

# COMPLETE SYMMETRIC POLYNOMIALS IN JUCYS-MURPHY ELEMENTS AND THE WEINGARTEN FUNCTION

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ABSTRACT. A connection is made between complete homogeneous symmetric polynomials in Jucys-Murphy elements and the unitary Weingarten function from random matrix theory. In particular we show that  $h_r(J_1, \dots, J_n)$ , the complete homogeneous symmetric polynomial of degree  $r$  in the JM elements, coincides with the  $r$ th term in the asymptotic expansion of the Weingarten function. We use this connection to determine precisely which conjugacy classes occur in the class basis resolution of  $h_r(J_1, \dots, J_n)$ , and to explicitly determine the coefficients of the classes of minimal height when  $r < n$ . These coefficients, which turn out to be products of Catalan numbers, are governed by the Moebius function of the non-crossing partition lattice  $NC(n)$ .

## 1. INTRODUCTION

Consider the tower

$$(1) \quad \{1\} = \mathbb{C}[S(1)] \subset \mathbb{C}[S(2)] \subset \dots \subset \mathbb{C}[S(n)] \subset \dots$$

of the group algebras of the symmetric groups, where  $\mathbb{C}[S(n-1)]$  is canonically embedded in  $\mathbb{C}[S(n)]$  as the linear span of permutations having  $n$  as a fixed point. Recently, Okounkov and Vershik [OV] have developed a novel approach to the representation theory of the symmetric groups in which a key role is played the the *Gelfand-Zetlin* algebra  $GZ(n)$ , which is by definition the algebra generated by  $Z(1), \dots, Z(n)$ , where  $Z(i)$  is the center of  $\mathbb{C}[S(i)]$ .

Okounkov and Vershik give a remarkably concrete description of the Gelfand-Zetlin algebra: they prove that  $GZ(n) = \mathbb{C}[J_1, \dots, J_n]$ , where the  $J_i$ 's are special elements of  $\mathbb{C}[S(n)]$  called the *Jucys-Murphy* elements. The JM elements are defined by  $J_1 := 0$  and

$$(2) \quad J_i = (1, i) + \dots + (i-1, i)$$

for  $2 \leq i \leq n$ . For example, the JM elements in  $\mathbb{C}[S(4)]$  are

$$\begin{aligned} J_1 &= 0 \\ J_2 &= (1, 2) \\ J_3 &= (1, 3) + (2, 3) \\ J_4 &= (1, 4) + (2, 4) + (3, 4). \end{aligned}$$

Given Okounkov and Vershik's characterization of  $GZ(n)$ , it is natural to ask for a characterization of  $Z(n)$  itself in terms of the JM elements. It was shown by Jucys [J] in 1974 that  $Z(n) = \text{Sym}[J_1, \dots, J_n]$ , the algebra of *symmetric* polynomials in

the JM elements. Jucys proved this remarkable fact by explicitly evaluating the elementary symmetric polynomials in JM elements.

For each partition (or Young diagram)  $\mu \vdash n$ , let  $\mathcal{C}_\mu$  denote the formal sum of all permutations in  $S(n)$  of cycle type  $\mu$ . Then  $\{\mathcal{C}_\mu\}_{\mu \vdash n}$  is a natural basis of  $Z(n)$ , called the *class basis*. If one thinks of  $Z(n)$  as the algebra of functions  $f : S(n) \rightarrow \mathbb{C}$  which are constant on conjugacy classes, then  $\mathcal{C}_\mu$  is the indicator function of the conjugacy class labelled by  $\mu$ .

**Theorem 1.1** ([J]). *For  $1 \leq r \leq n$ , let*

$$(3) \quad e_r(J_1, \dots, J_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} J_{i_1} J_{i_2} \dots J_{i_r}$$

*be the  $r$ th elementary symmetric polynomial in the JM elements, and put  $e_0 := 1$  (the identity element of  $\mathbb{C}[S(n)]$ ). Then for  $0 \leq r \leq n$ ,*

$$(4) \quad e_r(J_1, \dots, J_n) = \sum_{\substack{\mu \vdash n \\ \ell(\mu) = n-r}} \mathcal{C}_\mu.$$

In other words, Jucys' result states that  $e_r(J_1, \dots, J_n)$  is the indicator function of the set of permutations with exactly  $n - r$  cycles. For example, the class basis resolution of  $e_2(J_1, J_2, J_2, J_4)$  is the sum of all classes indexed by Young diagrams  $\mu \vdash 4$  with exactly 2 rows:

$$(5) \quad e_2(J_1, J_2, J_3, J_4) = \mathcal{C}_{(3,1)} + \mathcal{C}_{(2,2)}.$$

In symmetric function theory, the complete homogeneous symmetric polynomials are in many ways "dual" to the elementary symmetric polynomials. Recall that  $h_r \in \text{Sym}[x_1, \dots, x_n]$  are defined by  $h_0 := 1$  and

$$(6) \quad h_r := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n} x_{i_1} x_{i_2} \dots x_{i_r}$$

for  $r \geq 1$ . Clearly  $e_r = 0$  for  $r > n$ , but this is not the case for the  $h_r$ 's. Indeed one has the generating functions

$$(7) \quad E(t) = \sum_{r \geq 0} e_r t^r = (1 + x_1 t)(1 + x_2 t) \dots (1 + x_n t),$$

valid as an identity in  $\mathbb{C}[x_1, \dots, x_n, t]$ , and

$$(8) \quad H(t) = \sum_{r \geq 0} h_r t^r = (1 - x_1 t)^{-1} (1 - x_2 t)^{-1} \dots (1 - x_n t)^{-1},$$

valid as an identity in  $\mathbb{C}[x_1, \dots, x_n][[t]]$ . In particular, one has the identity  $E(t)H(-t) = 1$  in the latter ring.

Another example of the duality between  $e_r$  and  $h_r$  can be expressed in terms of antisymmetric and symmetric powers of matrices. If  $A$  is an  $n \times n$  matrix with eigenvalues  $x_1, \dots, x_n$ , then

$$(9) \quad e_r(x_1, \dots, x_n) = \text{Tr} \wedge^r A$$

while

$$(10) \quad h_r(x_1, \dots, x_n) = \text{Tr} \vee^r A,$$

see e.g. [B].

Given the dual nature of the elementary and complete homogeneous symmetric polynomials, one would hope for an analogue of Jucys result (Theorem 1.1) for  $h_r(J_1, \dots, J_n)$ . Below we will prove that, for any  $r \geq 0$ ,

$$(11) \quad h_r(J_1, \dots, J_n) = (-1)^r \sum_{\mu \vdash n} a_r(\mu) C_\mu,$$

where  $a_r$  is precisely the  $r$ th term in the asymptotic expansion of the *Weingarten function*, a special function which occurs in the theory of random unitary matrices.

Using known results about the Weingarten function, we characterize the conjugacy classes which appear in the class basis resolution of  $h_r(J_1, \dots, J_n)$ , and give exact formulas for some of the coefficients. The coefficients which we are able to determine explicitly correspond to “minimal classes,” and turn out to be governed by the Moebius function of the non-crossing partition lattice  $NC(n)$ .

## 2. THE WEINGARTEN FUNCTION

**2.1. Weingarten function in random matrix theory.** A random matrix may be described either “locally,” by giving the joint distribution of the matrix entries, or “globally” as a probability distribution on matrix space. For example, the Ginibre Unitary Ensemble is described locally as the class of random matrices whose entries are i.i.d. standard complex Gaussian random variables, while the Circular Unitary Ensemble is described globally in terms of the Haar probability measures on the unitary groups. In the latter case, a non-trivial problem is to recover the local description from the global one. This consists of determining the joint moments of the matrix entries of a Haar-distributed random unitary matrix, which is equivalent to evaluating all matrix integrals of the form

$$(12) \quad I_d(i, j, i', j') = \int_{\mathbb{U}_d} u_{i(1)j(1)} \dots u_{i(n)j(n)} \bar{u}_{i'(1)j'(1)} \dots \bar{u}_{i'(n)j'(n)} d\mathbf{U},$$

where

$$(13) \quad \mathbb{U}_d = \{\mathbf{U} \in GL_d(\mathbb{C}) : \mathbf{U}^* = \mathbf{U}^{-1}\}$$

is the unitary group equipped with normalized Haar measure  $d\mathbf{U}$ , and  $i, j, i', j' : \{1, \dots, n\} \rightarrow \{1, \dots, d\}$  are arbitrary functions. Of particular interest is the large  $d$  limit of the joint moments.

Considerable progress on this problem has been made by Collins [C] and Collins-Śniady [CS], who make use of the classical Schur-Weyl duality between representations of  $\mathbb{U}_d$  and  $S(n)$  in the tensors

$$(14) \quad \underbrace{\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d}_n,$$

to exhibit the existence of a central function  $Wg_d \in Z(n)$  with the following remarkable property:

$$(15) \quad I_d(i, j, i', j') = \sum_{\sigma, \tau \in S(n)} \prod_{k=1}^n \delta_{i(k)i'(\sigma(k))} \delta_{j(k)j'(\tau(k))} Wg_d(\tau\sigma^{-1}).$$

The function  $Wg_d$  is called the *Weingarten function*, and identity (15) is the *Weingarten convolution formula*. It is similar in spirit to the Wick formula for Gaussian

random variables. It follows immediately from (15) that  $Wg_d$  has the integral representation

$$(16) \quad Wg_d(\sigma) = \int_{U_d} u_{11} \dots u_{nn} \bar{u}_{1\sigma(1)} \dots \bar{u}_{n\sigma(n)} dU,$$

valid when  $d \geq n$ . In principle, the Weingarten convolution formula reduces the computation of the joint moments of the matrix entries of a random unitary matrix to the problem of explicitly describing the Weingarten function.

**2.2. Weingarten function and JM elements.** We will show that  $Wg_d$  is essentially the generating function for complete homogeneous symmetric polynomials in JM elements. Our approach is based on the following remarkable result due to Collins [C].

**Theorem 2.1** ([C]). *For any  $d \geq n$ ,  $Wg_d$  is an invertible element of  $Z(n)$ . The class basis resolution of  $Wg_d^{-1}$  is*

$$(17) \quad Wg_d^{-1} = \sum_{\mu \vdash n} d^{\ell(\mu)} \mathcal{C}_\mu.$$

Let  $H(t) \in \mathbb{C}[J_1, \dots, J_n][[t]]$  be the generating function for complete homogeneous symmetric polynomials in JM elements:

$$(18) \quad H(t) = \sum_{r \geq 0} h_r(J_1, \dots, J_n) t^r = (1 - J_1 t)^{-1} (1 - J_2 t)^{-1} \dots (1 - J_n t)^{-1}.$$

This is a formal power series in  $t$  with coefficients in the Gelfand-Zetlin subalgebra of  $\mathbb{C}[S(n)]$  (in fact in  $Z(n)$ ).

**Theorem 2.2** ([N2]). *For any  $d \geq n$ ,*

$$(19) \quad Wg_d = \frac{1}{d^n} H\left(-\frac{1}{d}\right) = (d + J_1)^{-1} (d + J_2)^{-1} \dots (d + J_n)^{-1}.$$

*Proof.* Observe that

$$\begin{aligned} d^n E\left(\frac{1}{d}\right) &= d^n \sum_{r=0}^n e_r(J_1, \dots, J_n) d^{-r} \\ &= \sum_{r=0}^n d^{n-r} \sum_{\substack{\mu \vdash n \\ \ell(\mu) = n-r}} \mathcal{C}_\mu \\ &= \sum_{\mu \vdash n} d^{\ell(\mu)} \mathcal{C}_\mu \\ &= Wg_d^{-1}, \end{aligned}$$

where the second line follows by applying Jucys' theorem. Since  $E(t)H(-t) = 1$ , the result follows.  $\square$

**2.3. Asymptotic expansion of  $Wg_d$  and the Moebius function.** Consider now the resolution of  $Wg_d$  with respect to the class basis of  $Z(n)$ :

$$(20) \quad Wg_d = \sum_{\mu \vdash n} Wg_d(\mu) \mathcal{C}_\mu.$$

Collins [C] demonstrated the existence of a sequence of central functions

$$(21) \quad \mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_r, \dots \in Z(\mathfrak{n})$$

with the property that

$$(22) \quad \text{Wg}_d(\mu) = \frac{\mathbf{a}_0(\mu)}{d^n} + \frac{\mathbf{a}_1(\mu)}{d^{n+1}} + \dots + \frac{\mathbf{a}_r(\mu)}{d^{n+r}} + \dots$$

for all  $\mu \vdash \mathfrak{n}$ . The series (22) is the *asymptotic expansion* of  $\text{Wg}_d$ .

Collins [C] has shown that the first order asymptotics of  $\text{Wg}_d$  are governed by the Moebius function of the non-crossing partition lattice  $\text{NC}(\mathfrak{n})$ . Recall that  $\text{NC}(\mathfrak{n})$  consists of the set of all non-crossing partitions of  $\{1, \dots, \mathfrak{n}\}$  under the reverse refinement partial order:  $\pi \leq \pi'$  in  $\text{NC}(\mathfrak{n})$  if and only if every block of  $\pi$  is contained in a block of  $\pi'$ .  $\text{NC}(\mathfrak{n})$  is a lattice with minimal element

$$(23) \quad \mathbf{0}_{\mathfrak{n}} = \{1\} \sqcup \{2\} \sqcup \dots \sqcup \{\mathfrak{n}\}$$

and maximal element

$$(24) \quad \mathbf{1}_{\mathfrak{n}} = \{1, 2, \dots, \mathfrak{n}\}.$$

Non-crossing partition lattices were first studied by Kreweras [K] from a purely combinatorial point of view. Later, Speicher discovered that non-crossing partition lattices play the same role in Voiculescu's free probability theory that full partition lattices play in classical probability, see [NS] for further information. The Moebius function of  $\text{NC}(\mathfrak{n})$  is determined as follows: for  $\pi = V_1 \sqcup \dots \sqcup V_\ell \in \text{NC}(\mathfrak{n})$ ,

$$(25) \quad \text{Moeb}([0_{\mathfrak{n}}, \pi]) = (-1)^{n-\ell} \prod_{i=1}^{\ell} \text{Cat}_{|V_i|-1},$$

where

$$(26) \quad \text{Cat}_{\mathfrak{N}} := \frac{1}{\mathfrak{N}+1} \binom{2\mathfrak{N}}{\mathfrak{N}}$$

is the Catalan number.

$\text{NC}(\mathfrak{n})$  is canonically embedded in the symmetric group  $S(\mathfrak{n})$  by mapping  $\pi = V_1 \sqcup \dots \sqcup V_\ell$  onto the permutation  $\text{perm}_\pi$  with cycle structure determined by  $V_1, \dots, V_\ell$  in the natural way. For example, the partition

$$(27) \quad \pi = \{1, 4, 5\} \sqcup \{2, 3\} \in \text{NC}(5)$$

is identified with the permutation

$$(28) \quad \text{perm}_\pi = (1, 4, 5)(2, 3) \in S(5).$$

Under this identification, the Moebius function becomes a central function on  $S(\mathfrak{n})$  whose class basis resolution is

$$(29) \quad \text{Moeb} = \sum_{\mu \vdash \mathfrak{n}} \text{Moeb}(\mu) \mathcal{C}_\mu,$$

where

$$(30) \quad \text{Moeb}(\mu) = \text{Moeb}(\mu_1, \dots, \mu_\ell) = (-1)^{n-\ell} \prod_{i=1}^{\ell} \text{Cat}_{\mu_i-1}$$

for each  $\mu \vdash \mathfrak{n}$ .

**Theorem 2.3** ([C]). *The coefficients  $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_r, \dots$  in the asymptotic expansion of  $\text{Wg}_d$  have the following properties:*

- for any  $r \geq 0$  and  $\mu \vdash n$ ,  $\mathbf{a}_r(\mu)$  is non-zero if and only if there exists an integer  $g \geq 0$  such that

$$(31) \quad \ell(\mu) = n - r + 2g.$$

- If  $\mu$  satisfies  $\ell(\mu) = n - r$ , then

$$(32) \quad \mathbf{a}_{n-\ell(\mu)}(\mu) = \text{Moeb}(\mu).$$

2.4. **The class expansion of  $\mathbf{h}_r(J_1, \dots, J_n)$ .** According to Theorem 2.2

$$(33) \quad \text{Wg}_d = \frac{1}{d^n} H\left(-\frac{1}{d}\right) = \sum_{r \geq 0} (-1)^r \mathbf{h}_r(J_1, \dots, J_n) \frac{1}{d^{n+r}}.$$

Since this is precisely the asymptotic expansion of  $\text{Wg}_d$ , we have the following result.

**Theorem 2.4.** *The coefficients in the asymptotic expansion of  $\text{Wg}_d$  are the complete homogeneous symmetric polynomials in JM elements:*

$$(34) \quad \mathbf{a}_r = (-1)^r \mathbf{h}_r(J_1, \dots, J_n).$$

*Equivalently the coefficients in the asymptotic expansion of  $\text{Wg}_d$  are the coefficients in the class basis resolution of complete homogeneous symmetric polynomials in JM elements:*

$$(35) \quad \mathbf{h}_r(J_1, \dots, J_n) = (-1)^r \sum_{\mu \vdash n} \mathbf{a}_r(\mu) \mathcal{C}_\mu.$$

Collins' theorem 2.3 on the asymptotic expansion of  $\text{Wg}_d(\mu)$  says that the coefficients  $\mathbf{a}_r(\mu)$  go down in steps of two, starting at  $\mathbf{a}_{n-\ell(\mu)}(\mu) = \text{Moeb}(\mu)$ .

**Corollary 2.5.** *The class basis resolution of  $\mathbf{h}_r(J_1, \dots, J_n)$  is of the form*

$$(36) \quad \mathbf{h}_r(J_1, \dots, J_n) = (-1)^r \sum_{g \geq 0} \sum_{\substack{\mu \vdash n \\ \ell(\mu) = n - r + 2g}} \mathbf{a}_r(\mu) \mathcal{C}_\mu.$$

When  $n > r \geq 1$ , this is

$$(37) \quad \mathbf{h}_r(J_1, \dots, J_n) = (-1)^r \sum_{\substack{\mu \vdash n \\ \ell(\mu) = n - r}} \text{Moeb}(\mu) \mathcal{C}_\mu + (-1)^r \sum_{g \geq 1} \sum_{\substack{\nu \vdash n \\ \ell(\nu) = n - r + 2g}} \mathbf{a}_r(\nu) \mathcal{C}_\nu.$$

As an example, consider the class basis resolution of  $\mathbf{h}_2(J_1, J_2, J_3, J_4)$  (recall that we computed  $\mathbf{e}_2(J_1, J_2, J_3, J_4)$  above as an example of Jucys' theorem). Since  $4 - 2 = 2$ , only classes of height 2 (corresponding to  $g = 0$ ) or 4 (corresponding to  $g = 1$ ) can occur in the resolution of  $\mathbf{h}_2(J_1, J_2, J_3, J_4)$ . We know that the coefficients of the "minimal"  $g = 0$  classes are given by the Moebius function of  $\text{NC}(n)$ . Thus

$$(38) \quad \mathbf{h}_2(J_1, J_2, J_3, J_4) = \underbrace{2\mathcal{C}_{(3,1)} + \mathcal{C}_{(2,2)}}_{g=0} + \underbrace{\mathbf{a}_2((1,1,1,1))\mathcal{C}_{(1,1,1,1)}}_{g=1}.$$

It is tempting to hypothesize that the coefficients  $\mathbf{a}_r(\mu)$  have a topological interpretation for arbitrary  $g \geq 0$ , as in [GJ] where it is found that the coefficients in the class basis resolution of transitive powers of JM elements are related to branched covers of the sphere. This is supported by the fact that the  $g = 0$  case in our setting is determined by the Moebius function of the lattice of partitions whose graphical representations can be embedded on the sphere.

## 3. CHARACTER EXPANSION AND CONTENT EVALUATION

**3.1. Character expansion via JM elements.** Another basis of the center  $Z(\mathfrak{n})$  of the group algebra  $\mathbb{C}[S(\mathfrak{n})]$  is provided by the set  $\{\chi^\lambda\}_{\lambda \vdash \mathfrak{n}}$  of irreducible characters of  $S(\mathfrak{n})$ . If  $f \in Z(\mathfrak{n})$  is a central function, its resolution

$$(39) \quad f = \sum_{\lambda \vdash \mathfrak{n}} f(\lambda) \chi^\lambda$$

with respect to the character basis of  $Z(\mathfrak{n})$  is called the *character expansion* of  $f$ . In this section we explicitly determine the character expansions of  $\text{Wg}_d$  and  $\mathfrak{h}_r(J_1, \dots, J_n)$ .

The character expansion of  $\text{Wg}_d$  can be deduced from Theorem 2.2 and the following fundamental property of JM elements, which is due to Jucys. Recall that the *content*  $c(\square)$  a cell  $\square$  in a Young diagram  $\lambda$  is the column index of  $\square$  subtract the row index of  $\square$ . For a Young diagram  $\lambda$ , we denote by  $A_\lambda$  the multiset (or “alphabet”) of its contents. For example, if  $\lambda = (4, 2, 1)$  then  $A_\lambda = \{0, 1, 2, 3, -1, 0, -2\}$ . For a symmetric polynomial  $f$  in  $\mathfrak{n}$  variables,  $f(A_\lambda)$  is the *content evaluation* of  $f$  at  $\lambda$ . For our example diagram this would be

$$(40) \quad f(0, 1, 2, 3, -1, 0, 2).$$

**Theorem 3.1** ([J]). *For any symmetric polynomial  $f$  in  $\mathfrak{n}$  variables,*

$$(41) \quad f(J_1, \dots, J_n) \chi^\lambda = f(A_\lambda) \chi^\lambda.$$

*That is,  $\chi^\lambda$  is an eigenvector for the linear operator “multiplication by  $f(J_1, \dots, J_n)$ ” on  $Z(\mathfrak{n})$  with corresponding eigenvalue  $f(A_\lambda)$ .*

Using Theorem 3.1 we can find the character expansion of the Weingarten function. Given a Young diagram  $\lambda \vdash \mathfrak{n}$ , let  $H_\lambda$  denote the product of hook-lengths of  $\lambda$ . Let

$$(42) \quad s_\lambda(1^d) = \frac{1}{H_\lambda} \prod_{\square \in \lambda} (d + c(\square))$$

be the Schur polynomial  $s_\lambda$  evaluated at  $(\underbrace{1, 1, \dots, 1}_d)$  (see [S]).

**Theorem 3.2.** *The character expansion of  $\text{Wg}_d \in \mathbb{C}[S(\mathfrak{n})]$  is*

$$(43) \quad \text{Wg}_d = \sum_{\lambda \vdash \mathfrak{n}} \frac{1}{H_\lambda^2 s_\lambda(1^d)} \chi^\lambda.$$

*Proof.* By the second orthogonality relation for the irreducible characters of a finite group, the character expansion of the unit  $1$  of  $Z(\mathfrak{n})$  is

$$1 = \frac{1}{\mathfrak{n}!} \sum_{\lambda \vdash \mathfrak{n}} \dim(\lambda) \chi^\lambda,$$

where

$$(44) \quad \dim(\lambda) = \frac{\mathfrak{n}!}{H_\lambda}$$

is the number of standard Young tableaux of shape  $\lambda$ . Therefore by Corollary 2.4 we have the following:

$$\begin{aligned}
\text{Wg}_d &= \sum_{r \geq 0} \frac{a_r}{d^{n+r}} \\
&= \sum_{r \geq 0} (-1)^r h_r(J_1, \dots, J_n) \left( \sum_{\lambda \vdash n} H_\lambda^{-1} \chi^\lambda \right) \frac{1}{d^{n+r}} \\
&= \sum_{\lambda \vdash n} H_\lambda^{-1} \left( \sum_{r \geq 0} (-1)^r h_r(A_\lambda) \frac{1}{d^{n+r}} \right) \chi^\lambda \\
&= \sum_{\lambda \vdash n} H_\lambda^{-1} \prod_{\square \in \lambda} (d + c(\square))^{-1} \chi^\lambda \\
&= \sum_{\lambda \vdash n} \frac{1}{H_\lambda^2 s_\lambda(1^d)} \chi^\lambda.
\end{aligned}$$

□

Theorem 3.2 was derived in [C] by a different method which does not involve JM elements. One benefit to our approach is that in the course of proving Theorem 3.2, we have shown that

$$(45) \quad a_r = (-1)^r \sum_{\lambda \vdash n} \frac{h_r(A_\lambda)}{H_\lambda} \chi^\lambda.$$

Thus we have the following.

**Theorem 3.3.** *The character expansion of  $h_r(J_1, \dots, J_n)$  is*

$$(46) \quad h_r(J_1, \dots, J_n) = \sum_{\lambda \vdash n} \frac{h_r(A_\lambda)}{H_\lambda} \chi^\lambda.$$

An interesting Corollary of this result is the formula

$$(47) \quad \text{Moeb}(\mu) = (-1)^{n-\ell(\mu)} \sum_{\lambda \vdash n} \frac{h_{n-\ell(\mu)}(A_\lambda)}{H_\lambda} \chi^\lambda(\mu)$$

for the Moebius function of  $\text{NC}(n)$ . This in turn gives an esthetically pleasing family of identities relating content evaluation, hook-products, characters, and Catalan numbers:

$$(48) \quad \sum_{\lambda \vdash n} \frac{h_{n-\ell(\mu)}(A_\lambda)}{H_\lambda} \chi^\lambda(\mu) = \prod_{i=1}^{\ell(\mu)} \text{Cat}_{\mu_i-1}.$$

#### 4. TABLES

Using Theorem 3.2, one can easily compute the asymptotic expansion of  $\text{Wg}_d \in \mathbb{Z}(n)$ , and thus the class basis resolution of  $h_r(J_1, \dots, J_n)$  for all  $r \geq 0$ , from the character table of  $\text{S}(n)$ . In this appendix we tabulate this data for  $n = 2, 3, 4$ .



4.1.  $n = 2$ .

- $\text{Wg}_d((1, 1)) = \frac{1}{d^2-1}$ 

$$= \frac{1}{d^2} + \frac{1}{d^4} + \frac{1}{d^6} + \frac{1}{d^8} + \frac{1}{d^{10}} + \frac{1}{d^{12}} \frac{1}{d^{14}} + \frac{1}{d^{16}} + \dots$$

- $\text{Wg}_d((2)) = \frac{-1}{d(d^2-1)}$ 

$$= -\frac{1}{d^3} - \frac{1}{d^5} - \frac{1}{d^7} - \frac{1}{d^9} - \frac{1}{d^{11}} - \frac{1}{d^{13}} - \frac{1}{d^{15}} - \dots$$

$$\begin{aligned} h_0(J_1, J_2) &= \mathcal{C}_{(1,1)} \\ h_1(J_1, J_2) &= \mathcal{C}_{(2)} \\ &\vdots \\ h_{2r}(J_1, J_2) &= \mathcal{C}_{(1,1)} \\ h_{2r+1}(J_1, J_2) &= \mathcal{C}_{(2)} \\ &\vdots \end{aligned}$$

4.2.  $n = 3$ .

- $\text{Wg}_d((1, 1, 1)) = \frac{d^2-2}{d(d^2-1)(d^2-4)}$ 

$$= \frac{1}{d^3} + \frac{3}{d^5} + \frac{11}{d^7} + \frac{43}{d^9} + \frac{171}{d^{11}} + \frac{683}{d^{13}} + \frac{2731}{d^{15}} + \dots$$

- $\text{Wg}_d((2, 1)) = \frac{-1}{(d^2-1)(d^2-4)}$ 

$$= -\frac{1}{d^4} - \frac{5}{d^6} - \frac{21}{d^8} - \frac{85}{d^{10}} - \frac{341}{d^{12}} - \frac{1365}{d^{14}} - \frac{5461}{d^{16}} - \dots$$

- $\text{Wg}_d((3)) = \frac{2}{d(d^2-1)(d^2-4)}$ 

$$= \frac{2}{d^5} + \frac{10}{d^7} + \frac{42}{d^9} + \frac{170}{d^{11}} + \frac{682}{d^{13}} + \frac{2730}{d^{15}} + \dots$$

$$\begin{aligned}
h_0(J_1, J_2, J_3) &= \mathcal{C}_{(1,1,1)} \\
h_1(J_1, J_2, J_3) &= \mathcal{C}_{(2,1)} \\
h_2(J_1, J_2, J_3) &= 3\mathcal{C}_{(1,1,1)} + 2\mathcal{C}_{(3)} \\
h_3(J_1, J_2, J_3) &= 5\mathcal{C}_{(2,1)} \\
h_4(J_1, J_2, J_3) &= 11\mathcal{C}_{(1,1,1)} + 10\mathcal{C}_{(3)} \\
h_5(J_1, J_2, J_3) &= 21\mathcal{C}_{(2,1)} \\
h_6(J_1, J_2, J_3) &= 43\mathcal{C}_{(1,1,1)} + 42\mathcal{C}_{(3)} \\
h_7(J_1, J_2, J_3) &= 85\mathcal{C}_{(2,1)} \\
h_8(J_1, J_2, J_3) &= 171\mathcal{C}_{(1,1,1)} + 170\mathcal{C}_{(3)} \\
h_9(J_1, J_2, J_3) &= 341\mathcal{C}_{(2,1)} \\
h_{10}(J_1, J_2, J_3) &= 683\mathcal{C}_{(1,1,1)} + 682\mathcal{C}_{(3)} \\
h_{11}(J_1, J_2, J_3) &= 1365\mathcal{C}_{(2,1)} \\
&\vdots
\end{aligned}$$

4.3.  $n = 4$ .

- $Wg_d((1, 1, 1, 1)) = \frac{6-8d^2+d^4}{d^2(-36+49d^2-14d^4+d^6)}$ 

$$= \frac{1}{d^4} + \frac{6}{d^6} + \frac{41}{d^8} + \frac{316}{d^{10}} + \frac{2631}{d^{12}} + \frac{22826}{d^{14}} + \frac{202021}{d^{16}} + \dots$$
- $Wg_d((2, 1, 1)) = \frac{1}{9d-10d^3+d^5}$ 

$$= \frac{1}{d^5} + \frac{10}{d^7} + \frac{91}{d^9} + \frac{820}{d^{11}} + \frac{7381}{d^{13}} + \frac{66430}{d^{15}} + \frac{597871}{d^{17}} + \dots$$
- $Wg_d((2, 2)) = \frac{6+d^2}{d^2(-36+49d^2-14d^4+d^6)}$ 

$$= \frac{1}{d^6} + \frac{20}{d^8} + \frac{231}{d^{10}} + \frac{2290}{d^{12}} + \frac{21461}{d^{14}} + \frac{196560}{d^{16}} + \frac{1782691}{d^{18}} + \dots$$
- $Wg_d((3, 1)) = \frac{-3+2d^2}{d^2(-36+49d^2-14d^4+d^6)}$ 

$$= \frac{2}{d^6} + \frac{25}{d^8} + \frac{252}{d^{10}} + \frac{2375}{d^{12}} + \frac{21802}{d^{14}} + \frac{197925}{d^{16}} + \frac{1788152}{d^{18}} + \dots$$
- $Wg_d((4)) = \frac{5}{d(-36+49d^2-14d^4+d^6)}$ 

$$= \frac{5}{d^7} + \frac{70}{d^9} + \frac{735}{d^{11}} + \frac{7040}{d^{13}} + \frac{65065}{d^{15}} + \frac{592410}{d^{17}} + \frac{5358995}{d^{19}} + \dots$$

$$\begin{aligned}
h_0(J_1, J_2, J_3, J_4) &= \mathcal{C}_{(1,1,1,1)} \\
h_1(J_1, J_2, J_3, J_4) &= \mathcal{C}_{(2,1,1)} \\
h_2(J_1, J_2, J_3, J_4) &= 6\mathcal{C}_{(1,1,1,1)} + \mathcal{C}_{(2,2)} + 2\mathcal{C}_{(3,1)} \\
h_3(J_1, J_2, J_3, J_4) &= 10\mathcal{C}_{(2,1,1)} + 5\mathcal{C}_{(4)} \\
h_4(J_1, J_2, J_3, J_4) &= 41\mathcal{C}_{(1,1,1,1)} + 20\mathcal{C}_{(2,2)} + 25\mathcal{C}_{(3,1)} \\
h_5(J_1, J_2, J_3, J_4) &= 91\mathcal{C}_{(2,1,1)} + 70\mathcal{C}_{(4)} \\
h_6(J_1, J_2, J_3, J_4) &= 316\mathcal{C}_{(1,1,1,1)} + 231\mathcal{C}_{(2,2)} + 252\mathcal{C}_{(3,1)} \\
h_7(J_1, J_2, J_3, J_4) &= 820\mathcal{C}_{(2,1,1)} + 735\mathcal{C}_{(4)} \\
h_8(J_1, J_2, J_3, J_4) &= 2631\mathcal{C}_{(1,1,1,1)} + 2290\mathcal{C}_{(2,2)} + 2375\mathcal{C}_{(3,1)} \\
h_9(J_1, J_2, J_3, J_4) &= 7381\mathcal{C}_{(2,1,1)} + 7040\mathcal{C}_{(4)} \\
h_{10}(J_1, J_2, J_3, J_4) &= 22826\mathcal{C}_{(1,1,1,1)} + 21461\mathcal{C}_{(2,2)} + 21802\mathcal{C}_{(3,1)} \\
h_{11}(J_1, J_2, J_3, J_4) &= 66430\mathcal{C}_{(2,1,1)} + 65065\mathcal{C}_{(4)} \\
h_{12}(J_1, J_2, J_3, J_4) &= 202021\mathcal{C}_{(1,1,1,1)} + 196560\mathcal{C}_{(2,2)} + 197925\mathcal{C}_{(3,1)}.
\end{aligned}$$

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