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The Combinatorial Relationship Between Trees, Cacti and Certain Connection Coefficients for the Symmetric Group

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A combinatorial bijection is given between pairs of permutations in S_n the product of which is a given n-cycle and two-coloured plane edge-rooted trees on n edges, when the numbers of cycles in the disjoint cycle representations of the permutations sum to $n+1$. Thus the corresponding connection coefficient for the symmetric group is determined by enumerating these trees with respect to appropriate characteristics. This is extended to the case of m-tuples of permutations in S_n the product of which is a given n-cycle, in which the combinatorial objects replacing trees are cacti of m-gons.

1. INTRODUCTION

Let $\lambda=(\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ are non-negative integers and $\lambda_1+\lambda_2+\cdots=n$. Then λ is a *partition* of n, denoted $\lambda \vdash n$. If m of the λ_i are positive we also write $\lambda=(\lambda_1, \ldots, \lambda_m)$, and say that λ has *m parts*, denoted by $l(\lambda)=m$. If k_j of the parts of λ are equal to j for $j \geqslant 1$, we can write $\lambda=1^{k_1}2^{k_2}\cdots$.

Associated with every permutation σ in S_n, the symmetric group on $\{1, \ldots, n\}$, is the partition of n the parts of which specify the lengths of the cycles in the disjoint cycle representation of σ. This partition is called the *cycle distribution* of σ. For $\alpha \vdash n$, the set of all permutations in S_n with cycle distribution α is a conjugacy class; let h^α be the size of this conjugacy class and let K_α be the formal sum of its elements. The set $\{K_\alpha \mid \alpha \vdash n\}$ is a basis for the centre of the group algebra $\mathbb{C}S_n$. Thus we can linearize the product of any elements in this set; so if $\alpha_i \vdash n$ for $i=1, \ldots, m$ we can write

$$K_{\alpha_1}\cdots K_{\alpha_m}=\sum_{\gamma \vdash n} c^{\gamma}_{\alpha_1,\ldots,\alpha_m}K_\gamma,$$

and the numbers $c^{\gamma}_{\alpha_1,\ldots,\alpha_m}$ are called *connection coefficients* for the symmetric group.

The need to determine connection coefficients for the symmetric group often arises in combinatorial problems. In certain cases involving involutions (cycle distribution $1^{n-2}2$) and n-cycles (cycle distribution n) explicit expressions are known for these coefficients (Bertram and Wei [1], Boccara [2], Jackson [6–8], Stanley [12] and Walkup [14]). Jackson and Stanley used the character theory for the symmetric group in their work.

Recently, Goupil and Bédard [4] have given an explicit expression for $c^{\beta}_{(n),\alpha}$ subject only to the restriction that $l(\alpha)+l(\beta)=n+1$. The argument used is inductive. In Section 2 of this paper we give a constructive proof to determine the equivalent connection coefficient $c^{(n)}_{\alpha,\beta}$. The equivalence is due to the fact that these coefficients satisfy some obvious symmetries, such as $c^{\gamma}_{\alpha\beta}=c^{\gamma}_{\beta\alpha}$, and

$$h^\gamma c^\gamma_{\alpha\beta}=h^\beta c^\beta_{\gamma\alpha}. \tag{1}$$

In our constructive approach, we first demonstrate (Theorem 2.1) that the pairs σ, ρ of permutations in S_n the product of which is a fixed permutation with cycle distribution (n) can be realized uniquely from two-coloured plane edge-rooted trees on n edges, precisely when the sum of the numbers of cycles in σ and ρ is $n+1$. Then we

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determine (Theorem 2.2) the connection coefficient $c^{(n)}_{\alpha,\beta}$ when $l(\alpha)+l(\beta)=n+1$ by enumerating the two-coloured trees.

In Section 3 this constructive method is extended to yield (Theorem 3.2) an explicit expression for $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$ where $l(\alpha_1)+\cdots+l(\alpha_m)=n+1$, for any $m\geq 2$. The combinatorial objects in the general case are cacti of m-gons (see, e.g., Harary and Palmer [5, p. 71]), and the case $m=2$ is shown to be isomorphic to the coloured trees in Section 2. However, they are treated separately because of the elegance of the case $m=2$, and so certain aspects of the general case can be simplified or omitted, by referring to the arguments for two-coloured trees.

In Section 4 related connection coefficients are discussed.

Throughout the paper we write $\mathbf{x}^{\mathbf{i}}$ for $\prod_j x_j^{i_j}$, where $\mathbf{x}$ and $\mathbf{i}$ are vectors with entries x_j and i_j, respectively, and $[M]F$ for the coefficient of the monomial M in the formal power series F.

2. Two-coloured Trees and Pairs of Permutations

In this section we consider two-coloured plane edge-rooted trees, using the colours white and black. Thus each edge joins a white vertex to a black vertex, and one edge is distinguished as the root edge. The unordered list of degrees of the white (black) vertices forms a partition called the white (respectively, black) *vertex distribution* of the tree.

The following result gives a combinatorial bijection between these trees and ordered pairs of permutations in S_n the product of which is a specified n-cycle, where the numbers of cycles in the permutations sum to $n+1$.

THEOREM 2.1. *Let α, β be partitions of n such that $l(\alpha)+l(\beta)=n+1$. Then there is a bijection between two-coloured plane edge-rooted trees on n edges with white vertex distribution α and black vertex distribution β, and pairs (σ,ρ), of permutations in S_n, with cycle distributions α, β respectively, such that $\sigma\rho=(1,2,\ldots,n)$.*

PROOF. Let t be a two-coloured plane edge-rooted tree on n edges, with white vertex distribution α and black vertex distribution β. Each edge joins a white and a black vertex, since the vertices of t are two-coloured black and white, so α and β are each partitions of n. In addition, there are $l(\alpha)$ white vertices and $l(\beta)$ black vertices, so $l(\alpha)+l(\beta)=n+1$.

Now t is the boundary of an unbounded region of the plane. Describe the boundary by moving along the edges, keeping the region on the left, beginning along the root edge from its incident white vertex to black vertex. Each edge is encountered twice, once from black vertex to white vertex and once from white vertex to black vertex. Assign the labels $1,2,\ldots,n$ to the edges as they are encountered in the white to black direction. As an example of this procedure for $n=14$, in Figure 1 is given a two-coloured rooted tree, with the root edge represented by a doubled line and with edges labelled.

The cyclic sequences of labels of the edges incident with the white vertices, in clockwise order, give the disjoint cycle decomposition of a permutation σ in S_n, with cycle distribution α. Similarly, there is a corresponding permutation ρ in S_n, with cycle distribution β, specified by the black vertices. Now the procedure to determine edge labels guarantees that i is mapped to $1+(i \bmod n)$ in $\sigma\rho$ for $i=1,\ldots,n$, so $\sigma\rho=(1,2,\ldots,n)$. For example, the permutations specified by the tree in Figure 1, with fixed points suppressed, are $\sigma=(1,5,6,11,14)$ $(2,3,4)$ $(8,9)$ and $\rho=(1,4)$ $(6,7,9,10)$ $(11,12,13)$, and actual multiplication confirms, in this case, that $\sigma\rho=(1,2,\ldots,14)$.

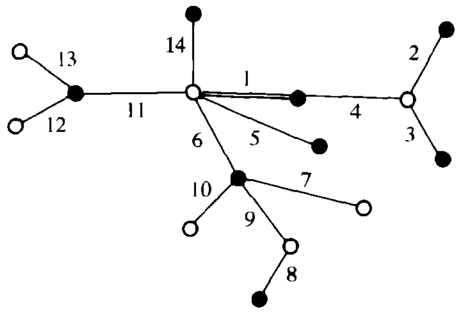

FIGURE 1. A two-coloured rooted tree and its edge labels.

To reverse this construction, consider a pair of permutations σ and ρ in S_n with cycle distributions α and β, respectively, (this means that α and β are partitions of n), such that $\sigma\rho = (1, 2, \ldots, n)$ and $l(\alpha) + l(\beta) = n + 1$. Now consider the graph the vertices of which correspond to the cycles of σ and ρ, with incident edges specified by elements of the cycle. This is a two-coloured graph since each edge is incident with a vertex (call it white) corresponding to a cycle of σ and a vertex (call it black) corresponding to a cycle of ρ.

Thus the graph has white vertex distribution α and black vertex distribution β, giving it $l(\alpha)$ white vertices and $l(\beta)$ black vertices for a total of $l(\alpha) + l(\beta) = n + 1$ vertices. Moreover, the graph is connected since $\sigma\rho = (1, 2, \ldots, n)$ means that the group generated by σ and ρ is transitive. Together, these imply that the graph is a tree.

It is a straightforward matter to embed the tree in the plane so that the cyclic sequence of edges incident clockwise at each vertex specifies the corresponding cycle. Moreover, in this embedding, a traversal of the edges keeping the unbounded region on the left and beginning at edge 1 will cause us to encounter edges $1, 2, \ldots, n$ in order of their traversal from white to black, since $\sigma\rho = (1, 2, \ldots, n)$. Now root the tree at edge 1 and remove the labels on the edges, to yield the required two-coloured plane edge-rooted tree on n edges with white vertex distribution α and black vertex distribution β. $\square$

This bijection allows us to calculate the connection coefficient $c_{\alpha,\beta}^{(n)}$ when $l(\alpha) + l(\beta) = n + 1$ by enumerating two-coloured edge-rooted trees on n vertices with white vertex distribution α and black vertex distribution β. This enumeration is carried out in the following result.

THEOREM 2.2. *Let* $\alpha = 1^{i_1}2^{i_2}\cdots$ *and* $\beta = 1^{k_1}2^{k_2}\cdots$ *be partitions of* $n \geqslant 1$, *with* $s = l(\alpha) = i_1 + i_2 + \cdots$, $r = l(\beta) = k_1 + k_2 + \cdots$. *If* $s + r = n + 1$ *then*

$$c_{\alpha,\beta}^{(n)} = n\frac{(s-1)!(r-1)!}{i_1!\,i_2!\cdots k_1!\,k_2!\cdots}.$$

PROOF. Let F be the generating function for two-coloured edge-rooted plane trees with white and black vertices marked by indeterminates w and b, respectively, and white and black vertices of degree i marked by w_i and b_i, respectively, for $i \geqslant 1$. Let $\mathbf{w} = (w_1, w_2, \ldots)$, $\mathbf{b} = (b_1, b_2, \ldots)$, $\mathbf{i} = (i_1, i_2, \ldots)$ and $\mathbf{k} = (k_1, k_2, \ldots)$. Then, from Theorem 2.1,

$$c_{\alpha,\beta}^{(n)} = [w^s b^r \mathbf{w}^{\mathbf{i}} \mathbf{b}^{\mathbf{k}}]F.$$

In order to determine F, we consider two associated generating functions. Let W and B be the generating functions for the sets $\mathcal{W}$ and $\mathcal{B}$ of two-coloured plane planted trees, the planted vertex of which has colour white and black, respectively. The planted vertex is not marked, but all other vertices are marked in W and B exactly as they are in F above. By considering the possible degrees of the vertex adjacent to the planted vertex, we deduce that W, B satisfy the simultaneous functional equations

$$B = w(w_1 + w_2 W + w_3 W^2 + \cdots),$$
$$W = b(b_1 + b_2 B + b_3 B^2 + \cdots).$$

Furthermore, we can construct uniquely every two-coloured plane edge-rooted tree by identifying the planted vertices in a tree in $\mathcal{W}$ and a tree in $\mathcal{B}$, suppressing this bivalent vertex and rooting the resulting two-coloured plane tree on this edge. Thus we deduce that $F = WB$. But WB can be determined by Lagrange's Implicit Function Theorem in the two variables w and b (see, e.g., [3, p. 21]). Let $\phi_1 = w_1 + w_2\lambda_2 + w_3\lambda_2^2 + \cdots$ and $\phi_2 = b_1 + b_2\lambda_1 + b_3\lambda_1^2 + \cdots$. Then

$$\begin{aligned}
c_{\alpha,\beta}^{(n)} &= [w^s b^r \mathbf{w}^\alpha \mathbf{b}^\beta] BW \\
&= [\lambda_1^s \lambda_2^r \mathbf{w}^\alpha \mathbf{b}^\beta] \lambda_1 \lambda_2 \phi_1^s \phi_2^r \begin{vmatrix} 1 & -\dfrac{\lambda_2}{\phi_1}\dfrac{\partial \phi_1}{\partial \lambda_2} \\ -\dfrac{\lambda_1}{\phi_2}\dfrac{\partial \phi_2}{\partial \lambda_1} & 1 \end{vmatrix} \\
&= [\lambda_1^{s-1}\lambda_2^{r-1}\mathbf{w}^\alpha \mathbf{b}^\beta] \left\{ \phi_1^s \phi_2^r - \left(\frac{\lambda_2}{s}\frac{\partial \phi_1^s}{\partial \lambda_2}\right)\left(\frac{\lambda_1}{r}\frac{\partial \phi_2^r}{\partial \lambda_1}\right)\right\} \\
&= \left\{1 - \left(\frac{r-1}{s}\right)\left(\frac{s-1}{r}\right)\right\}[\lambda_1^{s-1}\lambda_2^{r-1}\mathbf{w}^\alpha \mathbf{b}^\beta]\phi_1^s\phi_2^r, \qquad \text{since } [\lambda^m]\lambda\frac{\partial}{\partial\lambda}f(\lambda) = m[\lambda^m]f(\lambda), \\
&= \frac{n}{sr}[\lambda_1^{s-1}\lambda_2^{r-1}]\frac{s!}{i_1!\, i_2! \cdots}\lambda_2^{i_2+2i_3+\cdots}\frac{r!}{k_1!\, k_2!\cdots}\lambda_1^{k_2+2k_3+\cdots}, \qquad \text{since } r+s-1 = n.
\end{aligned}$$

But

$$i_2 + 2i_3 + \cdots = (i_1 + 2i_2 + 3i_3 + \cdots) - (i_1 + i_2 + \cdots) = n - s = r - 1$$

and

$$k_2 + 2k_3 + \cdots = (k_1 + 2k_2 + 3k_3 + \cdots) - (k_1 + k_2 + \cdots) = n - r = s - 1,$$

giving the result. □

Tutte [13] has previously determined the generating function W (and, equivalently, B) used in the above proof.

From (1), with $\gamma = (n)$, we have

$$c_{(n),\alpha}^{\beta} = \frac{h^{(n)}}{h^\beta} c_{\alpha,\beta}^{(n)}$$

so, for $l(\alpha) + l(\beta) = n + 1$, Theorem 2.2 is equivalent to

$$\begin{aligned}
c_{(n),\alpha}^{\beta} &= \frac{n!/n}{n!/k_1!\, k_2! \cdots 1^{k_1}2^{k_2}\cdots}\, n \frac{(s-1)!}{i_1!\, i_2!\cdots}\frac{(r-1)!}{k_1!\, k_2!\cdots} \\
&= \frac{(s-1)!(r-1)!\, 1^{k_1}2^{k_2}\cdots}{i_1!\, i_2!\cdots},
\end{aligned}$$

which was previously given by Goupil and Bédard ([4], Theorem 3), using an inductive argument.

3. Cacti of m-gons and m-tuples of Permutations

A cactus of m-gons, or *m-cactus,* is a connected graph in which every edge lies on exactly one cycle, which has length m, for fixed $m \geq 2$. A plane m-cactus is an embedding of an m-cactus in the plane so that every edge is incident with the unbounded region. Thus all bounded connected regions in the plane defined by a plane m-cactus are incident with m edges and m vertices, and are m-gons if the edges are represented by straight lines.

Now for a plane edge-rooted m-cactus we define a canonical m-colouring of the vertices with colours $1, \ldots, m$ as follows. Traverse the boundary of the unbounded region, keeping the unbounded region to the left. Assign colours $m, m-1, \ldots, 1$ in circular succession to the vertices encountered in this traversal, beginning by assigning colours 1 and m to the vertices incident with the root edge. Note that a vertex of degree $2k$ in the graph will thus be assigned a colour k times, but fortunately by construction it will be the same colour every time, so the colouring is well-defined.

For a coloured m-cactus the colour i vertex distribution is defined to be the partition giving the unordered list of numbers of m-gons incident with the vertices of colour i.

The following result gives a combinatorial bijection between m-cacti and m-tuples of permutations in S_n the product of which is a specified n-cycle, where the numbers of cycles in the permutations sum to $n+1$.

Theorem 3.1. *Let $\alpha_1, \ldots, \alpha_m$ be partitions of n such that $l(\alpha_1) + \cdots + l(\alpha_m) = n+1$. Then there is a bijection between plane edge-rooted m-cacti on n m-gons with colour i vertex distribution α_i, $i = 1, \ldots, m$, and m-tuples $(\sigma_1, \ldots, \sigma_m)$ of permutations in S_n with cycle distributions $\alpha_1, \ldots, \alpha_m$ respectively, such that $\sigma_1 \cdots \sigma_m = (1, 2, \ldots, n)$.*

Proof. Consider a plane edge-rooted m-cactus on n m-gons. Suppose that when the m-cactus is canonically coloured, the colour i vertex distribution is α_i for $i = 1, \ldots, m$. We assign the labels $1, \ldots, n$ to the m-gons in the order in which they are encountered, when colouring the vertices, incident to an edge which is incident with vertices of colours 1 and m. As an example of this procedure for $n = 9$, in Figure 2 is given a plane edge-rooted 3-cactus with vertices coloured and triangles labelled. The root edge is represented by a doubled line, the vertices of colour 1 by empty circles, colour 2 by stars, colour 3 by filled circles, and the labels assigned to the triangles appear inside the triangles.

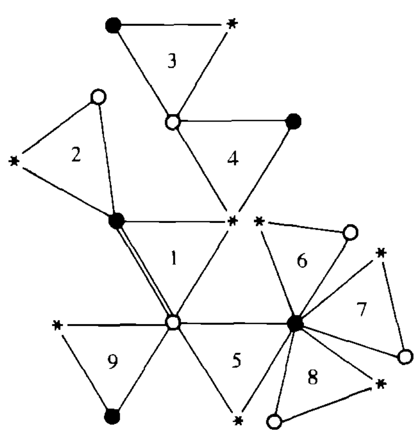

Figure 2. A plane edge-rooted 3-cactus with coloured vertices and labelled triangles.

The cyclic sequences of m-gons incident with the vertices of colour i, in clockwise order, give the disjoint cycle decomposition of permutation σ_i in S_n, with cycle distribution α_i, for $i = 1, \ldots, m$. But the procedure for labelling the m-gons ensures that i is mapped to $1 + (i \bmod n)$ in $\sigma_1 \cdots \sigma_m$ for $i = 1, \ldots, n$, so $\sigma_1 \cdots \sigma_m = (1, 2, \ldots, n)$. For example, the permutations specified by the 3-cactus in Figure 2, with fixed points suppressed, are $\sigma_1 = (1, 5, 9)(3, 4)$, $\sigma_2 = (1, 4)$ and $\sigma_3 = (1, 2)(5, 6, 7, 8)$, and actual multiplication confirms, in this case, that $\sigma_1\sigma_2\sigma_3 = (1, 2, \ldots, 9)$.

The proof that this construction is reversible is similar to that given for Theorem 2.1, and is left to the reader. □

Note that there is a straightforward bijection between two-coloured plane edge-rooted trees with white and black vertex distributions α_1 and α_2, respectively, and plane edge-rooted 2-cacti with colour i vertex distribution α_i, for $i = 1, 2$. To carry out this bijection, let colour 1 of a canonically coloured 2-cactus be white, and colour 2 be black, replace the digons containing the root edge by a single root edge and all other digons by single edges. The result is a two-coloured plane edge-rooted tree, and this is reversible.

Thus Theorem 3.1 reduces to Theorem 2.1 in the case $m = 2$.

The bijection in Theorem 3.1 allows us to calculate the connection coefficient $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$, where $l(\alpha_1) + \cdots + l(\alpha_m) = n + 1$ by enumerating plane edge-rooted m-cacti. This enumeration is carried out in the following result.

THEOREM 3.2. *Let $\alpha_i = 1^{k_{i1}}2^{k_{i2}} \cdots$ be a partition of $n \geq 1$ and $t_i = l(\alpha_i) = k_{i1} + k_{i2} + \cdots$, for $i = 1, \ldots, m$. If $t_1 + \cdots + t_m = n + 1$ then*

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = n^{m-1} \frac{\prod_{i=1}^m (t_i - 1)!}{\prod_{i=1}^m \prod_{j\geq 1} k_{ij}!}.$$

PROOF. Let Φ be the generating function for plane edge-rooted m-cacti with vertices of colour i marked by x_i, for $i = 1, \ldots, m$, and vertices of colour i incident with j m-gons marked by y_{ij} for $i = 1, \ldots, m$ and $j \geq 1$. Let $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{t} = (t_1, \ldots, t_m)$, $\mathbf{Y} = (y_{ij})_{m\times\infty}$ and $\mathbf{K} = (k_{ij})_{m\times\infty}$. Then, from Theorem 3.1, with $\mathbf{Y}^{\mathbf{K}} = \prod_{i=1}^m \prod_{j\geq 1} y_{ij}^{k_{ij}}$,

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = [\mathbf{x}^{\mathbf{t}}\mathbf{Y}^{\mathbf{K}}]\Phi.$$

In order to determine Φ, we define $\mathscr{C}_i$ to be the set containing a single vertex coloured i, together with plane edge-rooted m-cacti in which $i - 1$ has been added $(\bmod m)$ to the colours assigned in the canonical colouring, for $i = 1, \ldots, m$. Let C_i be the generating function for $\mathscr{C}_i$ in which the vertices are marked as in Φ, with the exception that the vertex of colour i incident with the root edge is marked by y_{ij+1}, instead of y_{ij}, when it is incident with j m-gons, for $i = 1, \ldots, m$. The single vertex element of $\mathscr{C}_i$ is marked by y_{i1} (as well as x_i).

Consider an arbitrary plane edge-rooted m-cactus. The m-gon containing the root edge contains one vertex of colour i for each $i = 1, \ldots, m$, so if the edges of this m-gon are removed, the resulting m-tuple of plane m-cacti is in $\mathscr{C}_1 \times \cdots \times \mathscr{C}_m$, where the rooting of these m-cacti makes this reversible. Moreover,

$$\Phi = C_1 \cdots C_m$$

since the root vertices of the elements of $\mathscr{C}_i$ are incident with an additional m-gon (the deleted one in the above decomposition) when counted in Φ.

It is a straightforward matter to determine a functional equation uniquely satisfied by the C_i. Consider an element of $\mathscr{C}_i$ the vertex of colour i of which incident with the root edge is incident with u m-gons. In addition to this vertex of colour i, each of these m-gons contains a single vertex of each other colour, so if the edges of these u m-gons are removed, the result is u $(m-1)$-tuples of elements of each $\mathscr{C}$ except $\mathscr{C}_i$, and the rooting makes this reversible. Thus

$$C_i = x_i \sum_{u\geq 0} y_{iu+1}\left(\frac{C_1\cdots C_m}{C_i}\right)^u, \qquad i=1,\ldots,m.$$

However, we can solve this system of functional equations for $\Phi = C_1\cdots C_m$ by Lagrange's Implicit Function Theorem in the m variables $x_1,\ldots,x_m$ (see, e.g., [3, p. 21]). Let

$$\phi_i = \sum_{u\geq 0} y_{iu+1}\left(\frac{\lambda_1\cdots\lambda_m}{\lambda_i}\right)^u, \qquad i=1,\ldots,m,$$

and $\phi=(\phi_1,\ldots,\phi_m)$, $\lambda=(\lambda_1,\ldots,\lambda_m)$, $\mathbf{1}=(1,\ldots,1)$. If $\|a_{ij}\|_{m\times m}$ is the determinant of the $m\times m$ matrix with (i,j)-entry a_{ij}, then

$$\begin{aligned} c^{(n)}_{\alpha_1,\ldots,\alpha_m} &= [\mathbf{x}^{\mathbf{t}}\mathbf{Y}^{\mathbf{K}}]C_1\cdots C_m \\ &= [\lambda^{\mathbf{t}}\mathbf{Y}^{\mathbf{K}}]\lambda^{\mathbf{1}}\phi^{\mathbf{t}}\left\|\delta_{ij} - \frac{\lambda_j}{\phi_i}\frac{\partial\phi_i}{\partial\lambda_j}\right\|_{m\times m} \\ &= [\lambda^{\mathbf{t}-\mathbf{1}}\mathbf{Y}^{\mathbf{K}}]\phi^{\mathbf{t}-\mathbf{1}}\left\|\phi_i\,\delta_{ij} - \lambda_j\frac{\partial\phi_i}{\partial\lambda_j}\right\|_{m\times m}. \end{aligned}$$

Now the determinant can be explicitly evaluated, by first noting that

$$\lambda_i\frac{\partial\phi_i}{\partial\lambda_i} = 0, \qquad i=1,\ldots,m,$$

and that

$$\lambda_j\frac{\partial\phi_i}{\partial\lambda_j} = \sum_{u\geq 0} u y_{iu+1}\left(\frac{\lambda_1\cdots\lambda_m}{\lambda_i}\right)^u, \qquad \text{for } j\neq i.$$

Thus, $\lambda_j(\partial\phi_i/\partial\lambda_j)$ is independent of j for $j\neq i$, and we call this common value ψ_i. But

$$\begin{aligned} \left\|\phi_i\,\delta_{ij} - \lambda_j\frac{\partial\phi_i}{\partial\lambda_j}\right\|_{m\times m} &= \|\phi_i\,\delta_{ij} - \psi_i(1-\delta_{ij})\|_{m\times m} \\ &= \|(\phi_i+\psi_i)\delta_{ij} - \psi_i\|_{m\times m} \\ &= \left\|\delta_{ij} - \frac{\psi_i}{\phi_i+\psi_i}\right\|_{m\times m}\prod_{i=1}^m(\phi_i+\psi_i) \\ &= \left\{1 - \sum_{u=1}^m\frac{\psi_u}{\phi_u+\psi_u}\right\}\prod_{i=1}^m(\phi_i+\psi_i) \end{aligned}$$

since $\det(I+A) = 1+\operatorname{trace}(A)$ when $\operatorname{rank}(A)=1$ (Sherman and Morrison [11]). Thus

$$\begin{aligned} c^{(n)}_{\alpha_1,\ldots,\alpha_m} &= [\lambda^{\mathbf{t}-\mathbf{1}}\mathbf{Y}^{\mathbf{K}}]\left\{\prod_{i=1}^m(\phi_i+\psi_i)\phi_i^{t_i-1} - \sum_{u=1}^m\psi_u\phi_u^{t_u-1}\prod_{i\neq u}(\phi_i+\psi_i)\phi_i^{t_i-1}\right\} \\ &= \frac{\prod_{i=1}^m(t_i-1)!}{\prod_{i=1}^m\prod_{j\geq 1}k_{ij}!}\left\{\prod_{i=1}^m\left(\sum_{j\geq 1}jk_{ij}\right) - \sum_{u=1}^m\left(\sum_{j\geq 1}(j-1)k_{uj}\right)\prod_{i\neq u}\left(\sum_{j\geq 1}jk_{ij}\right)\right\}, \end{aligned}$$

where $\sum_{j\geq 1}jk_{ij}=n$ and $\sum_{j\geq 1}k_{ij}=t_i$ for $i=1,\ldots,m$. The result follows, using the fact that $t_1+\cdots+t_m=(m-1)n+1$. □

4. Extensions

The connection coefficient determined in Section 2 is the case $l(\alpha)+l(\beta)=n+l(\gamma)$ and $l(\gamma)=1$ of $C^{\gamma}_{\alpha\beta}$. The case $l(\alpha)+l(\beta)=n+l(\gamma)$ in general can be handled by considering $l(\gamma)$-tuples of two-coloured plane edge-rooted trees, and yields a complicated summation over many partitions (the partitions distribute the vertices of given colour and degree amongst the trees). An equivalent result has been given in Goupil and Bédard [4, Theorem 4].

The restrictions satisfied by $l(\alpha)$, $l(\beta)$ and $l(\gamma)$ can be analysed most easily by considering the representation of a permutation as the product of a minimal number of transpositions. This minimal number, for the conjugacy class with cycle distribution α, was denoted by $\Lambda(\alpha)$ in the work of Goupil and Bédard. Indeed, they consider the graded lattice of conjugacy classes of S_n, with grading specified by Λ, and phrase all questions in terms of this lattice.

Consider the effect of multiplying a permutation σ by a transposition τ. If the elements of τ lie on the same cycle of σ, the cycle in σ will be split into two cycles in $\tau\sigma$. If the elements of τ lie on two different cycles of σ, those two cycles in σ will be joined into one cycle in $\tau\sigma$. Thus we find that $l(\gamma)\leqslant\Lambda(\alpha)+l(\beta)$. But $\Lambda(\alpha)=n-l(\alpha)$, and this gives

$$l(\alpha)+l(\gamma)\leqslant n+l(\beta).$$

Of course, the symmetry (1) of the connection coefficients also gives

$$l(\alpha)+l(\beta)\leqslant n+l(\gamma), \qquad l(\beta)+l(\gamma)\leqslant n+l(\alpha).$$

Moreover, the preceding analysis means that when one of these inequalities is strict, the resulting difference between the two sides is an even integer.

Thus the case $l(\alpha)+l(\beta)=n+l(\gamma)$ considered in this paper is extremal. The next case is

$$l(\alpha)+l(\beta)=n+l(\gamma)-2,$$

which can be handled, for $l(\gamma)\geqslant 2$, by considering an $(l(\gamma)-2)$-tuple of two-coloured plane edge-rooted trees, together with a connected two-coloured plane graph with a single cycle, of even length, with a root edge inside or on the cycle, and another root edge outside or on the cycle (these may coincide). If the edges of this latter unicursal component are labelled $1,\ldots,n$ by traversing the outer face beginning at the root edge outside or on the cycle as in Theorem 2.1, and $n+1,\ldots,n+k$ by traversing the inner face in a similar manner, then the product of the two permutations described in Theorem 2.1 is $(1,\ldots,n)(n+1,\ldots,n+k)$. Thus for this graph, the number of vertices minus the number of edges is $l(\gamma)-2$, so $l(\alpha)+l(\beta)-n=l(\gamma)-2$, as required. The reversibility of this construction follows as in Theorem 2.1.

It is a straightforward matter to express the generating function for this unicursal component in terms of the generating functions W and B considered in the proof of Theorem 2.2. We have been unable to determine a compact expression for the required coefficient from the Lagrange Theorem, though it does seem possible to deduce an efficient computational scheme. Similar comments can be made about m-tuples of permutations.

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