Moduli spaces in genus zero and inversion of power series

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Let $\mathcal{M}_{0,n}$ denote the moduli space Riemann surfaces of genus 0 with n ordered marked points. Its Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ is naturally partitioned into connected strata of the form

$$S \cong \mathcal{M}_{0,n_1} \times \cdots \times \mathcal{M}_{0,n_s},$$

indexed by the different topological types of stable curves with n marked points. The stable curves in the stratum above have s irreducible components and $s-1$ nodes; thus $\sum n_i = n + 2s - 2$.

This note provides a short proof of the following result, which shows that the universal formula for inversion of power series is encoded in the stratification of moduli space.

Theorem 1 *The formal inverse of* $f(x) = x - \sum_2^\infty a_n x^n/n!$ *is given by* $g(x) = x + \sum_2^\infty b_n x^n/n!$, *where*

$$b_n = \sum a_{n_1} \cdots a_{n_s} \times \left(\begin{array}{c} \text{the number of strata } S \subset \overline{\mathcal{M}}_{0,n+1} \\ \text{isomorphic to } \mathcal{M}_{0,n_1+1} \times \cdots \times \mathcal{M}_{0,n_s+1} \end{array} \right).$$

That is, $g(f(x)) = x$.

Here the coefficients of $f(x)$ and $g(x)$ are regarded as elements of the polynomial ring $\mathbb{Q}[a_2, a_3, \ldots]$, and the sum is over all $s \geq 1$ and all multi-indices $(n_1, \ldots, n_s)$ with $n_i \geq 2$.

Using basic properties of the Euler characteristic, we obtain:

Corollary 2 (Getzler) *The generating functions*

$$f(x) = x - \sum_{n=2}^\infty \chi(\mathcal{M}_{0,n+1}) \frac{x^n}{n!} \quad \text{and} \quad g(x) = x + \sum_{n=2}^\infty \chi(\overline{\mathcal{M}}_{0,n+1}) \frac{x^n}{n!}$$

are formal inverses of one another.

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It is easy to see that $a_n = \chi(\mathcal{M}_{0,n+1}) = (-1)^n(n-2)!$, using the fibration $\mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$. Thus by formally inverting $f(x)$, one can readily compute

$$\langle \chi(\overline{\mathcal{M}}_{0,n} \rangle_{n=3}^{\infty} = \langle 1, 2, 7, 34, 213, 1630, 14747, 153946, 1821473, \ldots \rangle.$$

Corollary 2 is a consequence of [Ge1, Thm. 5.9], stated explicitly in [LZ, Rmk. 4.5.3]. The development in [Ge1] uses operads and yields more information, such as Betti numbers for $\overline{\mathcal{M}}_{0,n}$. Theorem 1 shows that Corollary 2 holds for any generalized Euler characteristic on the Grothendieck ring of varieties over $\overline{\mathbb{Q}}$ (cf. [Bi]).

The proof of Theorem 1 will be based on simple properties of trees. Its aim is to provide an elementary entry point to the enumerative combinatorics of moduli spaces.

Trees. A tree τ is a finite, connected graph with no cycles; its vertices will be denoted $V(\tau)$. The degree function $d : V(\tau) \to \mathbb{N}$ gives the number of edges incident to each vertex. To each tree we associate the monomial

$$A(\tau) = \prod_{V(\tau)} A_{d(v)-1}$$

in the polynomial ring $\mathbb{Z}[A_1, A_2, A_3, \ldots]$, with the convention $A_0 = 1$.

A tree is *stable* if it has no vertices of degree 2. An *endpoint* of τ is a vertex with $d(v) = 1$. We say τ is *rooted* if it has a distinguished endpoint (the root). The number of endpoints of τ, other than its root, will be denoted $N(\tau)$. We always assume τ has *at least* one edge, so $N(\tau) \geq 1$; and the tree with *just one edge* is considered stable.

A *ribbon tree* is a rooted stable tree equipped with a cyclic ordering of the edges incident to each vertex. A ribbon structure records the same information as a planar embedding $\tau \hookrightarrow \mathbb{R}^2$ up to isotopy.

A *marked tree* is a rooted stable tree equipped with a labeling of its endpoints by the integers $1, 2, \ldots, N(\tau) + 1$. We require that the root is labeled 1.

Theorem 3 *The formal inverse of $F(x) = x - \sum_2^\infty A_n x^n$ is given by*

$$G(x) = \sum_{\text{ribbon } \tau} A(\tau) x^{N(\tau)}. \tag{1}$$

Here the sum is taken over all ribbon trees, up to isomorphism.

Proof. Suppose we are given ribbon trees $\tau_1, \ldots, \tau_d$ with $d \geq 2$. We can then construct a new ribbon tree τ by identifying the roots of these trees

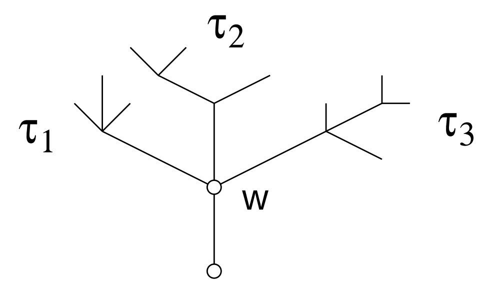

Figure 1. Three ribbon trees grafted together at their roots.

with a single vertex w, and adding a new edge leading from w to the root of τ (see Figure 1). The ribbon structure at w is determined by the ordering of the trees (τ_i), and by the condition that the root of τ lies between τ_d and τ_1.

Conversely, any ribbon tree with $N(\tau) \geq 2$ is obtained by applying this construction to the subtrees $(\tau_1, \ldots, \tau_d)$ leading away from the edge adjacent to its root. Taking into account the vertex w of degree $d+1$ where these trees are attached, we find:

$$A(\tau)x^{N(\tau)} = A_d \prod_{i=1}^{d} A(\tau_i)x^{N(\tau_i)}.$$

But the right hand side above is precisely one of the terms occurring in the expression $A_d G(x)^d$. Summing over all possible values for $d = d(w)$ we obtain

$$G(x) = x + \sum_{d=2}^{\infty} A_d G(x)^d,$$

where the first term accounts for the unique tree with $N(\tau) = 1$. Rearranging terms gives $F(G(x)) = x$. ∎

Corollary 4 *The formal inverse of* $f(x) = x - \sum_2^\infty a_n x^n/n!$ *is given by*

$$g(x) = \sum_{\text{marked } \tau} a(\tau) \frac{x^{N(\tau)}}{N(\tau)!}, \tag{2}$$

where $a(\tau) = \prod_{V(\tau)} a_{d(v)-1}$ *and* $a_0 = 1$.

Proof. The number of ribbon structures on a given stable rooted tree τ is given by $\prod(d(v)-1)!$. The group $\mathrm{Aut}(\tau)$ acts freely on the space of ribbon structures, so τ contributes $\prod(d(v)-1)!/|\mathrm{Aut}(\tau)|$ identical terms to equation (1) for $G(x)$. Similarly, τ contributes $N(\tau)!/|\mathrm{Aut}(\tau)|$ terms to equation (2) for $g(x)$. Setting $A_n = a_n/n!$, we find $F(x) = f(x)$ and

$$G(x) = \sum_{\text{marked } \tau} \frac{\prod(d(v)-1)!}{N(\tau)!} A(\tau) x^{N(\tau)} = g(x),$$

so $f(g(x)) = F(G(x)) = x$. ∎

Remark. The same reasoning shows (2) can be rewritten as

$$f^{-1}(x) = \sum_{\text{stable } \tau} \frac{N(\tau)+1}{|\mathrm{Aut}(\tau)|} a(\tau) x^{N(\tau)}.$$

For example, using the trees shown in Figure 2 we find

$$f^{-1}(x) = x + \frac{a_2}{2}x^2 + \frac{(a_3 + 3a_2^2)}{6}x^3 + \frac{(a_4 + 10a_2a_3 + 15a_2^3)}{24}x^4 + O(x^5).$$

For a quite different approach to Corollary 4, see [Ge2, Thm 1.3].

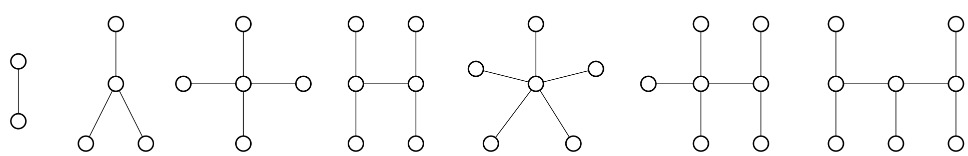

Figure 2. The stable trees with $N(\tau) \leq 4$.

Proof of Theorem 1. A stable curve $X \in \overline{\mathcal{M}}_{0,n+1}$ of genus zero determines a marked tree $t(X)$ whose interior vertices correspond to the irreducible components of X, and whose edges correspond to its nodes and labeled points. Conversely, any marked tree with $N(\tau) \geq 2$ can be realized by a stable curve, so the map

$$\tau \mapsto S(\tau) = \{X \in \overline{\mathcal{M}}_{0,N(\tau)+1} \ : \ t(X) \cong \tau\}$$

gives a bijection between marked trees with $N(\tau) \geq 2$ and the strata of moduli spaces. The desired inversion formula now follows from the preceding corollary. ∎

Proof of Corollary 2. Let $a_n = \chi(\mathcal{M}_{0,n+1})$. It is known that $\chi(X-Y)+\chi(Y) = \chi(X)$ whenever Y is a closed subvariety of a complex variety X [Ful, p.141, note 13], and that $\chi(A \times B) = \chi(A) \times \chi(B)$. The first property implies that $\chi(\overline{\mathcal{M}}_{0,n+1})$ is the sum of the Euler characteristics of its strata S, and the second implies that

$$\chi(S) = a_{n_1} \cdots a_{n_s}$$

whenever $S \cong \mathcal{M}_{0,n_1+1} \times \cdots \times \mathcal{M}_{0,n_s+1}$. Thus the stated relationship between generating functions follows from Theorem 1. ∎

Moduli space over $\mathbb{R}$. The real points of moduli space $\mathcal{M}_{0,n}(\mathbb{R})$ form a submanifold with $(n-1)!/2$ connected components, each homeomorphic to $\mathbb{R}^{n-3}$. Let M_n be the component of $\mathcal{M}_{0,n}(\mathbb{R})$ where the marked points can be chosen to lie in $\mathbb{R}$, with $x_1 < x_2 < \cdots < x_n$. Let $\overline{M}_n$ be the closure of M_n in $\overline{\mathcal{M}}_{0,n}$. The strata of $\overline{M}_n$ are encoded by ribbon trees, since the cyclic ordering of the points (x_i) is preserved under stable limits (cf. [De]). Thus in this setting, Theorem 3 yields:

Corollary 5 *The formal inverse of* $F(x) = x - \sum_2^\infty A_n x^n$ *is given by* $G(x) = x + \sum_2^\infty B_n x^n$*, where*

$$B_n = \sum A_{n_1} \cdots A_{n_s} \times \left(\begin{array}{c} \text{the number of strata } S \subset \overline{M}_{0,n+1} \\ \text{isomorphic to } M_{n_1+1} \times \cdots \times M_{n_s+1} \end{array} \right).$$

Notes and references. A compendium of results on trees, generating functions and inversion can be found in [St, Ch. 5]; see also [Ca]. For background on the many connections between graphs and moduli space, see e.g. [ACG, Ch. XVIII], [LZ] and the references therein.

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