomials have been studied as moments in [Bar11].

Note that each run of a permutation must follow a descent and therefore $\sharp(\operatorname{runs}(\sigma)) = d(\sigma)+1$, and Lemma 7.3 together with formula (7.2) yields the identity

$$\sum_{\substack{\pi \in \mathcal{P}_{\mathrm{irr}}(n) \\ |\pi| = k}} T_{\tilde{G}(\pi)}(1,0) = \left\langle {n-1 \atop k-1} \right\rangle. \tag{7.3}$$

As a special case of Corollary 1.6, we consider the Boolean Poisson distribution. We interpret the Eulerian polynomials as classical cumulants.

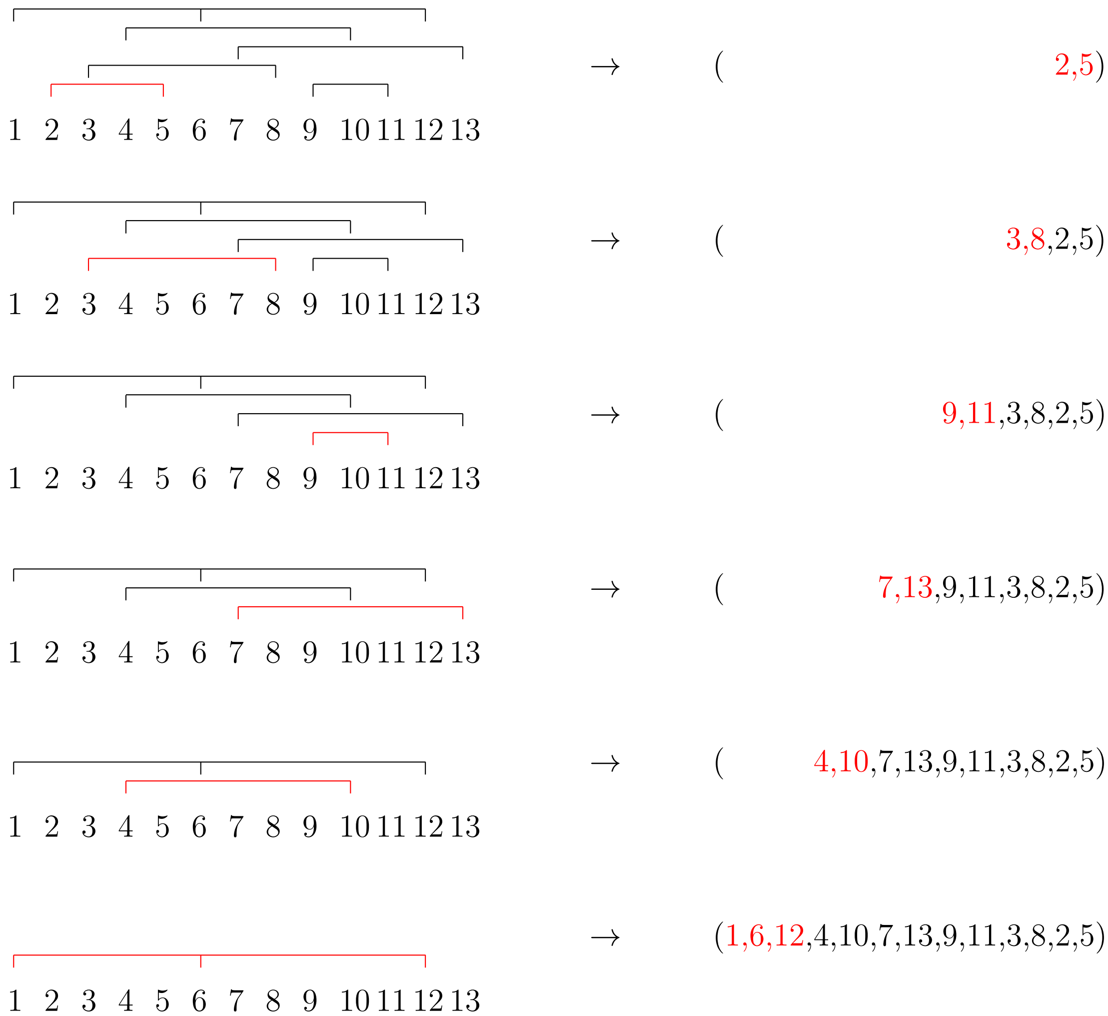


FIGURE 7. The inverse map for an interval heap on the partition $\pi = \{\{1, 6, 12\}, \{2, 5\}, \{3, 8\}, \{4, 10\}, \{7, 13\}, \{9, 11\}\}$

Proposition 7.4. *Consider the distribution with Boolean cumulants $b_n = x$ for all n. Then the classical cumulants are given by the Eulerian polynomials*

$$\kappa_n = xE_{n-1}(-x).$$

Proof. By the remarks above $b_{\mathrm{runs}(\sigma)} = x^{d(\sigma)+1}$, and hence from Theorem 1.6

$$\kappa_n = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=1}} (-1)^{d(\sigma)} x^{d(\sigma)+1}.$$

The desired formula follows from the natural bijection $\{\sigma \in \mathfrak{S}_n : \sigma(1) = 1\} \to \mathfrak{S}_{n-1}$, which preserves the number of descents. $\square$

Remark 7.5. (1) The following determinant formulas relating moments and classical cumulants are known:

$$\kappa_n = (-1)^{n-1}(n-1)! \begin{vmatrix} m_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ m_2 & m_1 & 1 & 0 & 0 & \cdots & 0 \\ \frac{m_3}{2!} & \frac{m_2}{2!} & m_1 & 1 & 0 & \cdots & 0 \\ \frac{m_4}{3!} & \frac{m_3}{3!} & \frac{m_2}{2!} & m_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \frac{m_n}{(n-1)!} & \frac{m_{n-1}}{(n-1)!} & \frac{m_{n-2}}{(n-2)!} & & \cdots & & m_1 \end{vmatrix}$$

and

$$m_n = \begin{vmatrix} \kappa_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ \kappa_2 & \kappa_1 & -2 & 0 & 0 & \cdots & 0 \\ \frac{\kappa_3}{2!} & \kappa_2 & \kappa_1 & -3 & 0 & \cdots & 0 \\ \frac{\kappa_4}{3!} & \frac{\kappa_3}{2!} & \kappa_2 & \kappa_1 & -4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \frac{\kappa_n}{(n-1)!} & \frac{\kappa_{n-1}}{(n-2)!} & \frac{\kappa_{n-2}}{(n-3)!} & & \cdots & & \kappa_1 \end{vmatrix}.$$

These formulas follow from Cramer's rule applied to the linear system obtained from the recursion formula

$$\sum_{k=1}^{n} \binom{n-1}{k-1} m_{n-k}\kappa_k = m_n, \qquad n \geq 1;$$

see [RS00].

(2) Similarly, comparison of coefficients in the identity $M(z)(1 - B(z)) = 1$ leads to the recursion

$$\sum_{k=1}^{n} m_{n-k} b_k = m_n, \qquad n \geq 1$$

and thus [DM00]

$$b_n = (-1)^{n-1} \begin{vmatrix} m_1 & 1 & 0 & \cdots & 0 \\ m_2 & m_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ m_{n-1} & m_{n-2} & \cdots & m_1 & 1 \\ m_n & m_{n-1} & \cdots & m_2 & m_1 \end{vmatrix}, \qquad m_n = \begin{vmatrix} b_1 & -1 & 0 & \cdots & 0 \\ b_2 & b_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ b_{n-1} & b_{n-2} & \cdots & b_1 & -1 \\ b_n & b_{n-1} & \cdots & b_2 & b_1 \end{vmatrix}.$$

8. Monotone-to-classical case:

Although we have not yet found a satisfactory description for the general coefficients appearing in the monotone-to-classical cumulant formula

$$K_n = \sum_{\pi \in \mathcal{P}(n)} \beta(\pi) H_\pi, \tag{8.1}$$

we report on some partial results including Theorem 1.7.

We first provide some general considerations on the recursive nature of this problem and then apply the approach of [JV13] to obtain some special cases.

We first observe that the equation

$$\sum_{\pi \in \mathcal{P}(n)} K_\pi(X_1, \ldots, X_n) = \varphi(X_1 \cdots X_n) = \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{\tau(\pi)!} H_\pi(X_1, \ldots, X_n) \tag{8.2}$$

can be transofmed into the form

$$K_n(X_1, \ldots, X_n) = -\sum_{\sigma \in \mathcal{P}(n), \sigma \neq \hat{1}_n} K_\sigma(X_1, \ldots, X_n) + \sum_{\pi \in \mathrm{NC}(n)} \frac{1}{\tau(\pi)!} H_\pi(X_1, \ldots, X_n). \tag{8.3}$$

which allows for a recursive algorithm to obtain the coefficients $\beta(\pi)$. More precisely, once β is known for all partitions of size $k < n$, the coefficient $\beta(\pi)$ of a partition $\pi \in \mathcal{P}(n)$ is obtained as follows: We take any partition $\sigma \in [\pi, \hat{1}_n)$ and express K_σ in terms of the monotone cumulants. If $\sigma = \{W_1, W_2, \ldots, W_s\}$, the coefficient of H_π in such an expression will be exactly $\beta(\pi|_{W_1}) \cdots \beta(\pi|_{W_s})$. We must do this for every $\sigma \geq \pi$ and if $\pi \in \mathrm{NC}(n)$ we must in addition consider the coefficient $(\tau(\pi)!)^{-1}$ on the right hand side of eq. (8.3) as well. Hence relation (8.3) above can be recast into the following recursion

$$\beta(\pi) = \begin{cases} \dfrac{1}{\tau(\pi)!} - \displaystyle\sum_{\substack{\sigma\in\mathcal{P}(n)\\ \sigma\in[\pi,\hat{1}_n)}} \prod_{W\in\sigma} \beta(\pi|_W) & \text{if } \pi\in \mathrm{NC}(n), \\ - \displaystyle\sum_{\substack{\sigma\in\mathcal{P}(n)\\ \sigma\in[\pi,\hat{1}_n)}} \prod_{W\in\sigma} \beta(\pi|_W) & \text{if } \pi\notin \mathrm{NC}(n). \end{cases} \tag{8.4}$$

We show first that $\beta(\pi) = 0$ for all $\pi \notin \mathcal{P}_{\mathrm{irr}}$ by induction on $|\pi|$. For $|\pi| = 1$ the assertion is trivial. Suppose that $\beta(\pi) = 0$ for all $\pi \notin \mathcal{P}_{\mathrm{irr}}$ with $|\pi| < k$.

Now let $|\pi| = k$ with $\pi \notin \mathcal{P}_{\mathrm{irr}}$. Let $\pi_1, \dots, \pi_s$, $s \geq 2$, be the irreducible components of π. By formula (8.4), we need to look at partitions $\sigma \in [\pi, 1_n)$. If a block V of σ contains blocks of π from different irreducible components, then $\pi|_V$ is reducible and hence $\beta(\pi|_V) = 0$ by induction hypothesis.

Therefore, a contribution to $\beta(\pi)$ can only come from partitions of the form $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_s$, with $\sigma_i \geq \pi_i$, and hence

$$\sum_{\substack{\sigma\in\mathcal{P}(n)\\ \sigma\in[\pi,\hat{1}_n)}} \prod_{W\in\sigma} \beta(\pi|_W) = \sum_{\substack{\sigma_1\cup\cdots\cup\sigma_s\in\mathcal{P}(n)\\ \sigma_i\geq\pi_i}} \prod_{W\in\sigma} \beta(\pi|_W) \tag{8.5}$$

$$= \prod_{i=1}^{s} \sum_{\sigma_i\geq\pi_i} \prod_{W\in\sigma_i} \beta(\pi_i|_W). \tag{8.6}$$

We now apply recursion (8.4) separately to each sum occuring in (8.5) and obtain

$$\sum_{\sigma_i\geq\pi_i} \prod_{W\in\sigma_i} \beta(\pi_i|_W) = \beta(\pi_i) + \sum_{\sigma_i\in[\pi_i,\hat{1})} \prod_{W\in\sigma_i} \beta(\pi_i|_W) = \begin{cases} \dfrac{1}{\tau(\pi_i)!} & \text{if } \pi_i\in\mathrm{NC}, \\ 0 & \text{if } \pi_i\notin\mathrm{NC}. \end{cases}$$

Now we observe that $\pi \in \mathrm{NC}$ if and only if each $\pi_i \in \mathrm{NC}$ and that $\tau(\pi)! = \tau(\pi_1)!\cdots\tau(\pi_s)!$ for $\pi \in \mathrm{NC}$, to conclude that $\beta(\pi) = 0$.

Remark 8.1. Note that, in order to show that β is supported on $\mathcal{P}_{\mathrm{irr}}$, we only used that, for any pair of cumulants $(A_n)_{n\geq1}$, $(C_n)_{n\geq1}$, we have:

$$\sum_{\pi\in\mathcal{P}(n)} \omega_1(\pi) A_\pi(\mathbf{X}) = \varphi(\mathbf{X}) = \sum_{\pi\in\mathcal{P}(n)} \omega_2(\pi) C_\pi(\mathbf{X}),$$

where, for $i = 1, 2$, the weights $\omega_i(\pi) = \omega_i(\pi_1)\cdots\omega(\pi_s)$ factorize according to the irreducible components $\pi_1, \dots, \pi_s$ of π. Following the proof that $\beta(\pi) = 0$ for $\pi \notin \mathcal{P}_{\mathrm{irr}}$, we get for some constants $(\alpha(\pi))_{\pi\in\mathcal{P}_{\mathrm{irr}}} \subset \mathbb{R}$ that

$$A_n(\mathbf{X}) = \sum_{\pi\in\mathcal{P}_{\mathrm{irr}}(n)} \alpha(\pi) C_\pi(\mathbf{X}).$$

This shows that all 12 cumulant formulas are supported on $\mathcal{P}_{\mathrm{irr}}$. Moreover, the fact that monotone, Boolean and free cumulants assign a weight $\omega(\pi) = 0$ to any crossing partition implies that the corresponding cumulant formulas will be actually supported on $\mathcal{P}_{\mathrm{irr}} \cap \mathrm{NC} = \mathrm{NC}_{\mathrm{irr}}$.

The classical and free cumulants both have weights that are invariant under cyclic rotations. This means that the corresponding $\alpha(\pi)$ will also be rotationally invariant. Hence $\alpha(\pi)$ can only be nonzero if all cyclic rotations of π remain in $\mathcal{P}_{\mathrm{irr}}(n)$. This means that $\pi \in \mathcal{P}_{\mathrm{conn}}$.

Because of the nature of the recursion (8.4), the dependence of the coefficient $\beta(\pi)$ is uniquely determined by crossing/nesting structure of the blocks of partitions contained in the interval $[\pi, \hat{1}_n]$. Hence we suggest the following refinement of the anti-interval graph which actually distinguishes between crossings and nestings (see fig. 3):

Definition 8.2. The anti-interval digraph $\overrightarrow{G}(\pi)$ is obtained from the interval graph by replacing every (non-directed) edge (V_i, V_j) of $\tilde{G}(\pi)$ by:

(1) a directed edge (V_i, V_j), if V_j nests inside V_i,
(2) a directed edge (V_j, V_i), if V_i nests inside V_j,
(3) a non-directed edge (V_i, V_j), otherwise (equivalently, if V_i and V_j cross).

It is not hard to see (by induction on $|\pi|$) that the digraph $\overrightarrow{G}(\pi) = (V, E)$ determines $\beta(\pi)$: In the recursion (8.4), the contribution of each $1_n > \sigma = \{W_1, \dots, W_s\} \geq \pi$, one should look at the subgraphs $\overrightarrow{G}(\pi)|_{W_i}$ of $\overrightarrow{G}(\pi)$ indicated by the blocks of σ and then just multiply all $\beta(\pi|_{W_i})$, which are known already since $1_n > \sigma$. So we may write $\beta(\pi) = \beta(\overrightarrow{G}(\pi))$.

Let us conclude with the proof of Theorem 1.7. First we will use the approach of [JV13] to obtain a rather explicit expression for $\beta(\pi)$ from which everything can be deduced.

Lemma 8.3.

$$\beta(\pi) = \sum_{\sigma \trianglerighteq \pi} \frac{\mu_{\mathcal{P}}(\sigma, \hat{1}_n)}{\tau(\pi|\sigma)!} \tag{8.7}$$

where $\tau(\pi|\sigma)! = \prod_{W\in\sigma} \tau(\pi|_W)!$ and the partial order relation $\pi \trianglelefteq \sigma$ on $\mathcal{P}$ refines the usual order $\pi \leq \sigma$ by the additional requirement that $\pi|_W$ is noncrossing for every block $W \in \sigma$.

Proof. We follow the proof of [JV13] and write

$$\begin{aligned}
K_n &= \sum_{\sigma\in\mathcal{P}(n)} \varphi_\sigma \, \mu_{\mathcal{P}}(\sigma, \hat{1}_n) \\
&= \sum_{\sigma\in\mathcal{P}(n)} \sum_{\pi\trianglelefteq\sigma} \frac{1}{\tau(\pi|\sigma)!} H_\pi \, \mu_{\mathcal{P}}(\sigma, \hat{1}_n) \\
&= \sum_{\pi\in\mathcal{P}(n)} H_\pi \sum_{\sigma\trianglerighteq\pi} \frac{\mu_{\mathcal{P}}(\sigma, \hat{1}_n)}{\tau(\pi|\sigma)!}.
\end{aligned}$$

□

Proof of Theorem 1.7. The first part follows from Weisner's lemma [BBR86, Prop. 6.3]. Let P be a lattice and $a, b, c \in P$, then

$$\sum_{x\wedge a=c} \mu(x, b) = \begin{cases} 0 & \text{if } a \ngeq b, \\ \mu(c, b) & \text{if } a \geq b. \end{cases}$$

Consider the function

$$f_\pi(\sigma) = \begin{cases} \frac{1}{\tau(\pi|\sigma)!} & \text{if } \pi \trianglelefteq \sigma, \\ 0 & \text{if } \pi \ntrianglelefteq \sigma. \end{cases}$$

Assume $\pi \in \mathcal{P}(n)$ is not irreducible, and let $\rho = \hat{\pi} \neq \hat{1}_n$ be its interval closure. Then it is easy to see that $f_\pi(\sigma) = f_\pi(\sigma \wedge \rho)$. Indeed, if the restriction $\pi|_b$ has a crossing for some $b \in \sigma$, then it occurs in the restriction $\pi_j|_b$ of some irreducible factor π_j of π. If $\pi|\sigma$ has no crossings, then all nesting trees are contained inside the irreducible factors and therefore occur inside the blocks of $\sigma \wedge \rho$.

Using this fact we can write

$$\begin{aligned}\beta(\pi) &= \sum_{\sigma \trianglerighteq \pi} \frac{\mu_{\mathcal{P}}(\sigma, \hat{1}_n)}{\tau(\pi|\sigma)!} \\ &= \sum_{\sigma \geq \pi} f_\pi(\sigma)\mu_{\mathcal{P}}(\sigma, \hat{1}_n) \\ &= \sum_{\sigma \geq \pi} f_\pi(\sigma \wedge \rho)\mu_{\mathcal{P}}(\sigma, \hat{1}_n) \\ &= \sum_{\tau \geq \pi} f_\pi(\tau) \sum_{\sigma \wedge \rho = \tau} \mu_{\mathcal{P}}(\sigma, \hat{1}_n) \\ &= 0.\end{aligned}$$

This proves the first statement.

The second part holds true because the involved trees $\tau(\pi|\sigma)$ are trivial and hence formula (8.7) becomes

$$\beta(\pi) = \sum_{\sigma \trianglerighteq \pi} \mu_{\mathcal{P}}(\sigma, \hat{1}_n) = (-1)^{|\pi|-1} T_{G(\pi)}(1, 0)$$

as in the proof of [JV13, Theorem 7.1]. Note that the assumptions imply that π is connected.

Finally, let $\pi \in \mathrm{NC}_{\mathrm{irr}}$ be an irreducible noncrossing partition of depth 2. This means that there is one outer block and $m := |\pi| - 1$ inner blocks and the nesting tree $\tau(\pi)$ consists of the root and m leaves.

We classify the partitions of $t = \tau(\pi)$ according to the number of elements in the block containing the root. There are $\binom{n}{k}$ different subsets b of t containing k vertices in addition to the root and for every set like this $(t|_b)! = k + 1$. Then the remaining vertices of t can be partitioned without affecting the factorial and as k ranges between 0 and n we obtain

$$\beta(t) = \sum_{k=0}^{m} \binom{m}{k} \sum_{\rho \in \mathcal{P}(m-k)} \frac{-|\rho|}{k+1} \mu_{\mathcal{P}}(\rho, \hat{1}_{m-k}) = -\sum_{k=0}^{m} \binom{m}{k} \frac{\alpha_{m-k}}{k+1}$$

where

$$\alpha_n = \sum_{\rho \in \mathcal{P}(n)} |\rho| \, \mu_{\mathcal{P}}(\rho, \hat{1}_n) = \frac{d}{dx} \sum_{\rho \in \mathcal{P}(n)} x^{|\rho|} \mu_{\mathcal{P}}(\rho, \hat{1}_n) \Big|_{x=1}.$$

The derivand in the last expression can be interpreted as the classical cumulant of order n of a standard Bernoulli law of weight x and the exponential generating function therefore is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\alpha_n}{n!} z^n &= \frac{d}{dx} \log\left(1 + \sum_{n=1}^{\infty} \frac{x}{n!} z^n\right)\Big|_{x=1} \\ &= \frac{d}{dx} \log(1 + x(e^z - 1))\Big|_{x=1} \\ &= \frac{e^z - 1}{1 + x(e^z - 1)}\Big|_{x=1} \\ &= 1 - e^{-z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} z^n\end{aligned}$$

and hence $\alpha_n = (-1)^{n-1}$. Thus, denoting by $\beta_m := \beta(t)$,

$$\beta_m = -\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{m-k-1}}{k+1}$$

and so

$$\sum_{m=0}^{\infty} \frac{\beta_m}{m!} z^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{(k+1)!} \frac{(-1)^{m-k}}{(m-k)!} z^m$$
$$= \frac{1}{z}(e^z - 1)e^{-z}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} z^m$$

and consequently $\beta(t) = \beta_m = \frac{(-1)^m}{m+1} = \frac{(-1)^{|\pi|-1}}{|\pi|}$. □

Remark 8.4. For the family $\pi_n = \{\{1, 2n\}, \{2, 2n-1\}, \dots, \{n, n+1\}\}$, $n \geq 1$, the sequence $(n!\beta(\pi_n))_n = (1, -1, 4, -33, 456, -9460, 274800, \dots)$ are the coefficients of the log-Bessel function ([Slo14, A101981]).

Indeed, for $\pi = \pi_n$ the nesting tree is just a line segment and

$$\beta(\pi) = \sum_{\sigma \geq \pi} \frac{\mu_{\mathcal{P}}(\sigma, \hat{1}_n)}{[\pi, \sigma]!}.$$

Here we use the fact that the interval $[\pi, \sigma]$ is isomorphic to a direct product $\prod \mathcal{P}(k)^{m_k}$ and $\tau(\pi|\sigma)! = [\pi, \sigma]! = \prod (k!)^{m_k}$. In other words $\beta = f * \mu_{\mathcal{P}}$, the convolution of the multiplicative functions on $\mathcal{P}$ associated to the sequences $f_n = \frac{1}{n!}$ and $\mu_n = (-1)^{n-1}(n-1)!$.

Recall [DRS72, Sta99] that the reduced incidence algebra of $\mathcal{P}$ incarnates the Faa di Bruno formula for exponential power series, i.e., if $A(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n$ and $B(z) = \sum_{n=1}^{\infty} \frac{b_n}{n!} z^n$ are the corresponding exponential generating functions and $c = a * b$, then the generating function of the sequence (c_n) is

$$\sum_{n=1}^{\infty} \frac{c_n}{n!} z^n = B(A(z)).$$

In our case $\beta = f * \mu_{\mathcal{P}}$, where

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{(n!)^2} = J_0(2i\sqrt{z}) - 1$$

is the Bessel function of first kind and

$$M(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \log(1+z)$$

is the logarithm, thus $\beta(\pi_n)$ are the coefficients of the log-Bessel function

$$\log J_0(2i\sqrt{z}).$$

The sequence $b_n = n!\beta(\pi_n)$ satisfies a recursion found by Carlitz [Car63], namely

$$b_{n+1} = \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} b_k b_{n+1-k}$$

and therefore $\beta_n = \beta(\pi_n)$ satisfies

$$\beta_{n+1} = \frac{b_{n+1}}{(n+1)!} = \sum_{k=1}^{n} \frac{1}{n+1} \frac{n!}{(n-k)!(k-1)!} \frac{b_k}{k!} \frac{b_{n+1-k}}{(n+1-k)!} = \sum_{k=1}^{n} \frac{n}{n+1} \binom{n-1}{k-1} \beta_k \beta_{n+1-k}.$$

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