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The Euler characteristic of the moduli space of curves

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Let Γ_g^1, $g \geqq 1$, be the mapping class group consisting of all isotopy classes of base-point and orientation preserving homeomorphisms of a closed, oriented surface F of genus g. Let $\chi(\Gamma_g^1)$ be its Euler characteristic in the sense of Wall, that is $\chi(\Gamma_g^1) = [\Gamma_g^1 : \Gamma]^{-1}\,\chi(E/\Gamma)$, where Γ is any torsion free subgroup of finite index in Γ_g^1 and E is a contractible space on which Γ acts freely and properly discontinuously. An example of such a space is the Teichmüller space $\mathscr{T}_g^1$, and $\chi(\Gamma_g^1)$ can be interpreted as the orbifold Euler characteristic of $\mathscr{T}_g^1/\Gamma_g^1 = \mathscr{M}_g^1$, the moduli space of curves of genus g with base point.

The purpose of this paper is to prove the following formula for $\chi(\Gamma_g^1)$:

Main theorem. $\chi(\Gamma_g^1) = \zeta(1-2g)$.

Here $\zeta(s)$ is the Riemann zeta function; its value at $s = 1-2g$ is a rational number, given by the well-known formula $\zeta(1-2g) = -B_{2g}/2g$, where B_{2g} is the $2g^{\text{th}}$ Bernoulli number.

If Γ_g denotes the mapping class group of a surface without base point, then if $g>1$, Γ_g is related to Γ_g^1 by an exact sequence

$$1 \to \pi_1(F) \to \Gamma_g^1 \to \Gamma_g \to 1$$

(for $g=1$ we have $\Gamma_1 \cong \Gamma_1^1 \cong \mathrm{SL}_2(\mathbb{Z})$), so there is an equivalent formulation

$$\chi(\Gamma_g) = \frac{1}{2-2g}\,\zeta(1-2g) = \frac{B_{2g}}{4g(g-1)} \qquad (g>1).$$

Again this may be interpreted as the Euler characteristic of $\mathscr{T}_g/\Gamma_g = \mathscr{M}_g$, thought of as an orbifold.

Note that $\zeta(1-2g) \sim (-1)^g \dfrac{(2g-1)!}{2^{2g-1}\pi^{2g}}$, so that $\chi(\Gamma_g^1)$ grows very rapidly in absolute value and alternately takes on positive and negative values. This implies that the Betti numbers of any torsion-free subgroup of finite index in Γ_g^1 grow very rapidly with g (more than exponentially). To make a similar statement about Γ_g^1 itself, we would like to know its true Euler characteristic, i.e. the number $e(\Gamma_g^1)=\sum(-1)^i \dim H_i(\Gamma_g^1;\mathbb{Q})$. We will show in §6 how to deduce a formula for $e(\Gamma_g^1)$ from the formula for $\chi(\Gamma_g^1)$, tabulate these numbers for small g, and show that $e(\Gamma_g^1)$ and $\chi(\Gamma_g^1)$ are asymptotically equal; we will also give analogous results for Γ_g. However, the formulas for $e(\Gamma_g^1)$ and $e(\Gamma_g)$ are much more complicated than those for $\chi(\Gamma_g^1)$ and $\chi(\Gamma_g)$ and will not be stated here. The fact that $e(\Gamma_g)\sim\chi(\Gamma_g)$ implies that the Betti numbers of Γ_g grow more than exponentially and that Γ_g has a lot of homology in dimensions congruent to $g-1$ modulo 2. The known constructions of homology classes for Γ_g [9, 10] yield only even-dimensional classes and give far fewer than our theorem indicates must be present. Analogy with the situation for $\mathrm{Sp}(2g;\mathbb{Z})$, where $\chi(\mathrm{Sp}(2g;\mathbb{Z}))=\zeta(-1)\zeta(-3)\dots\zeta(1-2g)$ [6] and yet the stable cohomology is small ($H^*(\mathrm{Sp};\mathbb{Q})\cong\mathbb{Q}[y_2,y_6,\dots]$, where y_{4i+2} is a polynomial generator in $H^{4i+2}(\mathrm{Sp};\mathbb{Q})$ [2]) suggests that the contribution to the large Euler characteristic from the stable part of the cohomology may be relatively small.

The formula for $\chi(\Gamma_g^1)$ will follow from two other theorems, which we now state.

For every positive integer $n>0$ let $\mathscr{P}_n$ denote a fixed $2n$-gon with its sides labeled $S_1,\dots,S_{2n}$ consecutively around its boundary. For $g\geqq 0$ denote by $\varepsilon_g(n)$ the number of ways of grouping the sides $S_1,\dots,S_{2n}$ into n pairs (each S_i occuring in one and only one pair) so that if each side is identified to the side it is paired to in such a way that the resulting surface is orientable, then that surface has genus g. Also define $\lambda_g(n)$ to be the number of such groupings which do not contain a configuration of the form

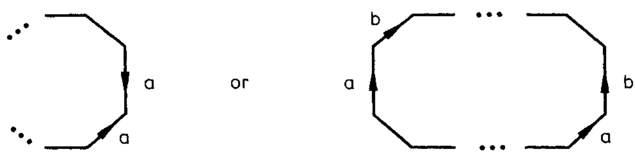

Fig. 1

The number $\varepsilon_g(n)$ is non-zero only for $n\geqq 2g$, while $\lambda_g(n)$ is non-zero only for $2g\leqq n\leqq 6g-3$. We will prove:

Theorem 1. $\chi(\Gamma_g^1)=\sum_{n=2g}^{6g-3}\frac{(-1)^{n-1}}{2n}\lambda_g(n)$.

Theorem 2. $\varepsilon_g(n)=\frac{(2n)!}{(n+1)!\,(n-2g)!}\times$ *Coefficient of* x^{2g} *in* $\left(\frac{x/2}{\tanh x/2}\right)^{n+1}$.

Since it is not hard to express $\lambda_g(n)$ in terms of $\varepsilon_g(n)$, these two results permit one to calculate $\chi(\Gamma_g^1)$; the result is the formula given above.

The proof of Theorem 1 is topological: it makes use of a contractible CW complex Y on which Γ_g^1 acts cellularly and virtually freely; the number $\frac{(-1)^{n-1}}{2n}\lambda_g(n)$ is the contribution to $\chi(\Gamma_g^1)$ of the cells of Y of dimension $6g-3-n$. The proof of Theorem 2 is combinatorial and rather indirect: We express the sum

Table 1

n	g	$\varepsilon_g(n)$
1	0	1
2	0	2
	1	1
3	0	5
	1	10
4	0	14
	1	70
	2	21
5	0	42
	1	420
	2	483
6	0	132
	1	2310
	2	6468
	3	1485
7	0	429
	1	12012
	2	66066
	3	56628
8	0	1430
	1	60060
	2	570570
	3	1169740
	4	225225
9	0	4862
	1	291720
	2	4390386
	3	17454580
	4	12317877
10	0	16796
	1	1385670
	2	31039008
	3	211083730
	4	351683046
	5	59520825
11	0	58786
	1	6466460
	2	205633428
	3	2198596400
	4	7034538511
	5	4304016990
12	0	208012
	1	29745716
	2	1293938646
	3	20465052608
	4	111159740692
	5	158959754226
	6	24325703325

The numbers $\varepsilon_g(n)$, $0 \leqq g \leqq n/2$

	n	$\lambda_g(n)$
$g=1$	2	1
	3	1
$g=2$	4	21
	5	168
	6	483
	7	651
	8	420
	9	105
$g=3$	6	1485
	7	25443
	8	173008
	9	635470
	10	1418835
	11	2023505
	12	1859858
$g=3$	13	1069068
	14	350350
	15	50050
$g=4$	8	225225
	9	6236802
	10	71110611
	11	456842386
	12	1882237357
	13	5321436120
	14	10718815107
	15	15679314651
	16	16740147996
	17	12934346997
	18	7051674630
	19	2575267695
	20	565815250
	21	56581525

The numbers $\lambda_g(n)$, $2g \leqq n \leqq 6g-3$

$$C(n,k) = \sum_{0 \leqq g \leqq n/2} \varepsilon_g(n)\, k^{n+1-2g}$$

as an integral over the k^2-dimensional space of $k \times k$ hermitian matrices and use some invariance properties of this integral to show that $C(n,k)$ equals $(2n-1)\cdot(2n-3)\cdot\ldots\cdot 5\cdot 3\cdot 1$ times a polynomial of degree $k-1$ in n; this polynomial is then identified from certain qualitative properties of the numbers

$\varepsilon_g(n)$. It would be nice to have a direct proof of Theorem 2. In particular, the formula of Theorem 2 implies, and is implied by, the recursion

$$(n+1)\,\varepsilon_g(n)=(4n-2)\,\varepsilon_g(n-1)+(2n-1)(n-1)(2n-3)\,\varepsilon_{g-1}(n-2)$$

(just differentiate with respect to x in Theorem 2); if one could give a direct geometrical proof of this recursion, one could circumvent many of the calculations in this paper.

A table of the values $\varepsilon_g(n)$ $(n\leqq 12)$ and $\lambda_g(n)$ $(g\leqq 4)$ is given on page 3.

§ 1. Construction of the CW-complex Y

Let F be a closed, oriented surface of genus g with basepoint p. The set of isotopy classes of orientation preserving homeomorphisms of F which fix p is a group under composition called the *mapping class group* and is denoted Γ_g^1. The *Teichmüller space* $\mathcal{T}_g^1$ is the space of all conformal equivalence classes of marked Riemann surfaces with basepoint or, equivalently, the space of all isometry classes of marked hyperbolic surfaces with basepoint. Γ_g^1 acts properly discontinuously on $\mathcal{T}_g^1$; the quotient is denoted $\mathcal{M}_g^1$ and called the *moduli space of curves* with basepoint. $\mathcal{M}_g^1$ is a V-manifold or orbifold: every point in $\mathcal{M}_g^1$ has a neighborhood modeled on $\mathbb{R}^{6g-4}$ modulo a finite group. In addition, Γ_g^1 is virtually torsion free (the subgroup $\Gamma_g^1[n]$ of all classes of maps which induce the identity on $H_1(F;\mathbb{Z}/n\mathbb{Z})$ is of finite index and torsion free for $n\geqq 3$), so $\mathcal{M}_g^1$ has a finite orbifold covering which is a manifold.

The *orbifold Euler characteristic* of Γ_g^1 is defined to be

$$\chi(\Gamma_g^1)=[\Gamma_g^1:\Gamma]^{-1}\cdot\chi(\Gamma),$$

where Γ is a torsion-free subgroup of finite index and $\chi(\Gamma)$ is the usual Euler characteristic of any $K(\Gamma,1)$ [14]. This is defined because Γ_g^1 has finite homological type (see, e.g. [8]). Suppose that Y is a CW-complex of dimension n on which Γ_g^1 acts cellularly such that the stabilizer of each cell of Y is a finite group (Y is then called a proper Γ_g^1-complex). Suppose further that the number of orbits of p-cells is finite for each p and that $\{\sigma_p^i\}$ is a set of representatives for these orbits. Then we have the following formula of Quillen ([13], Prop. 11):

$$\chi(\Gamma_g^1)=\sum_p(-1)^p\sum_i|G_p^i|^{-1}, \tag{1}$$

where $|G_p^i|$ denotes the order of the stabilizer of σ_p^i.

We now define one such complex Y. Fix the surface F and the basepoint p. Let $\alpha_1,\ldots,\alpha_n$ be a family of simple closed curves in F which intersect at p and nowhere else. Suppose that no α_i is null-homotopic and no two α_i are homotopic rel p (this implies that $n\leqq 6g-3$). The isotopy class of $\alpha_1,\ldots,\alpha_n$ is called an *arc-system* of rank $n-1$ in F. Define a simplicial complex A of dimension $6g-4$ by taking an $n-1$ simplex $\langle\alpha_1,\ldots,\alpha_n\rangle$ for each rank $n-1$ arc-system and identifying $\langle\alpha_1,\ldots,\alpha_n\rangle$ as a face of $\langle\beta_1,\ldots,\beta_m\rangle$ if there are representatives

$\{\alpha_i\}$, $\{\beta_j\}$ of the isotopy classes with $\{\alpha_i\} \subset \{\beta_j\}$. A cellular action of the group Γ_g^1 is defined by setting

$$[f] \cdot \langle \alpha_1, \ldots, \alpha_n \rangle = \langle f(\alpha_1), \ldots, f(\alpha_n) \rangle.$$

A family of curves $\alpha_1, \ldots, \alpha_n$ representing a rank $n-1$ arc-system is said to *fill up F* if each component of $F - \{\alpha_i\}$ is a 2-cell. Let $A_\infty \subset A$ be the subcomplex of all simplices $\langle \alpha_1, \ldots, \alpha_n \rangle$ such that $\alpha_1, \ldots, \alpha_n$ do not fill up F. The action of Γ_g^1 on A preserves A_∞, so Γ_g^1 acts on $A - A_\infty$.

In [7] it is proved that the simplicial complex A is contractible, and the argument applies directly to show that $A - A_\infty$ is also contractible. Another proof follows from the beautiful fact that $A - A_\infty$ is actually Γ_g^1*-equivariantly homeomorphic* to $\mathcal{T}_g^1$. A proof of this due to Mumford, based on a result of Strebel concerning quadratic differentials, is given in [8]. Another proof, based on an idea of Thurston and using hyperbolic geometry, is given in [3] (see also [11]).

The complex Y we need is the "dual" to A; its existence is based on the fact that $A - A_\infty$ is a manifold. Explicitly, Y has a $6g-3-n$ cell for each $n-1$ cell $\langle \alpha_1, \ldots, \alpha_n \rangle$ of A such that the α_i fill up F, and $\langle \alpha_1, \ldots, \alpha_n \rangle$ is a face of $\langle \beta_1, \ldots, \beta_m \rangle$ when there are representatives $\{\beta_j\} \subset \{\alpha_i\}$. The reason that the arc-systems which define Y must fill up F is explained in [8]; the point is that the link in A of a cell in A_∞ is contractible while that of a cell in $A - A_\infty$ is spherical. Since it takes at least $2g$ curves to fill up F, Y has dimension $4g-3$. The contractibility of Y follows from that of $A - A_\infty$.

We now apply formula (1) to Y to prove Theorem 1. The dual to an arc-system $\alpha_1, \ldots, \alpha_n$ which fills up F is a graph $\Omega \subset F$ with one vertex in each component of $F - \{\alpha_i\}$ and one edge transverse to each α_i. Splitting F along Ω gives a $2n$-gon $\mathcal{P}_n$ with its center at p. F is then identified with $\mathcal{P}_n/\sim$ where $\sim$ is an identification of the edges of $\mathcal{P}_n$ in pairs; the family $\alpha_1, \ldots, \alpha_n$ is easily recovered as in the example of Fig. 2.

Fig. 2

It is easy to see that the only restrictions on the identifications which may arise are:

Condition A: no edge may be identified with its neighbor,

Condition B: no adjacent pair of edges may be identified to another such pair in reverse order.

In A the dual edge would be null-homotopic and in B the dual edges would be homotopic rel p. These conditions are illustrated in Fig. 1.

As in the introduction, let $\lambda_g(n)$ be the number of ways of identifying the edges of a fixed $2n$-gon $\mathscr{P}_n$ in pairs so that the resulting surface is orientable of genus g and A and B are satisfied. We now prove Theorem 1.

The pairings of the edges of $\mathscr{P}_n$ occuring in the count for $\lambda_g(n)$ may be partitioned into equivalence classes, two pairings being equivalent if they differ by a rotation of $\mathscr{P}_n$. For example, $\lambda_2(4)=21$ and there are four classes, two of eight elements, one of four and one of one (Fig. 3).

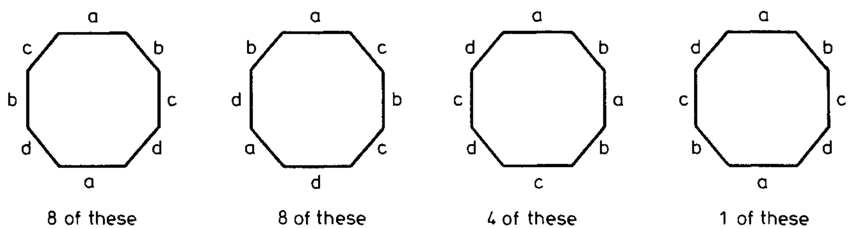

Fig. 3

Choose a representative for each equivalence class, pair the sides of $\mathscr{P}_n$ and identify the result with F so that the center of $\mathscr{P}_n$ is matched with p. This picks out a $6g-3-n$ cell σ^i for each class and $\{\sigma^i\}$ is a set of representatives for the action of Γ_g^1 on Y. If there are m elements in the equivalence class, the identification will have a cyclic symmetry of order $\frac{2n}{m}$ and the corresponding cell σ^i will have isotropy group which is cyclic of order $\frac{2n}{m}$. Counting $\left(\frac{2n}{m}\right)^{-1}$ for each σ^i gives the same answer as counting each of the m elements in each equivalence class with weight $1/2n$. Thus

$$\sum_i |G^i_{6g-3-n}|^{-1}=\lambda_g(n)/2n.$$

Theorem 1 now follows immediately from formula (1).

§ 2. Evaluation of $\sum(-1)^{n-1}\lambda_g(n)/2n$

In this section we assume Theorem 2 giving $\varepsilon_g(n)$ and deduce the main theorem. We have two tasks:

(i) to find the relationship between $\varepsilon_g(n)$ and $\lambda_g(n)$,
(ii) to calculate $\sum(-1)^{n-1}\lambda_g(n)/2n$.

Part (i) will be done in two steps: Define $\mu_g(n)$ to be the number of identifications of $\mathscr{P}_n$ which give a surface of genus g and satisfy condition A; then we will relate $\varepsilon_g(n)$ to $\mu_g(n)$ and $\mu_g(n)$ to $\lambda_g(n)$. Specifically, we have:

Lemma 1. $\varepsilon_g(n) = \sum_{i \geq 0} \binom{2n}{i} \mu_g(n-i),$

$$\mu_g(n) = \sum_{i \geq 0} \binom{n}{i} \lambda_g(n-i).$$

Proof. Let τ be an edge-pairing of $\mathscr{P}_n$ which does not satisfy A. Orient the boundary of $\mathscr{P}_n$ and number its vertices consecutively. Identifying a pair of adjacent edges which are paired by τ gives a map of $\mathscr{P}_n$ onto $\mathscr{P}_{n-1}$ (think of folding the identified edges inward, so that the vertex of $\mathscr{P}_n$ between the identified edges maps to an interior point of $\mathscr{P}_{n-1}$) and induces an edge pairing τ^1 on $\mathscr{P}_{n-1}$ with the genus of $\mathscr{P}_n/\tau$ equal to that of $\mathscr{P}_{n-1}/\tau^1$. Continuing this process eventually gives an edge-pairing τ^i of $\mathscr{P}_{n-i}$ which satisfies condition A for some $i \leqq n-2g$. Let $\varphi: \mathscr{P}_n \to \mathscr{P}_{n-i}$ be the quotient map; τ^i and φ determine τ and conversely. The intersection of φ (vertices of $\mathscr{P}_n$) with the interior of $\mathscr{P}_{n-i}$ is a finite set $\{w_1, \ldots, w_i\}$. For $1 \leqq j \leqq i$, let v_j be the lowest numbered vertex of $\mathscr{P}_n$ for which $\varphi(v_j) = w_j$. We claim that any collection of i vertices $v_1, \ldots, v_i$ may occur in this way, and that $\{v_j\}$ and τ^i determine τ. This will prove the first formula, since there are $\binom{2n}{i}$ choices for $\{v_j\}$.

Select i vertices of $\mathscr{P}_n$, $0 \leqq i \leqq n-2g$, and label them $v_1, \ldots, v_i$; also label the edges which proceed them $a_1, \ldots, a_i$ respectively. Each a_j must be identified with another edge b_j, defined as follows. If the edge after v_j is not labeled, pick it for b_j; do this for all possible j. If any b_j remain unchosen, proceed to the edge third after v_j and if it is unlabeled, call it b_j; again this should be done for all possible cases. Continue, selecting the fifth edge, seventh edge, etc. until all the b_j are chosen. An example is given in Fig. 4. Pairing a_j to b_j for each j, we have reversed the process above and established the claim.

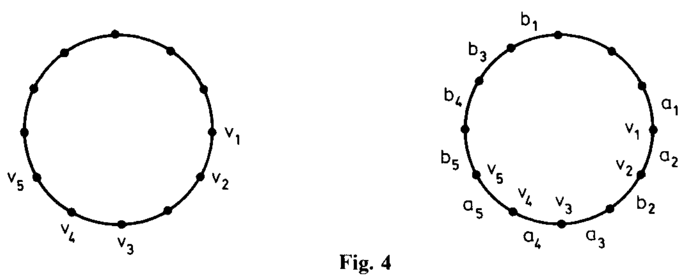

Fig. 4

For the second formula we proceed differently. Let τ be an edge-pairing of $\mathscr{P}_n$ which satisfies condition A but not condition B. Orient $\partial\mathscr{P}_n$ and number its edges consecutively. If e_i, e_{i+1}, e_j and e_{j+1} (indexed mod $2n$) are chosen so that τ pairs e_i to e_{j+1} and e_{i+1} to e_j, we may amalgamate e_i and e_{i+1} into one edge and e_j and e_{j+1} into another to get a new edge pairing τ^1 on $\mathscr{P}_{n-1}$ which still satisfies A. The genus of $\mathscr{P}_n/\tau$ and that of $\mathscr{P}_{n-1}/\tau^1$ are the same. Continuing eventually gives a pairing τ^i on $\mathscr{P}_{n-i}$ which satisfies both A and B.

To work backwards, orient $\partial\mathscr{P}_{n-i}$ and number its edges $f_1, \ldots, f_{2n-2i}$ consecutively. Let σ be an edge pairing of $\mathscr{P}_{n-i}$ and choose the lowest indexed edge in each pair as representative to get $f_1, f_{j_2}, \ldots, f_{j_{n-i}}$, $1<j_2<\ldots<j_{n-i}$. For any non-negative integers $m_1, \ldots, m_{n-i}$ which sum to i, divide f_{j_k} and $\sigma(f_{j_k})$ into m_k+1 edges by inserting m_k new vertices, and pair these in reverse order to agree with σ. We may identify the resulting $2n$-gon with $\mathscr{P}_n$ to give an edge-pairing τ with $\tau^i=\sigma$. There are m_1+1 choices of which edge to call e_1, but otherwise σ determines τ. It is easy to check that

$$\sum_{\substack{m_1+\ldots+m_{n-i}=i \\ m_j\geqq 0}} (m_1+1)=\binom{n}{i},$$

so the lemma is proved. □

For task (ii) we use:

Lemma 2. *Let $\{\varepsilon(n)\}_{n\geqq 0}$, $\{\mu(n)\}_{n\geqq 0}$, $\{\lambda(n)\}_{n\geqq 0}$ be three sequences related by*

$$\varepsilon(n)=\sum_{i\geqq 0}\binom{2n}{i}\mu(n-i), \qquad \mu(n)=\sum_{i\geqq 0}\binom{n}{i}\lambda(n-i), \tag{2}$$

and suppose that $\varepsilon(n)$ has the form

$$\varepsilon(n)=\binom{2n}{n+1}F(n) \tag{3}$$

for some polynomial F with $F(-1)=0$. Then the sum $\chi=\sum_n \frac{(-1)^{n-1}}{2n}\lambda(n)$ is finite (i.e. $\lambda(n)$ is zero for $n=0$ or n sufficiently large) and equals $F(0)$.

For the sequences $\varepsilon=\varepsilon_g$, $\mu=\mu_g$, $\lambda=\lambda_g$ $(g\geqq 1)$, the equations (2) are the content of Lemma 1. Here the conclusion that $\lambda(n)=0$ for $n=0$ or n sufficiently large is uninteresting since we know for geometric reasons that $\lambda_g(n)=0$ unless $2g\leqq n\leqq 6g-3$. On the other hand, the number χ of the lemma equals $\chi(\Gamma_g^1)$ by Theorem 1, and Theorem 2 gives (3) with

$$F(n)=(n-1)\cdot(n-2)\cdot\ldots\cdot(n-2g+1)\cdot C_{n,g}$$

where $C_{n,g}$ denotes the coefficient of x^{2g} in $\left(\frac{x/2}{\tanh x/2}\right)^{n+1}$. Clearly $F(n)$ is a polynomial (of degree $3g-1$) in n with $F(-1)=0$; the lemma then gives

$$\chi(\Gamma_g^1)=F(0)=-(2g-1)!\cdot C_{0,g}=-\frac{B_{2g}}{2g}$$

as desired. Thus it only remains to prove the lemma.

Clearly (3) implies $\varepsilon(0)=0$, and the relations (2) show that $\mu(0)=\lambda(0)=0$ also. (More generally, if $\varepsilon(n)$ vanishes for $n=0, 1, \ldots, n_0$, i.e., if $F(n)$ is divisible by $(n-1)\cdot(n-2)\cdot\ldots\cdot(n-n_0)$, then μ and λ also vanish for $n\leqq n_0$; this is the case for $\varepsilon=\varepsilon_g$ with $n_0=2g-1$.) To see that the sequence $\{\lambda(n)\}$ terminates and to compute χ, we introduce a fourth sequence of numbers $\{\kappa(n)\}$ as follows: Since $F(n)/(n+1)$ is a polynomial, say of degree $d-1$, it can be written as a linear combination of the polynomials 1, $n-1$, $(n-1)(n-2)$, ..., $(n-1)(n-2)\cdot\ldots\cdot(n-d+1)$. Write the coefficient of $(n-1)\cdot\ldots\cdot(n-r+1)$ as $\frac{r!}{(2r)!}\kappa(r)$ $\left(\text{the factor } \frac{r!}{(2r)!} \text{ is included for convenience}\right)$. Thus

$$F(n)=(n+1)\cdot\sum_{r=1}^{d}\frac{r!}{(2r)!}\kappa(r)\cdot(n-1)(n-2)\cdot\ldots\cdot(n-r+1),$$

$$\varepsilon(n)=\frac{(2n)!}{n!}\sum_{r=1}^{d}\frac{r!}{(2r)!}\frac{\kappa(r)}{(n-r)!}$$

with the usual convention $\frac{1}{(n-r)!}=0$ for $n<r$. The relationships between κ and ε, ε and μ, and μ and λ can be expressed most conveniently by introducing the generating functions

$$K(x)=\sum_{n\geqq 0}\kappa(n)\,x^n,\qquad E(x)=\sum_{n\geqq 0}\varepsilon(n)\,x^n,$$
$$M(x)=\sum_{n\geqq 0}\mu(n)\,x^n,\qquad L(x)=\sum_{n\geqq 0}\lambda(n)\,x^n.$$

Indeed,

$$\begin{aligned}E(x)&=\sum_{r=1}^{d}\frac{r!}{(2r)!}\kappa(r)\sum_{n\geqq r}\frac{(2n)!}{n!}\frac{x^n}{(n-r)!}\\&=\sum_{r=1}^{d}\frac{r!}{(2r)!}\kappa(r)\cdot x^r\frac{d^r}{dx^r}\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)\\&=\sum_{r=1}^{d}\frac{r!}{(2r)!}\kappa(r)\cdot x^r\frac{d^r}{dx^r}\left(\frac{1}{\sqrt{1-4x}}\right)\\&=\sum_{r=1}^{d}\kappa(r)\,x^r\frac{1}{(1-4x)^{r+1/2}}\\&=\frac{1}{\sqrt{1-4x}}K\left(\frac{x}{1-4x}\right);\end{aligned}$$

$$\begin{aligned}E(x)&=\sum_{n\geqq 0}x^n\sum_{i\geqq 0}\binom{2n}{i}\mu(n-i)\\&=\sum_{j\geqq 0}\mu(j)\sum_{i\geqq 0}\binom{2i+2j}{i}x^{i+j}\\&=\sum_{j\geqq 0}\mu(j)\,x^j\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^{2j}\end{aligned}$$

(here we have used the standard identity

$$\sum_{i\geq 0}\binom{2i+k}{i}x^i=\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k,$$

which is most easily verified by noting that $f_k=\sum_{i\geq 0}\binom{2i+k}{i}x^i$ satisfies

$$f_0=\frac{1}{\sqrt{1-4x}},\quad f_1=\frac{1}{2x}(f_0-1)\quad\text{and}\quad f_k=\frac{1}{x}(f_{k-1}-f_{k-2})\quad\text{for } k\geq 2\Bigg)$$

$$=\frac{1}{\sqrt{1-4x}}M\left(\frac{1-2x-\sqrt{1-4x}}{2x}\right);$$

$$\begin{aligned}M(x)&=\sum_{n\geq 0}x^n\sum_{i\geq 0}\binom{n}{i}\lambda(n-i);\\&=\sum_{j\geq 0}\lambda(j)\sum_{i\geq 0}\binom{i+j}{j}x^{i+j};\\&=\sum_{j\geq 0}\lambda(j)\frac{x^j}{(1-x)^{j+1}}\\&=\frac{1}{1-x}L\left(\frac{x}{1-x}\right).\end{aligned}$$

Combining these three formulas gives

$$L(x)=\frac{1}{1+x}M\left(\frac{x}{1+x}\right)=\frac{1}{(1+x)(1+2x)}E\left(\frac{x(1+x)}{(1+2x)^2}\right)=\frac{1}{1+x}K(x(1+x)).$$

Since K is a polynomial (of degree d) with constant term 0, this shows that L is also a polynomial (of degree $2d-1$) with constant term 0, proving the first assertion of the lemma. As to the value of χ, we find

$$\begin{aligned}\chi&=\sum_{n=1}^{2d-1}\frac{(-1)^{n-1}}{2n}\lambda(n)=-\frac{1}{2}\int_0^1\frac{L(-x)}{x}dx\\&=-\frac{1}{2}\int_0^1\frac{K(-x(1-x))}{x(1-x)}dx\\&=\frac{1}{2}\sum_{r=1}^{d}(-1)^{r-1}\kappa(r)\int_0^1 x^{r-1}(1-x)^{r-1}dx\\&=\sum_{r=1}^{d}(-1)^{r-1}\kappa(r)\frac{r!(r-1)!}{(2r)!}\qquad\text{(beta integral)}\\&=F(0),\end{aligned}$$

as desired. This completes the proof of Lemma 2.

Note that the κ's give the best coding of the information contained in the four equivalent series ε, μ, λ and κ, since the d numbers $\kappa(r)$ determine the $2d$

-1 values $\lambda(n)$ and the infinitely many values $\mu(n)$ and $\varepsilon(n)$. In the case of interest to us, namely $\varepsilon=\varepsilon_g$, $\mu=\mu_g$, $\lambda=\lambda_g$, all four sequences vanish for $n<2g$, and $d=3g-1$, so that the g numbers $\kappa(2g), \ldots, \kappa(3g-1)$ suffice to describe the $4g-2$ numbers $\lambda_g(2g), \ldots, \lambda_g(6g-3)$ and all the $\varepsilon_g(n)$, $n \geqq 2g$. We give a small table of the numbers $\kappa_g(n)$:

	$g=1$	2	3	4	5
$n-2g=0$	1	21	1485	225225	59520825
1		105	18018	4660227	1804142340
2			50050	29099070	18472089636
3				56581525	78082504500
4					117123756750

§ 3. Coloring the polygon

For fixed n, the numbers $\varepsilon_g(n)$ are non-zero only for $0 \leqq g \leqq n/2$. We take these as the coefficients of a polynomial

$$C(n, k) = \sum_{0 \leqq g \leqq n/2} \varepsilon_g(n)\, k^{n+1-2g}.$$

Thus the table in the introduction gives

$$\begin{aligned} &C(0, k) = k \\ &C(1, k) = k^2 \\ &C(2, k) = 2k^3 + k \\ &C(3, k) = 5k^4 + 10k^2 \\ &C(4, k) = 14k^5 + 70k^3 + 21k, \end{aligned}$$

while in another direction we have

$$C(n, 1) = (2n-1)!! \underset{\text{def}}{=} (2n-1) \cdot (2n-3) \cdot \ldots \cdot 5 \cdot 3 \cdot 1,$$

because $C(n, 1) = \sum_g \varepsilon_g(n)$ counts all ways of identifying sides of $\mathscr{P}_n$ in pairs, irrespective of the genus of the resulting surface. The number $C(n, k)$ can be interpreted as the number of pairs (ϕ, τ) consisting of a k-coloring ϕ of the vertices of $\mathscr{P}_n$ (i.e. a map from the set of vertices of $\mathscr{P}_n$ into a fixed set of cardinality k, called the set of *colors*) and an identification τ of the edges of $\mathscr{P}_n$ compatible with ϕ (i.e. two edges may be identified only if the left end of each has the same color as the right end of the other). Indeed, if we first do the identification τ, the number of inequivalent vertices is $n+1-2g$, where g is the genus of the resulting surface (because the surface has a cell-decomposition with one 2-cell and n 1-cells) and these can be colored in k^{n+1-2g} ways.

The functions $C(n, k)$ and $\varepsilon_g(n)$ clearly determine each other. We will prove the following result.

Theorem 3. *$C(n, k) = (2n-1)!!\, c(n, k)$, where $c(n, k)$ $(n, k \geqq 0)$ is defined by the generating function*

$$1+2\sum_{n=0}^{\infty} c(n,k)\,x^{n+1} = \left(\frac{1+x}{1-x}\right)^k \tag{4}$$

or by the recursion

$$c(n,k)=c(n,k-1)+c(n-1,k)+c(n-1,k-1) \qquad (n,k>0)$$

with boundary conditions $c(0,k)=k$, $c(n,0)=0$ $(n,k\geqq 0)$.

The recursion makes it easy to compute a table of values of $c(n,k)$:

	$n=0$	1	2	3	4	5
$k=0$	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	4	6	8	10	12
3	3	9	19	33	51	73
4	4	16	44	96	180	304
5	5	25	85	225	501	985

We can also use (4) to get closed formulae for $c(n,k)$, either by multiplying the binomial expansions of $(1+x)^k$ and $(1-x)^{-k}$ or by writing $\left(\frac{1+x}{1-x}\right)^k$ as $\left(1+\frac{2x}{1-x}\right)^k$ and expanding by the binomial theorem:

$$c(n,k)=\frac{1}{2}\sum_{l+m=n+1}\binom{k}{l}\binom{k+m-1}{m}=\sum_{l=1}^{k}2^{l-1}\binom{k}{l}\binom{n}{l-1}. \tag{5}$$

To see the equivalence of the two definitions of $c(n,k)$ in the theorem, note that the coefficients defined by (4) clearly satisfy the given boundary conditions, while the recursion follows from

$$\left(\frac{1+x}{1-x}\right)^k=(1+2x+2x^2+\dots)\left(\frac{1+x}{1-x}\right)^{k-1},$$

$$c(n,k)=c(n,k-1)+2\sum_{m=0}^{n-1}c(m,k-1)+1,$$

$$c(n,k)-c(n-1,k)=c(n,k-1)-c(n-1,k-1)+2c(n-1,k-1).$$

Theorem 3 will be proved in §4. Here we show how it implies Theorem 2. Differentiating (4) gives

$$\sum_{n=0}^{\infty}(n+1)\,c(n,k)\,x^n=\frac{k}{1-x^2}\left(\frac{1+x}{1-x}\right)^k$$

or

$$(n+1)\,c(n,k)=k\cdot\operatorname{Res}_{x=0}\left[\frac{1}{x^{n+1}}\left(\frac{1+x}{1-x}\right)^k\frac{dx}{1-x^2}\right].$$

Making the substitution $x=\tanh\frac{t}{2}$ gives

$$(n+1)\,c(n,k)=\tfrac{1}{2}k\cdot\operatorname{Res}_{t=0}\left[\left(\frac{1}{\tanh t/2}\right)^{n+1}e^{kt}\,dt\right]$$
$$=2^n k\cdot\text{Coefficient of } t^n \text{ in } e^{kt}\left(\frac{t/2}{\tanh t/2}\right)^{n+1}$$
$$=2^n k\cdot\sum_{r=0}^{n}\frac{k^r}{r!}\cdot\text{Coefficient of } t^{n-r} \text{ in } \left(\frac{t/2}{\tanh t/2}\right)^{n+1}.$$

Since $\frac{t/2}{\tanh t/2}$ is an even power series, the coefficient of t^{n-r} in $\left(\frac{t/2}{\tanh t/2}\right)^{n+1}$ is zero unless $n-r$ is an even number, $n-r=2g$. Hence the last equality $\left(\text{multiplied by } \frac{(2n-1)!!}{n+1}=\frac{(2n)!}{2^n(n+1)!}\right)$ can be written

$$(2n-1)!!\,c(n,k)$$
$$=\frac{(2n)!}{(n+1)!}\sum_{0\le g\le n/2}\frac{k^{n+1-2g}}{(n-2g)!}\cdot\textit{Coefficient of } t^{2g} \textit{ in } \left(\frac{t/2}{\tanh t/2}\right)^{n+1}.$$

The equivalence of Theorems 2 and 3 is now obvious.

§ 4. An integral formula for $C(n, k)$

In this section we carry out the heart of the combinatorial part of this paper, the evaluation of the numbers $C(n,k)$. Recall that $C(n,k)$ counts pairs (ϕ,τ) consisting of a k-coloring ϕ of the vertices of $\mathscr{P}_n$ and a compatible edge-identification τ. Performing first τ and then ϕ gave the formula $\sum_g \varepsilon_g(n)\,k^{n+1-2g}$ for $C(n,k)$. Performing ϕ first gives a different expression. There are k^{2n} possible k-colorings of the vertices of $\mathscr{P}_n$. Let ϕ be one of them, and for each $i,j\in\{1,\dots,k\}$ let n_{ij} be the number of edges of $\mathscr{P}_n$ whose left and right ends are colored with colors i and j, respectively. Thus $n_{ij}\geqq 0$, $\sum_{i,j=1}^{k} n_{ij}=2n$. The number of edge identifications τ compatible with ϕ depends only on the n_{ij} and not on the order in which the edges with coloring $i-j$ occur: If for some $i\neq j$ the numbers n_{ij} and n_{ji} are different, or if for some i the number n_{ii} is odd, then there are no identifications (because an edge colored $i-j$ must be identified with an edge $j-i$). If $n_{ij}=n_{ji}$ and $2\,|\,n_{ii}$ for all i and j, i.e. if $\mathscr{N}=(n_{ij})_{1\le i,j\le k}$ is an even symmetric matrix, then a moments's reflection shows that the number of edge identifications compatible with ϕ is $\prod_{i<j} n_{ij}!\cdot\prod_i (n_{ii}-1)!!$, where $(n-1)!!$ (n even) has the same meaning as in § 3. Thus

$$C(n,k)=\sum_{\mathscr{N}} c(\mathscr{N})\,\varepsilon(\mathscr{N}), \tag{6}$$

where the sum is over all $k\times k$ matrices $\mathcal{N}=(n_{ij})$ of non-negatives integers with $\sum n_{ij}=2n$, $c(\mathcal{N})$ is the number of k-colorings of $\mathcal{P}_n$ having n_{ij} edges colored $i-j$ for each i and j, and

$$\varepsilon(\mathcal{N})=\prod_{1\le i<j\le k}\begin{Bmatrix}0 & \text{if } n_{ij}\neq n_{ji}\\ n_{ij}! & \text{if } n_{ij}=n_{ji}\end{Bmatrix}\cdot\prod_{i=1}^{k}\begin{Bmatrix}0 & \text{if } n_{ii} \text{ is odd}\\ (n_{ii}-1)!! & \text{if } n_{ii} \text{ is even}\end{Bmatrix}.$$

The number $c(\mathcal{N})$ is given by the generating function

$$\operatorname{tr}(Z^{2n})=\sum_{\mathcal{N}} c(\mathcal{N})\,Z^{\mathcal{N}} \tag{7}$$

where $Z=(z_{ij})_{1\le i,j\le k}$ is a $k\times k$ matrix of independent variables and $Z^{\mathcal{N}}$ denotes $\prod_{i,j} z_{ij}^{n_{ij}}$. This follows directly from the definition of matrix multiplication and of the trace, since

$$\operatorname{tr}(Z^{2n})=\sum_{i_1=1}^{k}\cdots\sum_{i_{2n}=1}^{k} z_{i_1 i_2}\, z_{i_2 i_3}\cdots z_{i_{2n-1} i_{2n}}\, z_{i_{2n} i_1}$$

and we can think of each term of the summation as corresponding to the coloring of the vertices of $\mathcal{P}_n$ by colors $i_1, \ldots, i_{2n}$.

To proceed further we express the function $\varepsilon(\mathcal{N})$ as a multiple integral. For two integers $n, m\ge 0$ we have

$$\delta_{nm}=\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta,\qquad n!=\int_0^{\infty} t^n e^{-t}\,dt$$

and therefore, setting $t=r^2$ and shifting to polar coordinates,

$$\begin{aligned}\delta_{nm}\,n! &=\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} r^n e^{in\theta} r^m e^{-im\theta}\, r\,dr\,d\theta\\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+iy)^n(x-iy)^m e^{-x^2-y^2}\,dx\,dy.\end{aligned}$$

Similarly the function $(n-1)!!$ (n even) can be represented by

$$\begin{aligned}(n-1)!! &=2^{n/2}\left(\frac{n}{2}-\frac{1}{2}\right)\left(\frac{n}{2}-\frac{3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)=2^{n/2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})}\\ &=\frac{2^{n/2}}{\sqrt{\pi}}\int_0^{\infty} t^{\frac{n-1}{2}} e^{-t}\,dt\\ &=\sqrt{\frac{2}{\pi}}\int_0^{\infty} x^n e^{-x^2/2}\,dx \qquad (t=x^2/2)\end{aligned}$$

so we have the integral representation

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2}\,dx=\begin{cases}0 & n \text{ odd},\\ (n-1)!! & n \text{ even}.\end{cases} \tag{8}$$

Hence

$$\begin{aligned}\varepsilon(\mathcal{N})&=2^{-k/2}\,\pi^{-k^2/2}\prod_{i<j}\left(\iint_{-\infty}^{\infty}(x+iy)^{n_{ij}}(x-iy)^{n_{ji}}\,e^{-x^2-y^2}\,dx\,dy\right)\\ &\quad\cdot\prod_{i=1}^{k}\left(\int_{-\infty}^{\infty} x^{n_{ii}}\,e^{-x^2/2}\,dx\right)\\ &=2^{-k/2}\,\pi^{-k^2/2}\int_{H_k} Z^{\mathcal{N}}\,e^{-\frac{1}{2}\operatorname{tr}(Z^2)}\,d\mu_H,\end{aligned}$$

where H_k is the k^2-dimensional euclidean space with variables $x_{ij}(i\leqq j)$, $y_{ij}(i<j)$, Z the hermitian ($Z^t=\bar{Z}$) matrix

$$Z=(z_{ij}),\qquad z_{ij}=\begin{cases}x_{ii} & i=j,\\ x_{ij}+\sqrt{-1}\,y_{ij} & i<j,\\ x_{ij}-\sqrt{-1}\,y_{ij} & i>j,\end{cases}$$

and $d\mu_H=\prod_{i\leqq j} dx_{ij}\prod_{i<j} dy_{ij}$ the euclidean volume. (Note that $\operatorname{tr}(Z^2)=\sum_{i,j}|z_{ij}|^2>0$ because Z is hermitian.) Combining this formula with (6) and (7), we obtain:

Proposition 1.

$$C(n,k)=2^{-k/2}\,\pi^{-k^2/2}\int_{H_k}\operatorname{tr}(Z^{2n})\,e^{-\frac{1}{2}\operatorname{tr}(Z^2)}\,d\mu_H.$$

We now apply the following general result:

Proposition 2. *Let F be an integrable function on H_k which is invariant under the action of the unitary group*

$$U_k=\{u\in \mathrm{GL}(k,\mathbb{C})\,|\,u^t\bar{u}=1\},$$

i.e. $F(u^{-1}Zu)=F(Z)$ for $u\in U_k$. Then

$$\int_{H_k} F(Z)\,d\mu_H=c_k\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} F\begin{pmatrix}t_1 & & 0\\ & \ddots & \\ 0 & & t_k\end{pmatrix}\prod_{1\leqq i<j\leqq k}(t_i-t_j)^2\,dt_1\dots dt_k$$

where $c_k=\dfrac{\pi^{k(k-1)/2}}{k!\,(k-1)!\dots 1!}$.

Proof. Let T_k be the set of diagonal matrices of size k with real entries. Any matrix in H_k is conjugate under U_k to an element of T_k, say $Z=u^{-1}tu$. For almost all t (namely, those with distinct non-zero entries), the choice of u in this formula is unique up to left multiplication with an element of $\Delta_k\cdot W$, where Δ_k is the set of diagonal elements of U_k $\left(\text{i.e. elements }\begin{pmatrix}e^{i\theta_1} & & 0\\ & \ddots & \\ 0 & & e^{i\theta_k}\end{pmatrix}\text{ with }\theta_i\in\mathbb{R}\right)$

and W is the group of $k\times k$ permutation matrices. Hence the map

$$\begin{aligned} T_k\times \Delta_k\backslash U_k &\to H_k \\ (t,u) &\mapsto Z=u^{-1}tu \end{aligned}$$

is generically a covering of degree $k!$. Differentiating the formula $Z=u^{-1}tu={}^t\bar{u}tu$ gives

$$dZ={}^t\overline{du}\cdot tu+u^{-1}\cdot dt\cdot u+u^{-1}t\cdot du=u^{-1}(dt+t\Omega+{}^t\bar{\Omega}t)u$$

where dt, du and dZ are $k\times k$ matrices of differentials and $\Omega=du\cdot u^{-1}$. Differentiating the equation $u{}^t\bar{u}=1$ shows that Ω is skew-hermitian, i.e. $\Omega=(\omega_{ij})$ with $\bar{\omega}_{ij}=-\omega_{ji}$. Hence the matrix $dt+t\Omega+{}^t\bar{\Omega}t=dt+t\Omega-\Omega t$ has diagonal entries dt_i and off-diagonal entries $(t_i-t_j)\omega_{ij}$, so

$$d\mu_H=\prod_{i<j}(t_i-t_j)^2\ \ d\mu_{\Delta\backslash U}\cdot d\mu_T$$

where $d\mu_H$ is the Euclidean volume element on H_k introduced above, $d\mu_T=dt_1\dots dt_k$ is the Euclidian volume element on $T_k\cong\mathbb{R}^k$, and $d\mu_{\Delta\backslash U}=|\omega_{12}\wedge\bar{\omega}_{12}\wedge\dots\wedge\omega_{k-1k}\wedge\bar{\omega}_{k-1k}|$. (Since Ω is clearly invariant under right translation by U_k, $d\mu_{\Delta\backslash U}$ is the measure on $\Delta_k\backslash U_k$ induced by Haar measure). We have proved the formula

$$\int_{H_k}F(Z)\,d\mu_H=\frac{1}{k!}\int_{T_k}\int_{\Delta_k\backslash U_k}F(u^{-1}tu)\prod_{i<j}(t_i-t_j)^2 d\mu_{\Delta\backslash U}\,d\mu_T$$

for any integrable function F on H_k; the proposition follows by specializing to the case where F is U_k-invariant, with

$$c_k=\frac{1}{k!}\int_{\Delta_k\backslash U_k}d\mu_{\Delta\backslash U}=\frac{1}{k!}\operatorname{vol}(\Delta_k\backslash U_k).$$

This volume can be computed by integrating $e^{-\frac{1}{2}\operatorname{tr}(X{}^t\bar{X})}$ over $M_k(\mathbb{C})$ and observing that any $X\in M_k(\mathbb{C})$ can be uniquely decomposed as the product of a unitary and an upper triangular matrix. Alternatively, we can obtain c_k by taking $F(Z)=e^{-\frac{1}{2}\operatorname{tr}(Z^2)}$ in Proposition 2 and evaluating on the right by a formula of Selberg. The result is as given in the proposition.

Combining Propositions 1 and 2 we get

$$C(n,k)=c_k'\int_{\mathbb{R}^k}(t_1^{2n}+\dots+t_k^{2n})\,e^{-\frac{1}{2}(t_1^2+\dots+t_k^2)}\prod_{1\le i<j\le k}(t_i-t_j)^2\,dt_1\dots dt_k$$

with $c_k'=2^{-k/2}\pi^{-k^2/2}c_k$. Since the function $e^{-\frac{1}{2}\Sigma t_i^2}\prod(t_i-t_j)^2$ is symmetric in all the t_i, we can replace $t_1^{2n}+\dots+t_k^{2n}$ by kt_1^{2n} without changing the value of the integral. Expand $\prod_{i<j}(t_i-t_j)^2$ as a monic polynomial in t_1, say $\sum_{r=0}^{2k-2}a_r(t_2,\dots,t_k)t_1^r$ with $a_{2k-2}=\prod_{2\le i<j\le k}(t_i-t_j)^2$, and perform the integration over t_1 using (8). This gives

$$C(n, k)=\sum_{r=0}^{k-1} \alpha_{k,r}(2n+2r-1)!!$$

with

$$\alpha_{k,r}=kc'_k\cdot\sqrt{2\pi}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} a_{2r}(t_2,\dots,t_k)\,e^{-\frac{1}{2}(t_2^2+\dots+t_k^2)}\,dt_2\dots dt_k.$$

For $r=k-1$, $\alpha_{k,r}$ can be evaluated by Proposition 2:

$$\begin{aligned}\alpha_{k,k-1}&=kc'_k\sqrt{2\pi}\cdot c_{k-1}^{-1}\int_{H_{k-1}} e^{-\frac{1}{2}\operatorname{tr}(Z^2)}\,d\mu_{H_{k-1}}\\&=k\cdot 2^{-\frac{k}{2}}\pi^{-\frac{k^2}{2}}c_k\cdot\sqrt{2\pi}\cdot c_{k-1}^{-1}\cdot 2^{\frac{k-1}{2}}\pi^{\frac{(k-1)^2}{2}}\\&=\frac{1}{(k-1)!}.\end{aligned}$$

Since $(2n+2r-1)!!$ equals $(2n-1)!!$ times a monic polynomial in $2n$ of degree r, this proves

$$C(n, k)=(2n-1)!!\,c'(n, k) \tag{9}$$

where $c'(n,k)$ is a polynomial in n of degree $k-1$ with leading term $\frac{(2n)^{k-1}}{(k-1)!}$. It remains only to identify this polynomial as $c(n, k)$.

To do this, we let $C_0(n, k)$ be the number of pairs (ϕ, τ) consisting of a *surjective* k-coloring ϕ and a compatible edge coloring τ of $\mathscr{P}_n$, i.e. $C_0(n, k)$ is defined like $C(n, k)$ but with the extra requirement that all k colors are used. Since any k-coloring uses exactly l colors for some $l\leq k$, and these colors may be chosen in exactly $\binom{k}{l}$ ways, we have

$$C(n, k)=\sum_{l=0}^{k}\binom{k}{l}\,C_0(n, l). \tag{10}$$

This can be inverted to give

$$C_0(n, k)=\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}\,C(n, l).$$

Hence (9) gives

$$C_0(n, k)=(2n-1)!!\,c'_0(n, k) \tag{11}$$

where $c'_0(n, k)=\sum(-1)^{k-l}\binom{k}{l}c'(n, l)$ is again a polynomial of degree $k-1$ in n with leading term $\frac{(2n)^{k-1}}{(k-1)!}$. But $C_0(n, k)$ vanishes for $k<n+1$ since no identification τ of $\mathscr{P}_n$ has more than $n+1$ inequivalent vertices (the number of inequivalent vertices was $n+1-2g$, where g is the genus of $\mathscr{P}_n(\tau)$ and hence no coloring compatible with τ can involve more than $n+1$ colors). Therefore

$c'_0(n,k)$ is a polynomial with leading term $\frac{(2n)^{k-1}}{(k-1)!}$ which has zeros at $n=0,1,\dots,k-2$, i.e. $c'_0(n,k)=2^{k-1}\binom{n}{k-1}$. Substituting this into (10) and (11) gives

$$C(n,k)=(2n-1)!!\sum_{l\geqq 1} 2^{l-1}\binom{k}{l}\binom{n}{l-1},$$

which (by (5)) is equivalent to the assertion of Theorem 3. This completes the proof of Theorem 3 and hence of Theorem 2 and the Main Theorem.

Note that if we had used Proposition 2 without knowing the constant before the integral, then the same argument would have proved the formula

$$C(n,k)=(2n-1)!!\sum_{l\geqq 1} \gamma_l\binom{k}{l}\binom{n}{l-1}$$

with *some* constants γ_l depending only on l. This formula with n fixed and k variable gives

$$C(n,k)=(2n-1)!!\,\gamma_{n+1}\frac{k^{n+1}}{(n+1)!}+O(k^n);$$

in view of the definition of $C(n,k)$, this means that $\varepsilon_0(n)=\frac{(2n-1)!!}{(n+1)!}\gamma_{n+1}$; and the proof that $\gamma_{n+1}=2^n$ (and consequently that $C(n,k)=(2n-1)!!\,c(n,k)$) could have been completed by using the direct computation of $\varepsilon_0(n)$ which we will give in § 5.

§ 5. Interlude: Recursions for $\varepsilon_g(n)$

In this section we discuss some recursion formulas which have a geometric origin. In principle these recursions determine $\varepsilon_g(n)$; unfortunately we were not able to solve them in closed form.

Let F_0^k be a compact surface of genus 0 with k boundary components and divide the i^{th} boundary component into n_i edges. We define $f_g(n_1,\dots,n_k)$ to be the number of ways of identifying these edges to obtain a closed, orientable, connected surface of genus g. Clearly $f_g(n_1,\dots,n_k)$ is symmetric in the variables, $f_g(n_1,\dots,n_k)=0$ unless $n_1+\dots+n_k$ is even, and $f_g(2n)=\varepsilon_g(n)$.

Let ∂_1 be the first boundary component of F and let e_1 be a fixed edge in ∂_1. If e_1 is identified to another edge e_j of ∂_1 which is separated from it by $j-1$ other edges the result is a surface of genus 0 with $k+1$ boundary components having $j-1,\ n_1-j-1,\ n_2,\dots,n_k$ edges respectively. If e_1 is identified to an edge on the i^{th} boundary component, $i>1$, the result has genus 1 and $k-1$ boundary components with $n_1+n_i-2,\ n_2,\dots,\hat{n}_i,\dots,n_k$ edges. In either case the identifications can be continued until a closed surface is obtained. This gives the recursive formula

$$f_g(n_1, \ldots, n_k) = \sum_{a+b=n_1-2} f_g(a, b, n_2, \ldots, n_k)$$
$$+ \sum_{i=2}^{k} n_i f_{g-1}(n_1+n_i-2, n_2, \ldots, \hat{n}_i, \ldots, n_k).$$

For $g=0$ this reduces to

$$f_0(n_1, \ldots, n_k) = \sum_{a+b=n_1-2} f_0(a, b, n_2, \ldots, n_k)$$

and one sees by induction (or geometrically) that $f_0(n_1, \ldots, n_k)=0$ unless the n_i are even and in that case

$$f_0(n_1, \ldots, n_k) = \prod_{i=1}^{k} f_0(n_i) = \prod_{i=1}^{k} \varepsilon_0\left(\frac{n_i}{2}\right).$$

The recursion then implies

$$\varepsilon_0(n) = \sum_{a+b=n-1} \varepsilon_0(a)\, \varepsilon_0(b).$$

Using the initial condition $\varepsilon_0(0)=1$, this may be solved to show $\varepsilon_0(n)$ is the n^{th} Catalan number $C(n)=\binom{2n}{n}\Big/(n+1)$; indeed, the recursion translates immediately to the formula $e(x)=1+xe(x)^2$ for the generating frunction $e(x) = \sum_{n \geq 0} \varepsilon_0(n)\, x^n$, so $e(x)=\dfrac{1-\sqrt{1-4x}}{2x} = \sum_{n \geq 0} C(n)\, x^n$ by the binomial theorem.

More generally, one sees by induction that $f_g(n_1, \ldots, n_k)$ vanishes if more than $2g$ of the n_i are odd. Thus for $g=1$ there are two cases, according as none or two of the n_i are odd, for $g=2$ there are three cases, etc. For genus one we find

$$f_1(2n_1, \ldots, 2n_k) = \left(\sum_{i=1}^{k} \frac{1}{2} \binom{n_i+1}{3} + \sum_{i<j} \frac{n_i(n_i+1)\, n_j(n_j+1)}{n_i+n_j} \right) \prod_{i=1}^{k} C(n_i),$$

$$f_1(2n_1+1, 2n_2+1, 2n_3, \ldots, 2n_k) = \frac{(2n_1+1)(n_1+1)(2n_2+1)(n_2+1)}{n_1+n_2+1} \prod_{i=1}^{k} C(n_i).$$

The formulas for higher genus are considerably more complicated and we were not able to give a direct proof of Theorem 2 in this way.

§6. The true Euler characteristic of Γ_g and Γ_g^1

Let Γ_g^s be the mapping class group defined like Γ_g^1 but with s points $q_1, \ldots, q_s$ fixed (individually) rather than just one; $\Gamma_g^0 = \Gamma_g$. For $2g-2+s \leq 0$ we have $\Gamma_g^s = \Gamma_g^{s+1}$, while for $2g-2+s>0$ we have an exact sequence

$$1 \to \pi_1(F - \{q_1, \ldots, q_s\}) \to \Gamma_g^{s+1} \to \Gamma_g^s \to 1$$

so that $\chi(\Gamma_g^{s+1}) = \chi(\Gamma_g^s)\cdot(2-2g-s)$. This gives the formulas

$$\chi(\Gamma_0^s) = \begin{cases} 1 & s \leqq 3 \\ (-1)^{s-3}(s-3)! & s \geqq 3, \end{cases}$$

$$\chi(\Gamma_1^s) = \begin{cases} -\dfrac{1}{12} & s \leqq 1 \\ \dfrac{(-1)^s(s-1)!}{12} & s \geqq 1, \end{cases}$$

$$\chi(\Gamma_g^s) = (-1)^s \frac{(2g+s-3)!}{2g(2g-2)!} B_{2g} \qquad g \geqq 2,\ s \geqq 0.$$

In this section we explain how to get the values for the ordinary, as opposed to orbifold, Euler characteristics $e(\Gamma_g^0)$ and $e(\Gamma_g^1)$ in terms of the numbers $\chi(\Gamma_g^s)$.

Define a group Γ to be *geometrically WFL* if there is a contractible, finite dimensional, proper Γ-complex Y such that there are only finitely many cells of $Y \bmod \Gamma$. Such a group is automatically WFL (virtually torsion-free such that for any torsion-free subgroup $\hat{\Gamma} < \Gamma$ of finite index there is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\hat{\Gamma}$; see [4], p. 226). Suppose that (i) Γ has finitely many conjugacy classes of elements of finite order and (ii) for every element σ of finite order in Γ the centralizer Z_σ of σ is geometrically WFL (including $\Gamma = Z_1$). A theorem of Brown [5] then says that

$$e(\Gamma) = \sum_{\langle\sigma\rangle} \chi(Z_\sigma),$$

where the sum is taken over all conjugacy classes $\langle\sigma\rangle$ of elements of finite order in Γ.

The mapping class groups Γ_g^s are well-known to be virtually torsion-free (see, e.g. [8]). Furthermore it is shown in [8] that Γ_g^s for $s \geqq 1$ is geometrically WFL (when $s=1$ an example of a Γ_g^1-complex is the complex Y of section 1). An alternative proof of this which works for all $s \geqq 0$ goes as follows.

Let $\mathcal{M}_g^s$ be the moduli space of all isometry classes of hyperbolic metrics (complete, finite area) on a surface F of genus g with s punctures. Also let $\mathcal{T}_g^s$ be the Teichmüller space of all equivalence classes of marked hyperbolic metrics on F. Then Γ_g^s acts properly discontinuously on $\mathcal{T}_g^s$ with quotient $\mathcal{M}_g^s$. A result of Mumford (see e.g. [1]) says that for all $\varepsilon > 0$, the subspace $\mathcal{M}_g^s(\varepsilon) \subset \mathcal{M}_g^s$ of all metrics for which the length of every closed geodesic is at least ε is compact. Furthermore, for ε small enough $\mathcal{M}_g^s(\varepsilon)$ is a deformation retract of $\mathcal{M}_g^s$.

Let $\mathcal{T}_g^s(\varepsilon)$ be the inverse image of $\mathcal{M}_g^s(\varepsilon)$ in $\mathcal{T}_g^s$, so that Γ_g^s acts on $\mathcal{T}_g^s(\varepsilon)$ with quotient $\mathcal{M}_g^s(\varepsilon)$. Choose a finite triangulation of $\mathcal{M}_g^s(\varepsilon)$ which is compatible with the stratification of $\mathcal{M}_g^s$ by symmetry types; that is, if Δ is an open k-simplex of $\mathcal{M}_g^s$ and $[X_1]$, $[X_2]$ are points of Δ, then the symmetry groups of the surfaces X_1, X_2 are the same. This triangulation will then lift to $\mathcal{T}_g^s(\varepsilon)$ which becomes the complex desired. Hence Γ_g^s is geometrically WFL for all g, s (actually, this proof requires $2g-2+s > 0$; the other cases are well-known).

Now, in order to apply Brown's theorem to $\Gamma = \Gamma_g^0$ or Γ_g^1 we must compute the centralizers of elements of finite order in Γ and show they are geometrically WFL. Consider Γ_g^1 first and let σ have finite order. A result of Nielsen [12] says that σ may be represented by a periodic homeomorphism f of F of order k which fixes the basepoint p. The quotient F/f is an orbifold of genus h with singular points $p_0, \ldots, p_s$; the p_i are the ramification points of the branched covering $\psi_f: F \to F/f$. Since f fixes p, $\psi_f(p)$ is a singular point, say $\psi_f(p) = p_0$. If B_0 denotes $F/f - \{p_i\}$ and F_0 denotes $\psi_f^{-1}(B_0)$, the covering $F_0 \to B_0$ is determined by a map $\omega_\sigma: H_1(B_0) \to \mathbb{Z}/k\mathbb{Z}$. Let γ_i, $0 \leqq i \leqq s$, denote the class in $H_1(B_0)$ represented by a circle around p_i. Define $\Gamma(F/f)$ to be the group of all isotopy classes of homeomorphisms f_1 of F/f which fix p_0, fix $\{p_1, \ldots, p_s\}$, may permute p_i and p_j $(i, j \geqq 1)$ when $\omega_\sigma(\gamma_i) = \omega_\sigma(\gamma_j)$, and satisfy $\omega_\sigma \circ f_1 = \omega_\sigma$.

Lemma 3. *There is an exact sequence*

$$1 \to \mathbb{Z}/k\mathbb{Z} \to N_\sigma \to \Gamma(F/f) \to 1$$

where N_σ is the normalizer of σ in Γ_g^1. The groups $\Gamma(F/f)$, N_σ and Z_σ are all geometrically WFL; in particular, $\chi(\Gamma(F/f))$, $\chi(N_\sigma)$ and $\chi(Z_\sigma)$ are all defined.

Proof. Let $\hat{\Gamma}_h^s$ denote the mapping class group defined as usual but with one basepoint p fixed and s other points $p_1, \ldots, p_s$ fixed setwise. Then $\hat{\Gamma}_h^s$ acts on $Y = \mathscr{T}_h^{s+1}(\varepsilon)$ (or on the complex Y constructed in [8]) and Y is structurally finite. Now $\Gamma(F/f)$ is a subgroup of finite index of $\hat{\Gamma}_h^s$ so it acts on Y and is therefore geometrically WFL. Furthermore, the exact sequence of the lemma gives an action of N_σ and Z_σ on Y so they are geometrically WFL. Thus it remains only to construct the exact sequence.

Let $\tau \in N_\sigma$ and write $\langle \sigma, \tau \rangle$ for the subgroup of Γ_g^1 generated by σ and τ. There is a short exact sequence

$$1 \to \mathbb{Z}/k\mathbb{Z} \to \langle \sigma, \tau \rangle \to \mathbb{Z}/n\mathbb{Z} \to 1$$

where τ has order $n \geqq 0$. The Nielsen conjecture is known for such groups when $n > 0$ by a result of Zieschang ([15], Theorem 54.7). It is also true when $n = 0$ by an argument of Kerckhoff (private communication) which is based on an analysis of the action of $\langle \sigma, \tau \rangle$ on Teichmüller space. This means we may find diffeomorphisms f representing σ and f_0 representing τ with $f^k = 1$, $f_0^n = 1$ and $f_0 f f_0^{-1} = f^r$ for some r. The map f_0 descends to a homeomorphism of F/f. This gives the map $N_\sigma \to \Gamma(F/f)$.

An element of $\Gamma(F/f)$ clearly lifts to an element of N_σ and the identity in $\Gamma(F/f)$ is covered only by powers of σ. The lemma follows. □

Now we turn to the computation of $e(\Gamma_g^1)$. Looking more closely at the branched cover $\psi: F \to F/f$, let the singular point p_i of F/f have type $n_i > 1$. By the Riemann-Hurwitz formula

$$2 - 2g = k\left(2 - 2h - (s+1) + \sum_{i=0}^{s} 1/n_i\right);$$

here h is again the genus of F/f and each n_i divides k. Let $r_i=\omega_\sigma(\gamma_i)$ (recall $\omega_\sigma\colon H_1(B_0)\to\mathbb{Z}/k\mathbb{Z}$ determines the covering $F_0\to B_0$ and γ_i is the class in $H_1(B_0)$ of a circle around p_i); $r_i\in\mathbb{Z}/k\mathbb{Z}$ and we have

$$(r_i,k)=\frac{k}{n_i}\ (0\leqq i\leqq s),\qquad \sum_{i=0}^{s} r_i\equiv 0 \bmod k. \tag{12}$$

It is easy to see that the existence of the data $\{h,k,s,n_i,\omega\}$ satisfying (12) is necessary and sufficient for the existence of σ in Γ_g^1. A map $f\colon F_g^1\to F_g^1$ with data $\{h,k,s,n_i,\omega\}$ is conjugate to a power of the map f' with data $\{h',k',s',n_i',\omega'\}$ if and only if $h=h'$, $k=k'$, $s=s'$, $\{n_i\}=\{n_i'\}$ and there is an automorphism λ of $H_1(B_0)$ such that $\omega'\circ\lambda=\omega$, $\{\lambda(\gamma_i)\}=\{\gamma_i'\}$ and whenever $\lambda(\gamma_i)=\gamma_j'$, $n_i=n_j'$.

To pass from Z_σ to N_σ, suppose Z_σ has index l in N_σ; then $\chi(Z_\sigma)=l\cdot\chi(N_\sigma)$. The map σ is conjugate to exactly l of its powers, so if S denotes a set of representatives of the conjugacy classes of $\{\sigma^n\colon (n,k)=1\}$ in Γ_g^1, then

$$\sum_{\tau\in S}\chi(Z_\tau)=\varphi(k)\cdot\chi(N_\sigma)$$

where φ is the Euler phi-function.

The lemma above allows us to pass form N_σ to $\Gamma(F/f)$; we have

$$\chi(N_\sigma)=\frac{1}{k}\cdot\chi(\Gamma(F/f)).$$

Finally, to pass from $\Gamma(F/f))$ to Γ_h^{s+1}, let Ω_B be the set of characters $H_1(B_0)\to\mathbb{Z}/k\mathbb{Z}$ which satisfy (12). The group $\hat{\Gamma}_h^s$ acts on Ω_B and the stabilizer of the element $\omega_\sigma\in\Omega_B$ corresponding as above to $f\colon F\to F$ is easily identified with $\Gamma(F/f)$. Therefore the orbit $\mathcal{O}(\omega_\sigma)$ has order $[\hat{\Gamma}_h^s\colon\Gamma(F/f)]$ and we have

$$\chi(\Gamma(F/f))=\#\mathcal{O}(\omega_\sigma)\cdot\chi(\hat{\Gamma}_h^s)=\#\mathcal{O}(\omega_\sigma)\cdot\frac{\chi(\Gamma_h^{s+1})}{s!}$$

since Γ_h^{s+1} is a subgroup of $\hat{\Gamma}_h^s$ of index $s!$.

To put this all together, fix the orbifold B and let Λ be the collection of conjugacy classes $\langle\sigma\rangle$ with σ the isotopy class of a map f with F/f isomorphic to B as an orbifold. Normalize by setting $r_0=1$; then

$$\begin{aligned}\sum_{\langle\sigma\rangle\in\Lambda}\chi(Z_\sigma)&=\frac{\varphi(k)}{k}\sum_{\langle\sigma\rangle\in\Lambda}\chi(\Gamma(F/f))\\&=\frac{\varphi(k)}{k}\cdot\frac{\chi(\Gamma_h^{s+1})}{s!}\cdot\sum_{\langle\sigma\rangle\in\Lambda}\#\mathcal{O}(\omega_\sigma)\\&=\frac{\varphi(k)}{k}\cdot\frac{\chi(\Gamma_h^{s+1})}{s!}\cdot\#\Omega_B.\end{aligned}$$

Since a character is determined by its value on $H_1(B_0)$, the cardinality of Ω_B equals k^{2h} (corresponding to the values on $\operatorname{Im}(H_1(B)\to H_1(B_0))$) times the number of $(s+1)$-tuples $(r_0,\dots,r_s)\in(\mathbb{Z}/k\mathbb{Z})^{s+1}$ with $r_0=1$ satisfying (12). Writing l_i for $\frac{k}{n_i}$, we have:

Theorem 4. *The Euler characteristic of Γ_g^1 is given by*

$$e(\Gamma_g^1) = \sum_{\substack{k \geqq 1,\, h \geqq 0,\, s \geqq 0 \\ l_1, \dots, l_s \mid k,\, l_i \neq k \\ 2g-1 = k(2h-1+s) - l_1 - \dots - l_s}} \frac{\varphi(k)}{k} \cdot \frac{\chi(\Gamma_h^{s+1})}{s!} \cdot k^{2h} N^1(k; l_1, \dots, l_s)$$

where

$$N^1(k; l_1, \dots, l_s) = \#\{(r_1, \dots, r_s) \in (\mathbb{Z}/k\mathbb{Z})^s \mid 1 + r_1 + \dots + r_s \equiv 0 (\operatorname{mod} k), (k, r_i) = l_i\}.$$

Similar arguments work for Γ_g except that to guarantee that the cover of B is connected we must add the requirement that the character $\omega\colon H_1(B_0) \to \mathbb{Z}/k\mathbb{Z}$ be surjective (this was automatic before because r_0 was prime to k). If a_i $(1 \leqq i \leqq 2h)$ are the values of ω on a basis of $\operatorname{Im}(H_1(B) \to H_1(B_0))$, and r_i $(1 \leqq i \leqq s)$ are as before the values on the γ_i, then this condition is simply g.c.d. $(a_1, \dots, a_{2h}, r_1, \dots, r_s, k) = 1$. Set $l_i = (k, r_i)$ as before; then for fixed $r_1, \dots, r_{2h}$ we must count the number of $2h$-tuples in $(\mathbb{Z}/h\mathbb{Z})^{2h}$ whose greatest common divisor is prime to $(l_1, \dots, l_s)$, and this number is clearly $k^{2h} \cdot \prod_{p \mid (l_1, \dots, l_s)} (1 - p^{-2h})$. Hence we have

Theorem 5.

$$e(\Gamma_g) = \sum_{\substack{k \geqq 1,\, h \geqq 0,\, s \geqq 0 \\ l_1, \dots, l_s \mid k,\, l_i \neq k \\ 2g-2 = k(2h-2+s) - l_1 - \dots - l_s}} \frac{1}{k} \cdot \frac{\chi(\Gamma_h^s)}{s!} k^{2h} \prod_{p \mid (l_1, \dots, l_s)} (1 - p^{-2h}) \cdot N(k; l_1, \dots, l_s),$$

where

$$N(k; l_1, \dots, l_s) = \#\{(r_1, \dots, r_s) \in (\mathbb{Z}/k\mathbb{Z})^s \mid r_1 + \dots + r_s \equiv 0 (\operatorname{mod} k), (r_i, k) = l_i\}.$$

Theorems 4 and 5 are already sufficient to compute $e(\Gamma_g^1)$ and $e(\Gamma_g)$ numerically. The computation of $e(\Gamma_g^1)$ for $g \leqq 3$ is illustrated in Table 2 (here we list the l_i in increasing order and include a multiplicity to count the permutations). As g grows, however, the number of terms to be considered becomes very large, so we would like to have closed formulas for the functions N^1 and N. Clearly $\varphi(k) N^1(k; l_1, \dots, l_s) = N(k; 1, l_1, \dots, l_s)$, so it suffices to treat N. Using the identity

$$\frac{1}{k} \sum_{\zeta^k = 1} \zeta^r = \begin{cases} 1 & \text{if } r \equiv 0 (\operatorname{mod} k) \\ 0 & \text{otherwise,} \end{cases}$$

we find

$$\begin{aligned} N(k; l_1, \dots, l_s) &= \frac{1}{k} \sum_{\zeta^k = 1} \sum_{\substack{r_1 \bmod k \\ (r_1, k) = l_1}} \cdots \sum_{\substack{r_s \bmod k \\ (r_s, k) = l_s}} \zeta^{r_1 + \dots + r_s} \\ &= \frac{1}{k} \sum_{\zeta^k = 1} \prod_{i=1}^{s} \Bigg(\sum_{\substack{r \bmod k \\ (r, k) = l_i}} \zeta^r \Bigg). \end{aligned}$$

Table 2. Computation of $e(\Gamma_g^1)$

	k	h	s	$l_1, \ldots, l_s$	Number of permutations	·	$\frac{\chi(\Gamma_h^{s+1}) \cdot \varphi(k) \cdot k^{2h-1}}{s!}$	·	$N^1(k; l_1, \ldots, l_s)$	=
$g=1$	1	1	0	–	1	·	−1/12	·	1	−1/12
	2	0	3	1, 1, 1	1	·	−1/12	·	1	−1/12
	3	0	2	1, 1	1	·	1/3	·	1	1/3
	4	0	2	1, 2	2	·	1/4	·	1	1/2
	6	0	2	2, 3	2	·	1/6	·	1	1/3
										1
$g=2$	1	2	0	–	1	·	1/120	·	1	1/120
	2	1	1	1	1	·	1/6	·	1	1/6
	2	0	5	1, 1, 1, 1, 1	1	·	−1/40	·	1	−1/40
	3	0	3	1, 1, 1	1	·	−1/9	·	3	−1/3
	4	0	3	1, 2, 2	3	·	−1/12	·	1	−1/4
	5	0	2	1, 1	1	·	2/5	·	3	6/5
	6	0	2	1, 2	2	·	1/6	·	1	1/3
	8	0	2	1, 4	2	·	1/4	·	1	1/2
	10	0	2	2, 5	2	·	1/5	·	1	2/5
										2
$g=3$	1	3	0	–	1	·	−1/252	·	1	−1/252
	2	1	3	1, 1, 1	1	·	1/6	·	1	1/6
	2	0	7	1, 1, 1, 1, 1, 1, 1	1	·	−1/84	·	1	−1/84
	3	1	1	1	1	·	1/2	·	1	1/2
	3	0	4	1, 1, 1, 1	1	·	1/18	·	5	5/18
	4	0	3	1, 1, 1	1	·	−1/12	·	4	−1/3
	4	0	4	1, 2, 2, 2	4	·	1/24	·	1	1/6
	6	0	3	1, 3, 3	3	·	−1/18	·	1	−1/6
	6	0	3	2, 2, 3	3	·	−1/18	·	1	−1/6
	7	0	2	1, 1	1	·	3/7	·	5	15/7
	8	0	2	1, 2	2	·	1/4	·	2	1
	9	0	2	1, 3	2	·	1/3	·	2	4/3
	12	0	2	1, 6	2	·	1/6	·	1	1/3
	12	0	2	3, 4	2	·	1/6	·	1	1/3
	14	0	2	2, 7	2	·	3/14	·	1	3/7
										6

Now for $l|k$ and ζ a primitive d^{th} root of unity, $d|k$, we have by an elementary calculation

$$\sum_{\substack{r \bmod k \\ (r,k)=l}} \zeta^r = \mu\left(\frac{d}{(d,l)}\right) \frac{\varphi(k/l)}{\varphi(d/(d,l))}$$

where φ and μ are the Möbius and Euler functions (Ramanujan sum). Denote this expression by $c(k,l,d)$. Since for each $d|k$ there are $\varphi(d)$ primitive d^{th} roots of unity among the k^{th} roots of unity, this gives the closed formulas

$$N(k; l_1, \ldots, l_s) = \frac{1}{k} \sum_{d|k} \varphi(d) \prod_{i=1}^{s} c(k, l_i, d)$$

and (since $c(k, 1, d) = \mu(d)\,\varphi(k)/\varphi(d)$)

$$N^1(k; l_1, \ldots, l_s) = \frac{1}{k} \sum_{d|k} \mu(d) \prod_{i=1}^{s} c(k, l_i, d).$$

These formulas can be used to calculate N and N^1 rapidly. Substituting the above expressions for $c(k, l, d)$, we find

$$N(k; l_1, \ldots, l_s) = \frac{1}{k} \varphi\left(\frac{k}{l_1}\right) \ldots \varphi\left(\frac{k}{l_s}\right) \sum_{d|k} \varphi(d) \prod_{i=1}^{s} \frac{\mu(d/(d, l_i))}{\varphi(d/(d, l_i))},$$

$$N^1(k; l_1, \ldots, l_s) = \frac{1}{k} \varphi\left(\frac{k}{l_1}\right) \ldots \varphi\left(\frac{k}{l_s}\right) \sum_{d|k} \mu(d) \prod_{i=1}^{s} \frac{\mu(d/(d, l_i))}{\varphi(d/(d, l_i))}.$$

We can simplify further by noting that the expressions in the sums are multiplicative in d, so that the sums can be written as products over prime divisors of k, viz.

$$\sum_{d|k} \mu(d) \prod_{i=1}^{s} \frac{\mu(d/(d, l_i))}{\varphi(d/(d, l_i))} = \prod_{p|k} \left(1 - \prod_{i=1}^{s} \frac{\mu(p/(p, l_i))}{\varphi(p/(p, l_i))}\right) = \prod_{p|k} \left(1 - \left(\frac{-1}{p-1}\right)^{\nu_p}\right)$$

(ν_p = number of i for which $p \nmid l_i$) and similarly

$$\sum_{d|k} \varphi(d) \prod_{i=1}^{s} \frac{\mu(d/(d, l_i))}{\varphi(d/(d, l_i))} = \prod_{p|k} p^{\lambda_p} \left(1 - \left(\frac{-1}{p-1}\right)^{\mu_p}\right)$$

(λ_p = largest λ such that $p^\lambda | l_i$ for all i, μ_p = number of i for which $p^{\lambda_p+1} \nmid l_i$). In particular $N^1(k; l_1, \ldots, l_s) = 0$ if $\nu_p = 0$ for some p and $N(k; l_1, \ldots, l_s) = 0$ if $\mu_p = 0$ for some i; these properties, of course, are clear from the definitions of N^1 and N.

Finally, we recast Theorems 4 and 5 into a more convenient form using generating functions. In Theorem 4, we have $k(2h-1+s) = 2g-1+l_1+\ldots+l_s \geqq 2g-1 \geqq 1$, so we cannot have $h=s=0$ or $h=0$, $s=1$. Conversely, given any $k \geqq 1$ and $s, h \geqq 0$ with $s+2h \geqq 2$, and any proper divisors $l_1, \ldots, l_s$ of k with $N^1(k; l_1, \ldots, l_s) \neq 0$, we have $k(2h-1+s) - l_1 - \ldots - l_s = 2g-1$ for some integer $g \geqq 1$. Indeed, the left-hand side is $\geqq 0$ because $l_i \leqq k/2$ and $(s, h) \neq (0, 0)$, $(1, 0)$, and odd because

$$k \text{ odd} \Rightarrow k(2h-1+s) - l_1 - \ldots - l_s \equiv 1 + s - s \equiv 1 \pmod 2$$

$$k \text{ even}, N^1(k; l_1, \ldots, l_s) \neq 0 \Rightarrow \nu_2 \text{ odd} \Rightarrow k(2h-1+s) - l_1 - \ldots - l_s \equiv \nu_2 \equiv 1.$$

Hence Theorem 4 can be rewritten as the formal power series identity

$$\sum_{g \geqq 1} e(\Gamma_g^1) t^{2g-1} = \sum_{\substack{k \geqq 1 \\ h, s \geqq 0 \\ s+2h \geqq 2}} \varphi(k) \frac{\chi(\Gamma_h^{s+1})}{s!} k^{2h-1} t^{k(2h-1)} \cdot \sum_{\substack{l_1, \ldots, l_s | k \\ l_i \neq k}} N^1(k; l_1, \ldots, l_s) t^{(k-l_1)+\ldots+(k-l_s)}.$$

Substituting the formula for N^1 given previously, we find that the inner sum equals

$$\frac{1}{k}\sum_{d|k}\mu(d)\sum_{\substack{l_1,\dots,l_s|k\\ l_i\neq k}} c(k,l_1,d)\,t^{k-l_1}\dots c(k,l_s,d)\,t^{k-l_s}=\frac{1}{k}\sum_{d|k}\mu(d)\Big(\sum_{\substack{l|k\\ l\neq k}} c(k,l,d)\,t^{k-l}\Big)^s.$$

Thus Theorem 4 is equivalent to

Theorem 4′. *The numbers $e(\Gamma_g^1)$ are given by the generating function*

$$\sum_{g\geq 1} e(\Gamma_g^1)\,t^{2g-1}=\sum_{\substack{d,k\geq 1\\ d|k}}\ \sum_{\substack{h,s\geq 0\\ s+2h\geq 2}} \frac{\chi(\Gamma_h^{s+1})}{s!}\,\mu(d)\,\varphi(k)\,k^{2h-2}\,\beta_{k,d}(t)^s\,t^{k(2h-1)}$$

where

$$\beta_{k,d}(t)=\sum_{r=1}^{k-1} e^{\frac{2\pi i r}{d}}\,t^{k-(k,r)}=\sum_{\substack{l|k\\ l\neq k}}\mu\left(\frac{d}{(d,l)}\right)\frac{\varphi(k/l)}{\varphi(d/(d,l))}\,t^{k-l}\in\mathbb{Z}[t].$$

The generating function in Theorem 4′ can be written

$$\sum_{k\geq 1}\frac{\varphi(k)}{k}\sum_{d|k}\mu(d)\,\Phi^1(\beta_{k,d}(t),\,kt^k)$$

where

$$\Phi^1(X,Y)=\sum_{\substack{h,s\geq 0\\ s+2h\geq 2}}\frac{1}{s!}\,\chi(\Gamma_h^{s+1})\,X^s\,Y^{2h-1}.$$

By the formulas for $\chi(\Gamma_g^s)$ at the beginning of this section, we have

$$\Phi^1(X,Y)=\sum_{s\geq 2}\frac{(-1)^s}{s(s-1)}\,X^s\,Y^{-1}+\sum_{\substack{h\geq 1\\ s\geq 0}}(-1)^{s-1}\binom{s+2h-2}{s}\frac{B_{2h}}{2h}\,X^s\,Y^{2h-1}$$

$$=\frac{1}{Y}\,((1+X)\log(1+X)-X)+\mathscr{B}\left(\frac{Y}{1+X}\right)$$

where $\mathscr{B}(T)=-\sum_{h\geq 1}\frac{B_{2h}}{2h}\,T^{2h-1}\in\mathbb{Q}[[T]]$. The power series $\mathscr{B}(T)$ is familiar from Stirling's formula for $\log\Gamma(x)$, which when differentiated says

$$\frac{\Gamma'(x)}{\Gamma(x)}\sim\log x-\frac{1}{2x}+\frac{1}{x}\,\mathscr{B}\left(\frac{1}{x}\right)\qquad(x\to\infty).$$

However, it is not clear whether these remarks can be used to simplify the power series on the right-hand side of Theorem 4′ and, in particular, to show directly that its coefficients are integers.

For Γ_g the situation is similar but more complicated. Here we find

$$\sum_{g \geq 1} e(\Gamma_g)\, t^{2g-2} = \sum_{\substack{k \geq 1 \\ h, s \geq 0 \\ s+2h \geq 3}} \frac{\chi(\Gamma_h^s)}{s!} k^{2h-1} t^{k(2h-2)} \cdot \sum_{\substack{l_1, \ldots, l_s | k \\ l_i \neq k}} \left(\sum_{p | (l_1, \ldots, l_s)} \left(1 - \frac{1}{p^{2h}}\right) \cdot N(k; l_1, \ldots, l_s)\, t^{k-l_1+\ldots+k-l_s} \right)$$

and now the inner sum equals

$$\sum_{\substack{l_1, \ldots, l_s | k \\ l_i \neq k}} \left(\sum_{m | (l_1, \ldots, l_s)} \frac{\mu(m)}{m^{2h}} \right) N(k; l_1, \ldots, l_s)\, t^{k-l_1+\ldots+k-l_s}$$

$$= \sum_{m | k} \frac{\mu(m)}{m^{2h}} \sum_{\substack{l_1, \ldots, l_s | k \\ l_i \neq k \\ m | l_i}} N(k; l_1, \ldots, l_s)\, t^{k-l_1+\ldots+k-l_s}$$

$$= \frac{1}{k} \sum_{m | k} \frac{\mu(m)}{m^{2h}} \sum_{d | k} \varphi(d) \left(\sum_{\substack{l | k \\ l \neq k \\ m | l}} c(k, l, d)\, t^{k-l} \right)^s.$$

Also

$$\sum_{\substack{l | k \\ l \neq k \\ m | l}} c(k, l, d)\, t^{k-l} = \sum_{\substack{l | k' \\ l \neq k'}} c(k, lm, d)\, t^{m(k'-l)} \qquad \left(k' = \frac{k}{m}\right)$$

$$= \sum_{\substack{l | k' \\ l \neq k'}} \mu\left(\frac{d}{(d, lm)}\right) \frac{\varphi(k'/l)}{\mu(d/(d, lm))}\, t^{m(k'-l)}$$

$$= \sum_{\substack{l | k' \\ l \neq k'}} \mu\left(\frac{d'}{(d', l)}\right) \frac{\varphi(k'/l)}{\mu(d'/(d', l))}\, t^{m(k'-l)} \qquad \left(d' = \frac{d}{(d, m)}\right)$$

$$= \beta_{k', d'}(t^m).$$

Hence Theorem 5 is equivalent to

Theorem 5′. *The numbers $e(\Gamma_g)$ are given by the generating function*

$$\sum_{g \geq 1} e(\Gamma_g)\, t^{2g-2} = \sum_{\substack{k \geq 1 \\ m, d | k}} \sum_{\substack{h, s \geq 0 \\ s+2h \geq 3}} \frac{\chi(\Gamma_h^s)}{s!} \frac{\mu(m)}{m^2} \varphi(d) \left(\frac{k}{m} t^k\right)^{2h-2} \beta_{\frac{k}{m}, \frac{d}{(d,m)}}(t^m)^s.$$

The expression on the right can also be put in the form

$$\sum_{k \geq 1} \sum_{m, d | k} \frac{\mu(m)}{m^2} \varphi(d)\, \Phi\left(\beta_{\frac{k}{m}, \frac{d}{(d,m)}}(t^m), \frac{k}{m} t^k\right),$$

where

$$\Phi(X,Y)=\sum_{\substack{h,s\geqq 0\\ s+2h\geqq 3}} \frac{\chi(\Gamma_h^s)}{s!} X^s Y^{2h-2}$$

$$=\sum_{s\geqq 3} \frac{(-1)^{s-1}}{s(s-1)(s-2)} X^s + \sum_{s\geqq 1} \frac{(-1)^s}{12s} X^s Y^2$$

$$+\sum_{h\geqq 2} \frac{B_{2h}}{2h(2h-2)} \left(\frac{Y}{1+X}\right)^{2h-2}.$$

Theorems 4′ and 5′ are much more convenient for computation than Theorems 4 and 5, since we no longer have the summations over s-tuples $(l_1, \ldots, l_s)$. Using them, we found the following values for $g \leqq 15$:

g	$e(\Gamma_g)$	$e(\Gamma_g^1)$
1	1	1
2	1	2
3	3	6
4	2	2
5	3	6
6	4	8
7	1	8
8	−6	−34
9	45	164
10	−86	−350
11	173	118
12	−100	4206
13	2641	−43770
14	−48311	919838
15	717766	−20261676

For comparison, the orbifold characteristics for genus 15 are

$$\chi(\Gamma_{15})=716167.5514\ldots, \qquad \chi(\Gamma_{15}^1)=-20052695.7966\ldots.$$

In general the terms of Theorems 4 or 5 with $k=1$ give numbers $\chi(\Gamma_g^1)$, $\chi(\Gamma_g)$ which grow roughly like g^{2g} (the exact asymptotic formulas were given in the introduction), while the terms with $k \geqq 2$ grow roughly like $g^{2g/k}$. Thus for $\Gamma = \Gamma_g$ or Γ_g^1 the formula for $e(\Gamma)$ consists of a very large main term $\chi(\Gamma)$ and an error term of about half as many digits. In particular $e(\Gamma) \sim \chi(\Gamma)$, so the Euler characteristics of both Γ_g and Γ_g' grow more than exponentially rapidly with g and take on positive and negative values infinitely often, indicating that Γ_g and Γ_g^1 have some very large Betti numbers and that these occur in both odd and even dimensions.

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