Trees, maps and Hurwitz numbers

GILLES SCHAEFFER CNRS & École Polytechnique

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Séminaire Lotharingien de Combinatoire, mars 2012

General summary of the 3 lectures:

Factorizations, maps and ramified coverings

Orientations and decompositions of maps into trees

Applications to Hurwitz numbers

First lecture

Factorizations, maps and ramified coverings

Permutations, factorizations and increasing maps

Hurwitz original motivation, ramified coverings

Ramified coverings provide bijections "for free"

Permutations, factorizations, increasing maps

Permutation factorizations

Permutations in cycle notation: $\sigma = (1,2,5)(3,6)(4)(7) = (1,2,5)(3,6)$

Cycle type = distribution of cycle lengths: $\lambda(\sigma) = 1^2\,2\,3$

Transpositions = permutations with type $\lambda = 2\,1^{n-1}$: $\tau = (2,5)$.

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Factorisation in tranpositions = decomposition as product of transpositions

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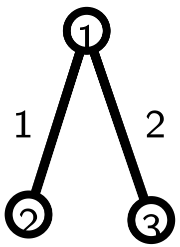
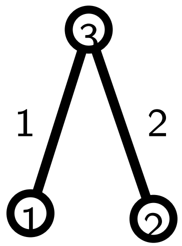
The graph of a factorization $\tau_1 \ldots \tau_m = \sigma \in S_n$:

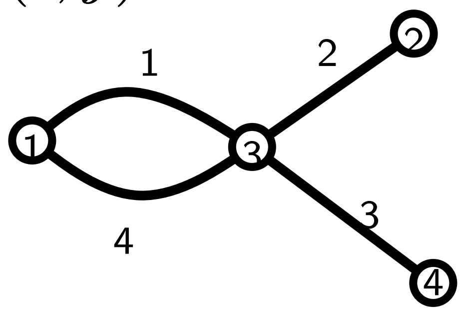
– vertices represent the permuted elements: $\{1, \ldots, n\}$

– edges represent transpositions:

an edge (i,j) with index k if $\tau_k = (i,j)$







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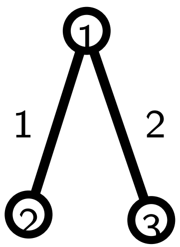

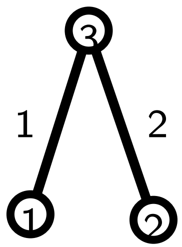

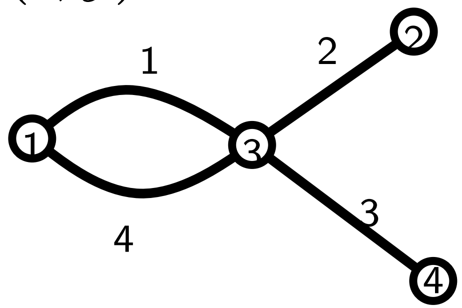

Transitive factorization = connected graph

Factorizations of n-cycles (Dénes 1959)

Lemma. If σ has ℓ cycles then $\sigma' = \sigma \cdot (i, j)$ has

- $\ell - 1$ cycles if i and j are in different cycles of σ
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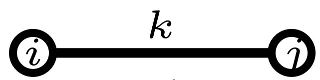
- The product $\tau_1 \dots \tau_{n-1}$ is a n-cycle if and only if the associated graph is a tree.



$\Updownarrow$

$\tau_k = (i, j)$

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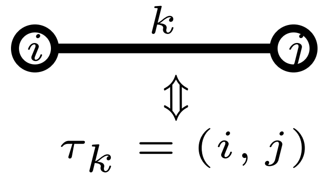

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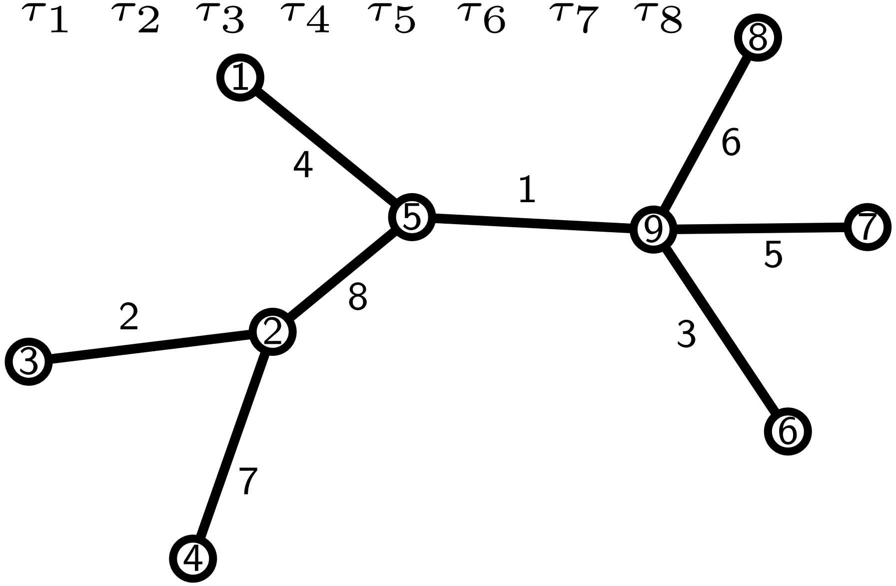

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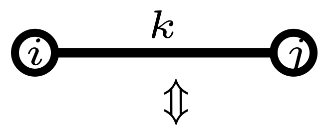

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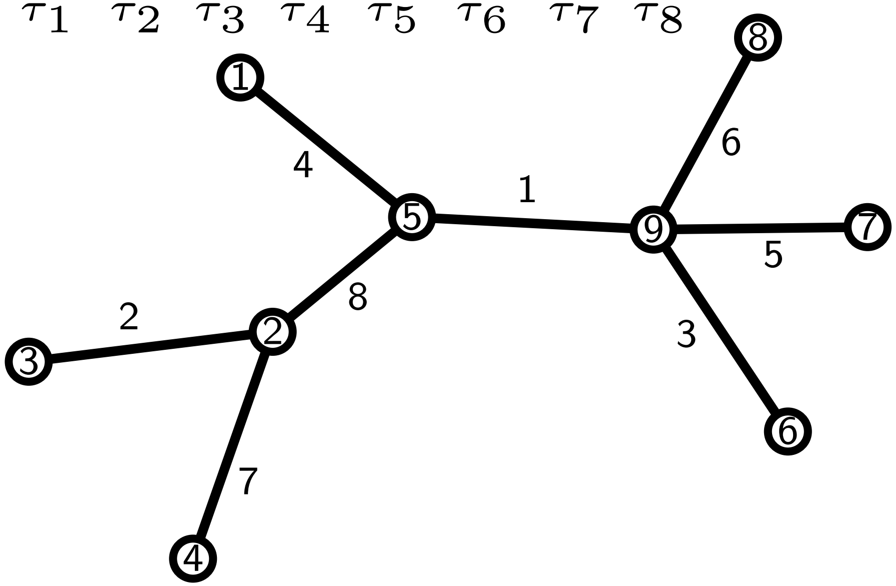

Cayley trees with n nodes (non-embedded) n^{n-2}

edge indexing $(n-1)!$

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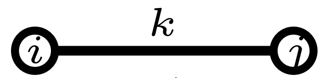

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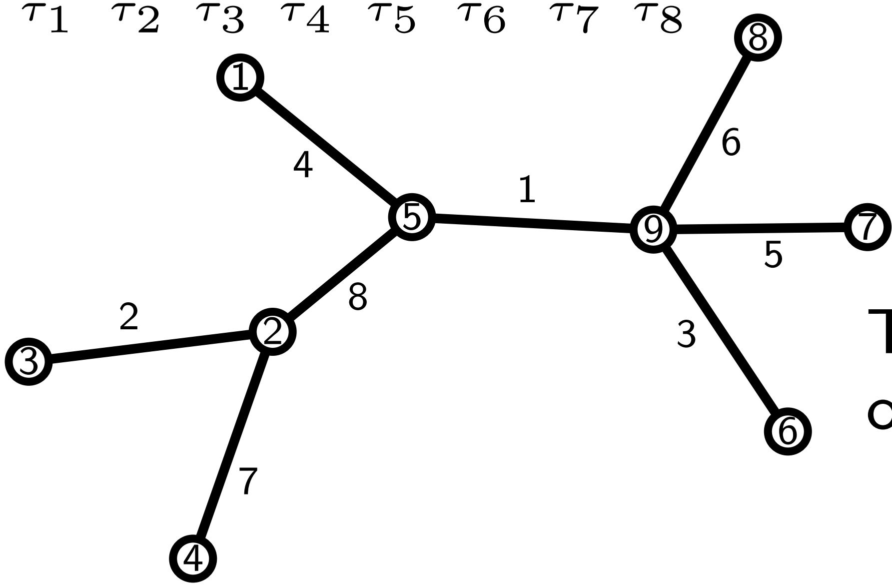

Cayley trees with n nodes (non-embedded) $\quad n^{n-2}$

edge indexing $\quad (n-1)!$

The number of minimal factorizations of n-cycles is $n^{n-2} \cdot (n-1)!$

Minimal factorizations

Proposition: Let $\lambda = 1^{\ell_1} \dots n^{\ell_n}$ with $\sum_i \ell_i = \ell$. A minimal factorization of a permutation of cycle type λ has $m = n - \ell$ factors.

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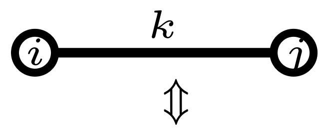

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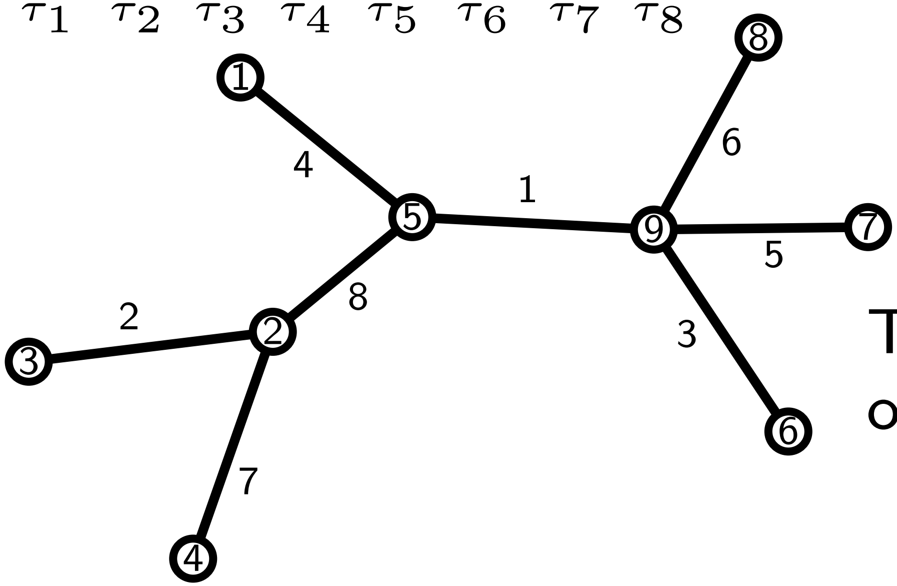

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Their number is $\frac{n!}{\prod_i \ell_i! i^{\ell_i}} \prod_i (i^{i-2})^{\ell_i} \frac{m!}{\prod_i (i-1)!^{\ell_i}} = m! n! \prod_i \frac{1}{\ell_i!} \left(\frac{i^{i-2}}{i!} \right)^{\ell_i}$.

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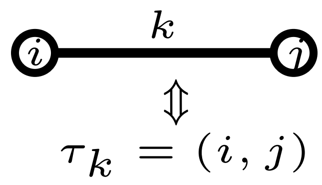

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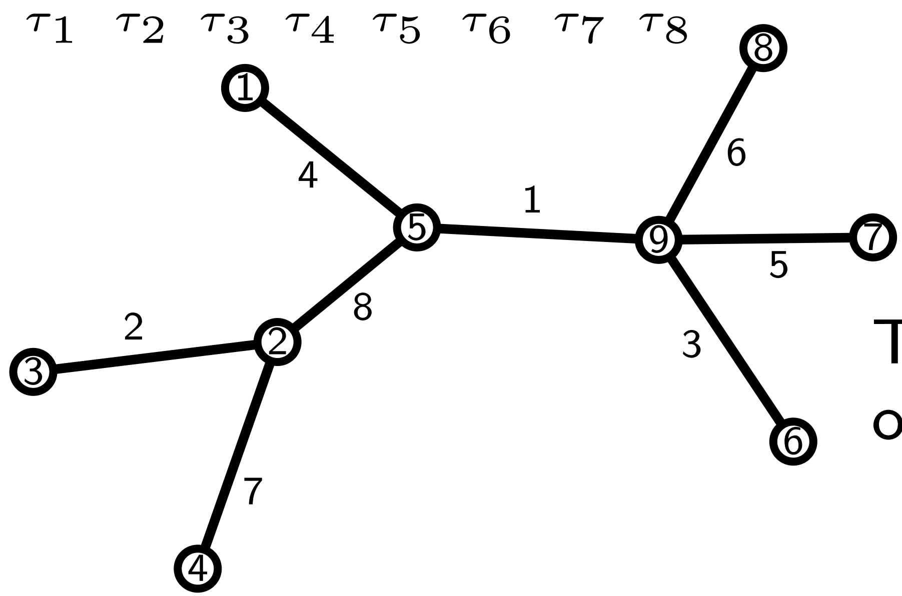

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The number of minimal factorizations of n-cycles is $n^{n-2} \cdot (n-1)!$

Hurwitz formula for the number of minimal transitive factorizations in transpositions

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition n, and $\ell = \sum_i \ell_i$.
The number of m-uples of transpositions $(\tau_1, \ldots, \tau_m)$ such that

- (product cycle type) $\tau_1 \cdots \tau_m = \sigma$ has cycle type λ
- (transitivity) the associated graph is connected
- (minimality) the number of factors is $m = n + \ell - 2$

is

$$n^{\ell-3} \cdot m! \cdot n! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left(\frac{i^i}{i!}\right)^{\ell_i}$$

Proofs:

(Hurwitz 1891, Strehl 1996) (recurrences, Abel identities)
(Goulden–Jackson 1997) (gfs and differential eqns)
(Lando–Zvonkine 1999) (geometry of LL mapping)
(Bousquet-Mélou–Schaeffer 2000) (bijection + inclusion/exclusion)

$\lambda = n$, factorizations of n-cycles: $n^{n-2} \cdot (n-1)!$

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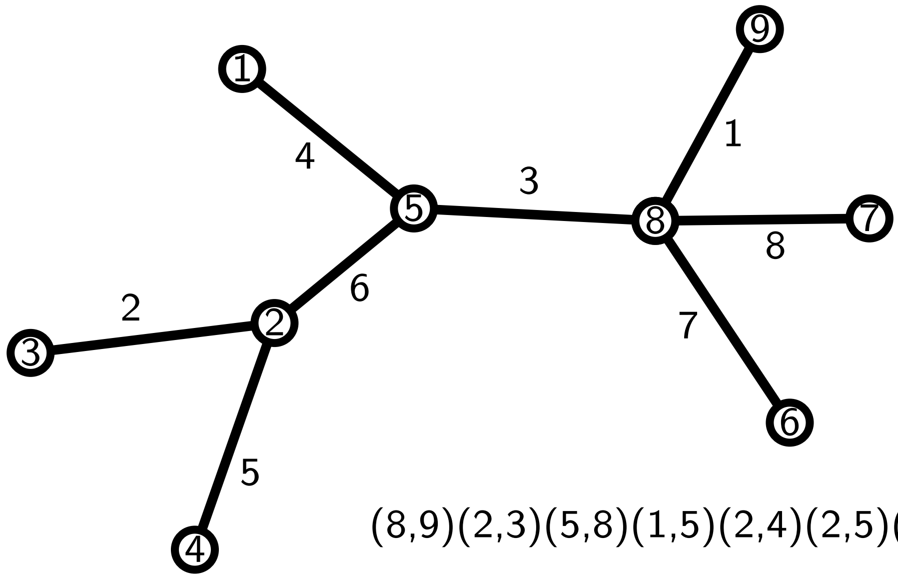
Combinatorial interpretation and proof?

Computation of the product and increasing embedding

How do we compute the product directly on the graph



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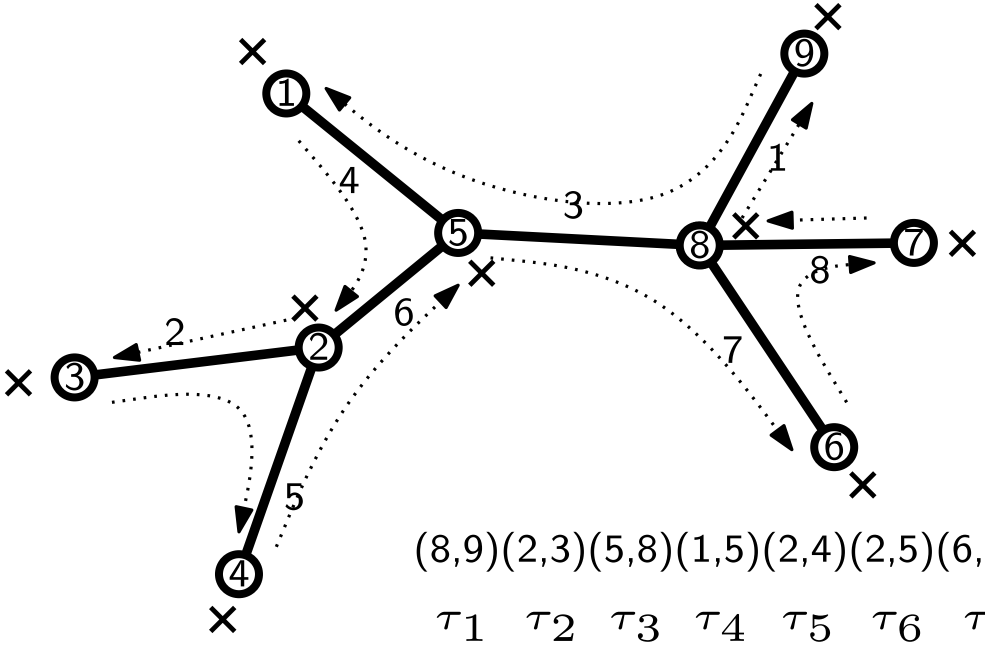

$1 \longrightarrow 5 \longrightarrow 2$

$2 \longrightarrow 3$

$\vdots$

$9 \longrightarrow 8 \longrightarrow 5 \longrightarrow 1$

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$i \quad \cdots\cdots\cdots\cdots\cdots\cdots \blacktriangleright \quad \sigma(i)$

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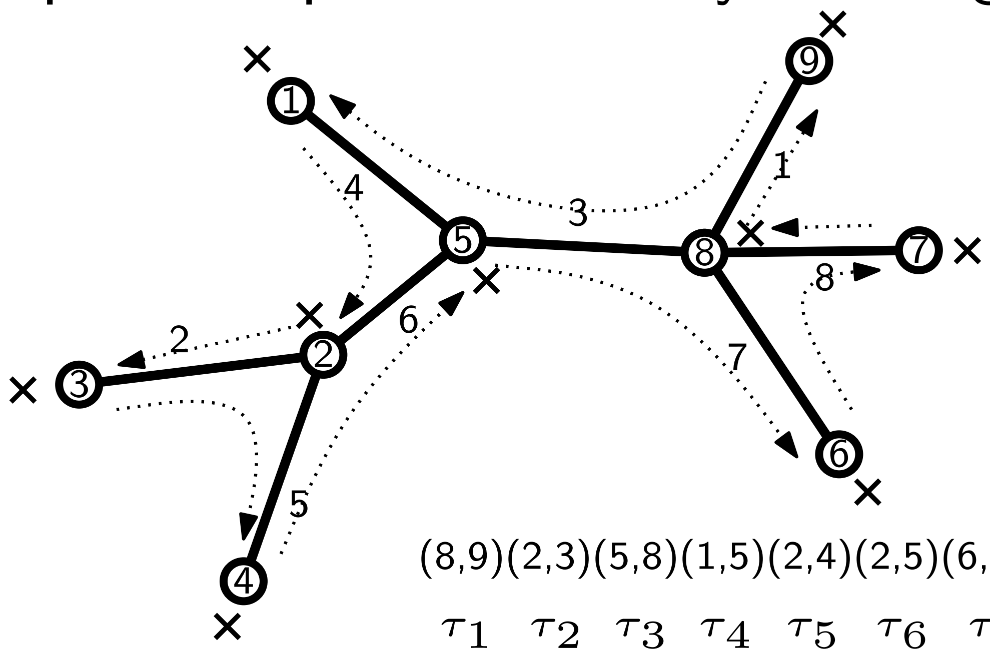

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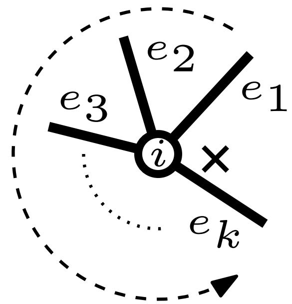
$i \longrightarrow \sigma(i)$

The computation is performed along the boundary of the graph because I have consistently drawn edges increasingly in counterclockwise direction around each vertex

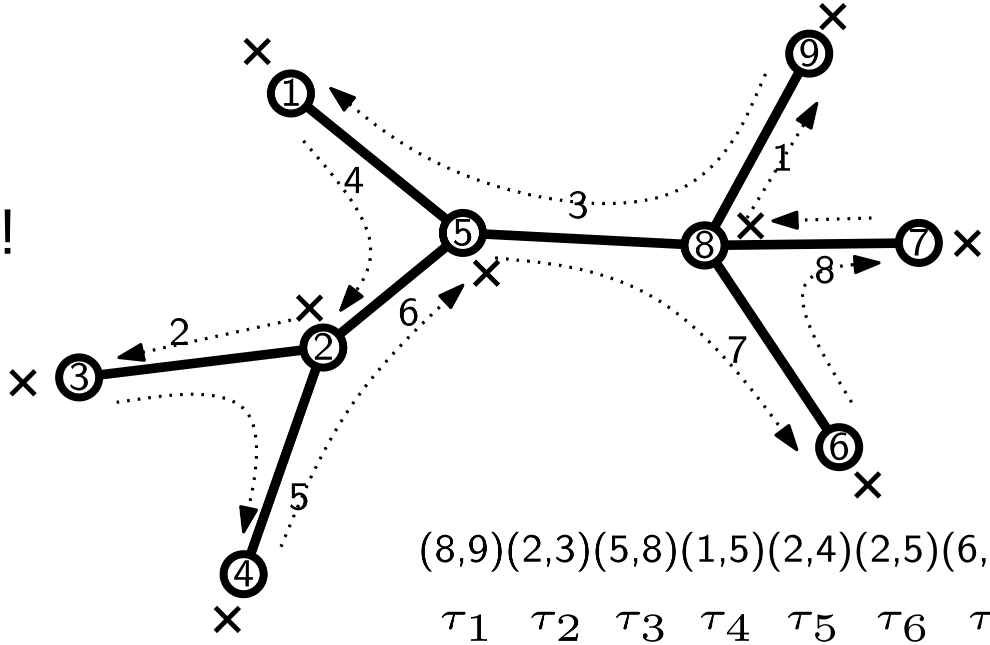
$$e_1 < e_2 < \ldots < e_k$$



Computation of the product and increasing embedding

How do we compute the product directly on the graph

Stop on crosses!



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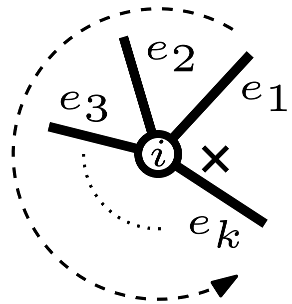

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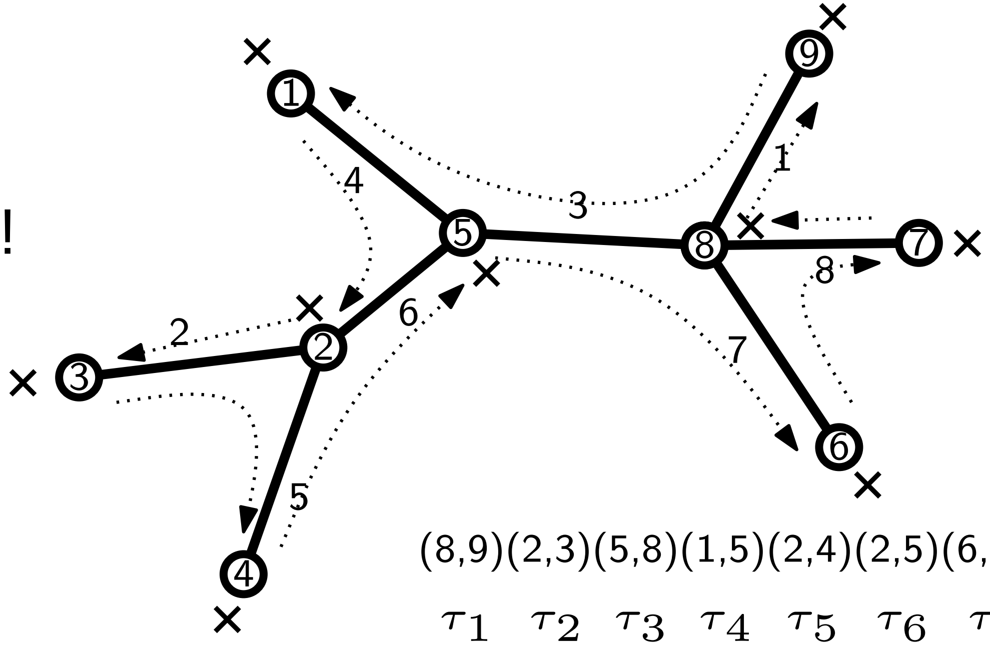

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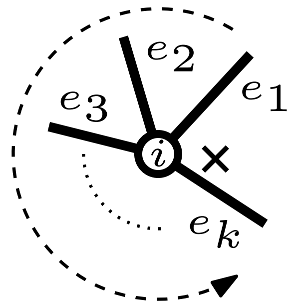
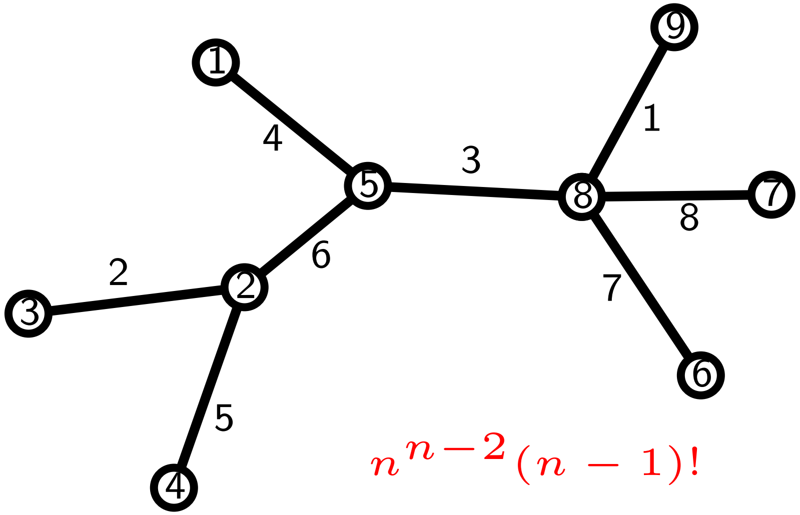
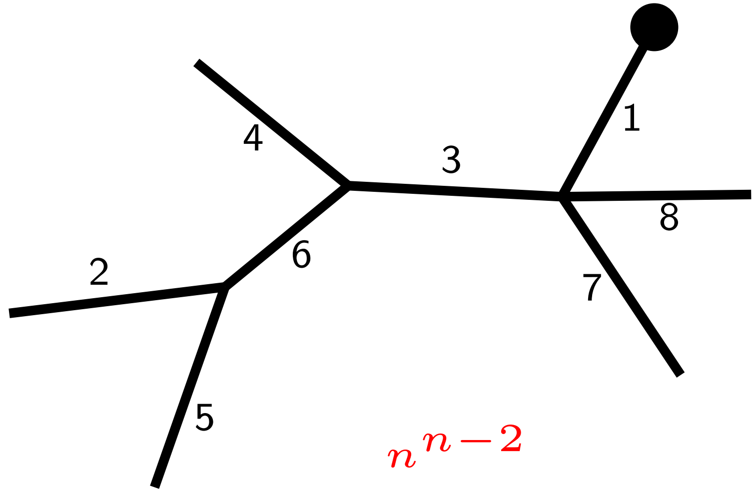


Any tree with indexed edges has a unique such increasing embedding

Moszkowski's proof
for factorizations of an n-cycle



$n^{n-2}(n-1)!$



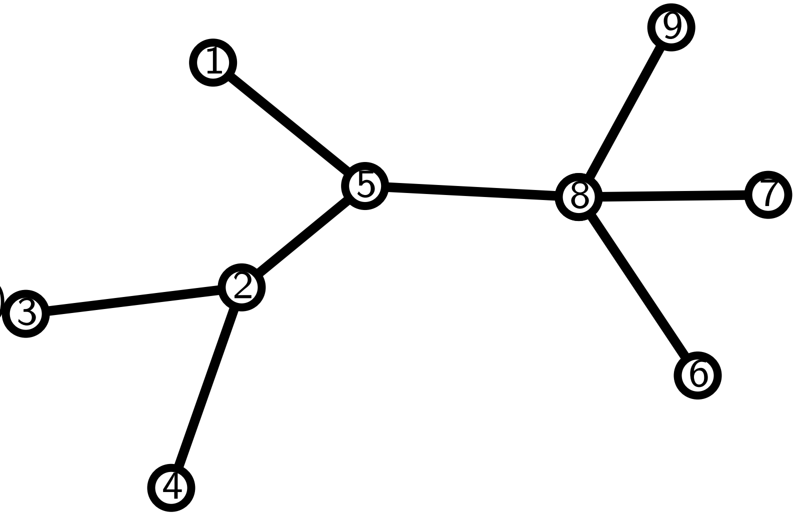
n^{n-2}

pointed indexed tree
with n vertices

(admit a unique embedding
as increasing plane trees)

n^{n-2}

Cayley tree
with n vertices
(non embedded)



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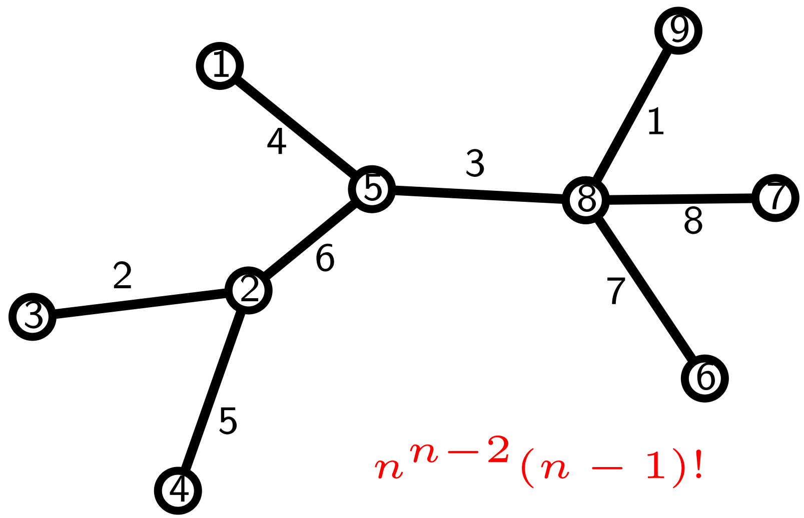

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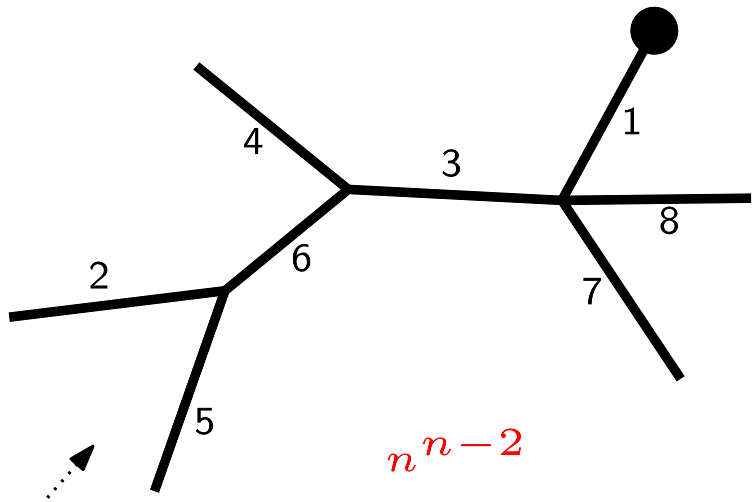

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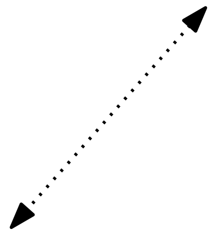

n^{n-2}

Cayley tree
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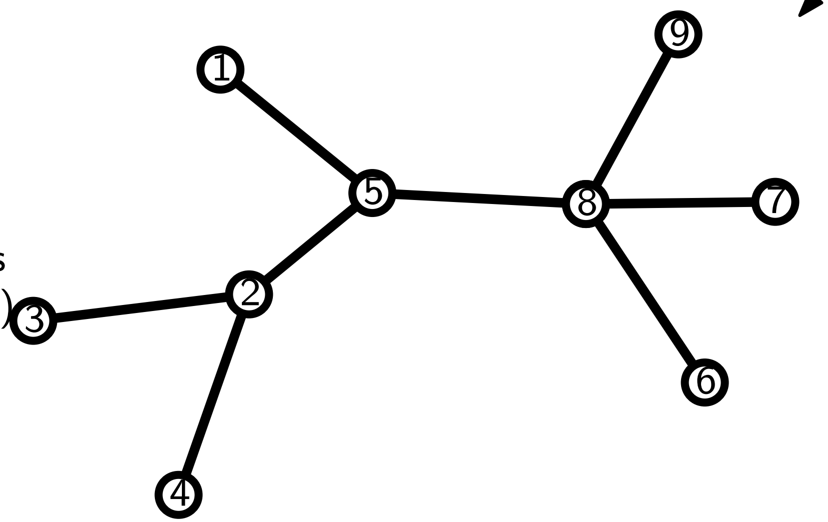
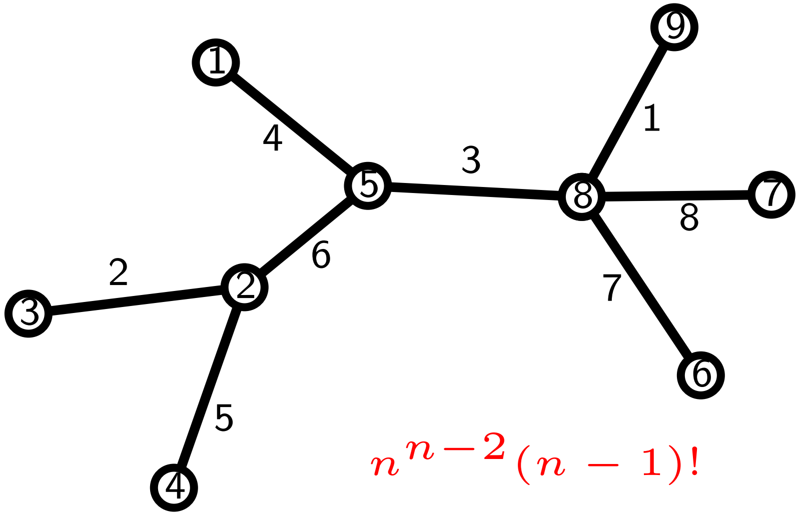
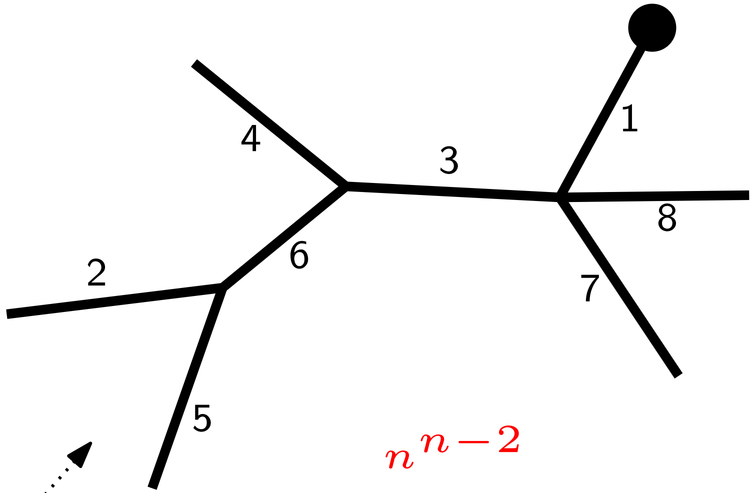


Moszkowski's proof for factorizations of an n-cycle



$n^{n-2}(n-1)!$

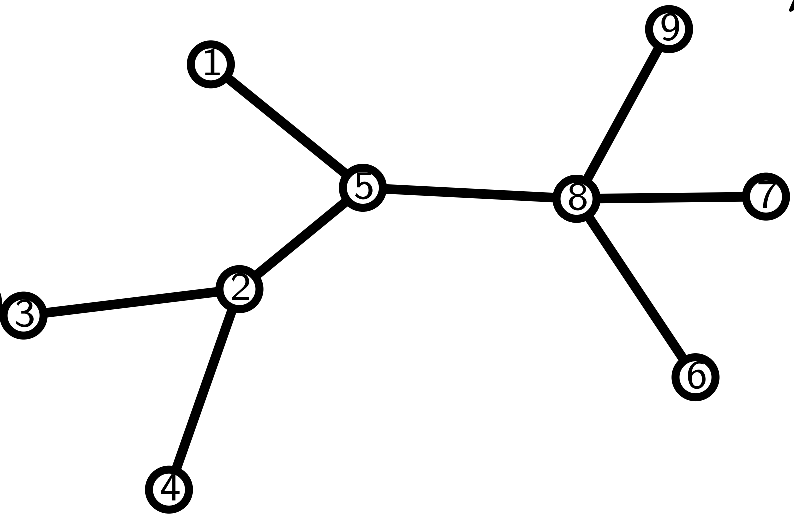


n^{n-2}

pointed indexed tree
with n vertices

(admit a unique embedding
as increasing plane trees)

n^{n-2}
Cayley tree
with n vertices
(non embedded)



Theorem (Moszkowski, 1989). Bijection between Cayley trees with n vertices and minimal factorizations in transpositions of $(1, 2, \dots, n)$.

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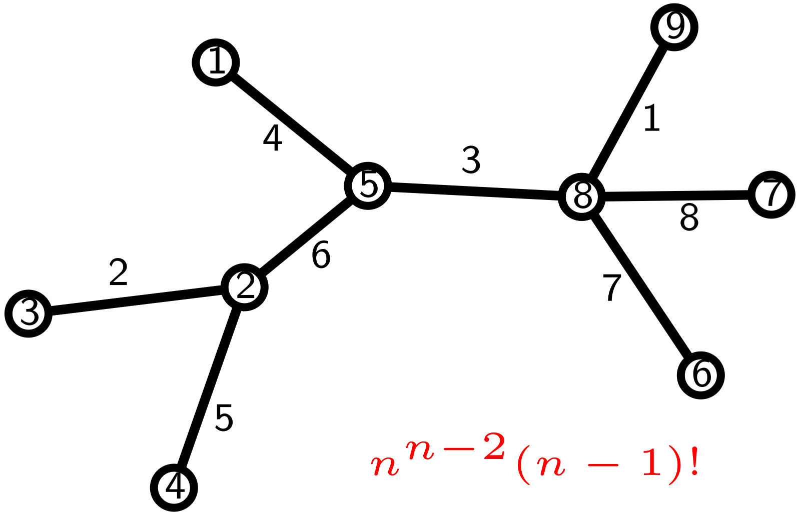

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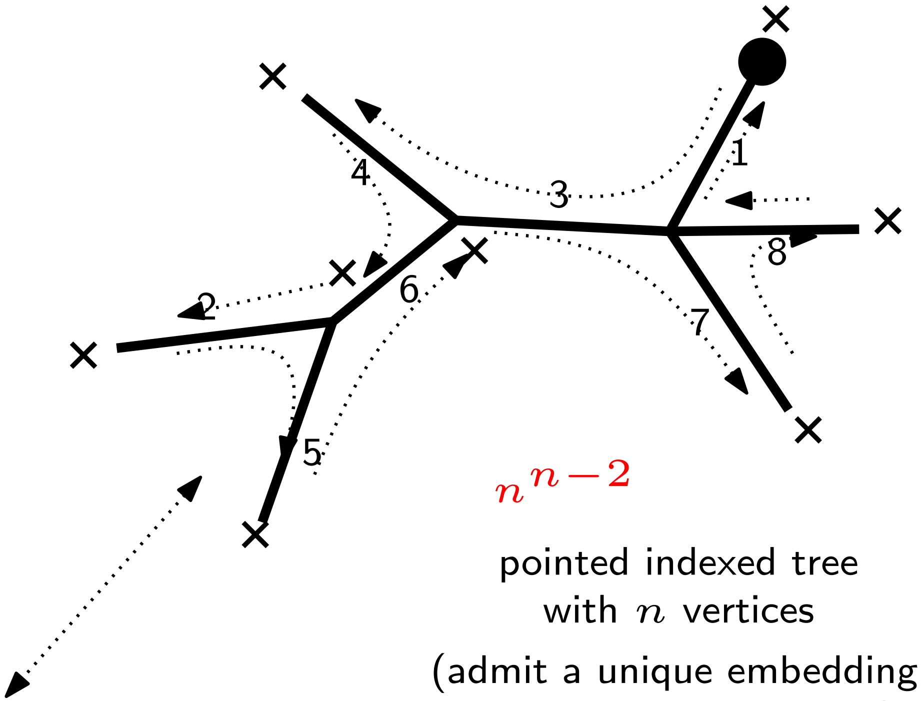

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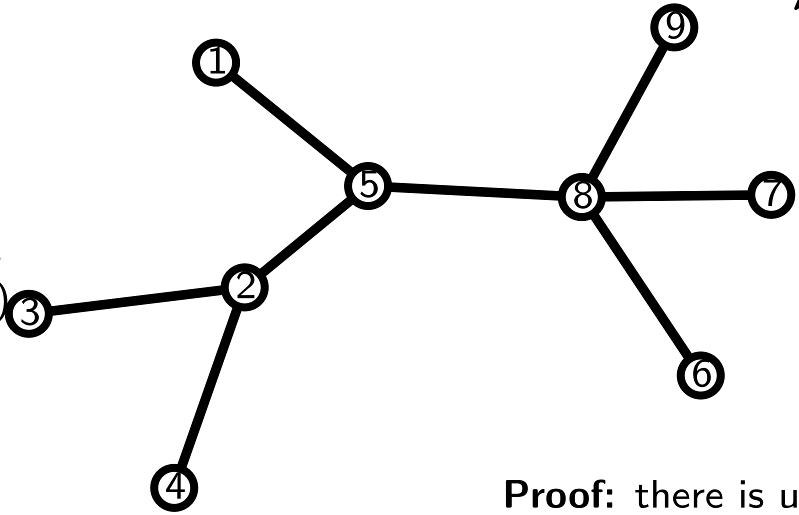

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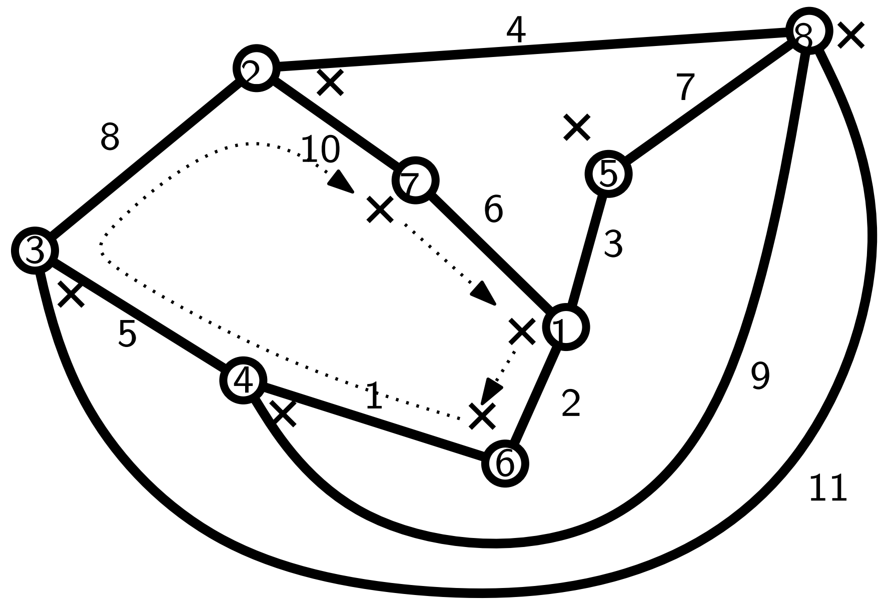
Proof: there is unique way to put labels so that the computation works!

Increasing maps

Moszkowski's bijection extends to all types of transitive factorizations in transpositions (with type λ, non necessarily minimal)

(4,6)(1,6)(1,5)(2,8)(3,4)(1,7)(5,8)(2,3)(4,8)(2,7)(3,8)= (1,6,7)(2,5)(3)(4)(8)

$$\begin{array}{l} 1 \longrightarrow 6 \\ 6 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 7 \\ 7 \longrightarrow 1 \end{array}$$



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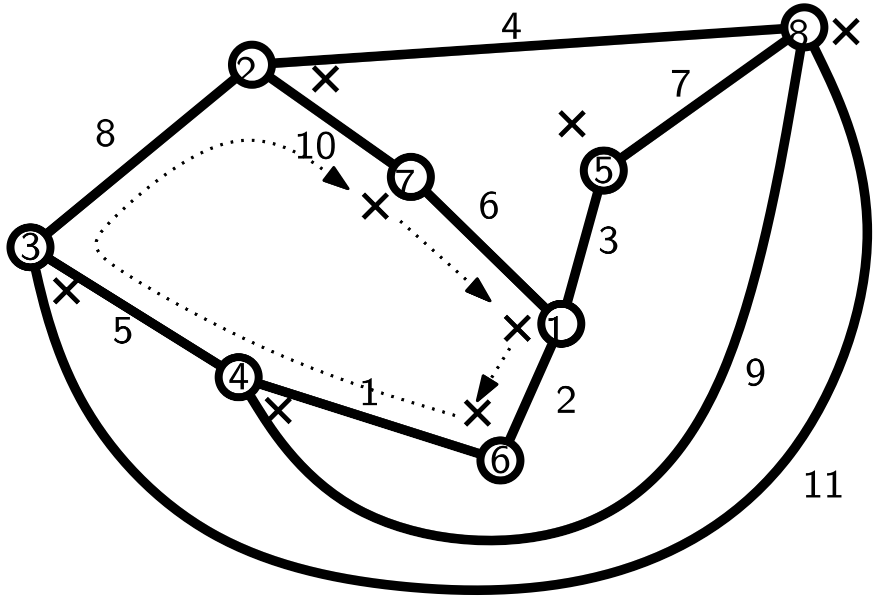

Increasing map: embedded graph with

- n labeled vertices $\{1, \ldots, n\}$
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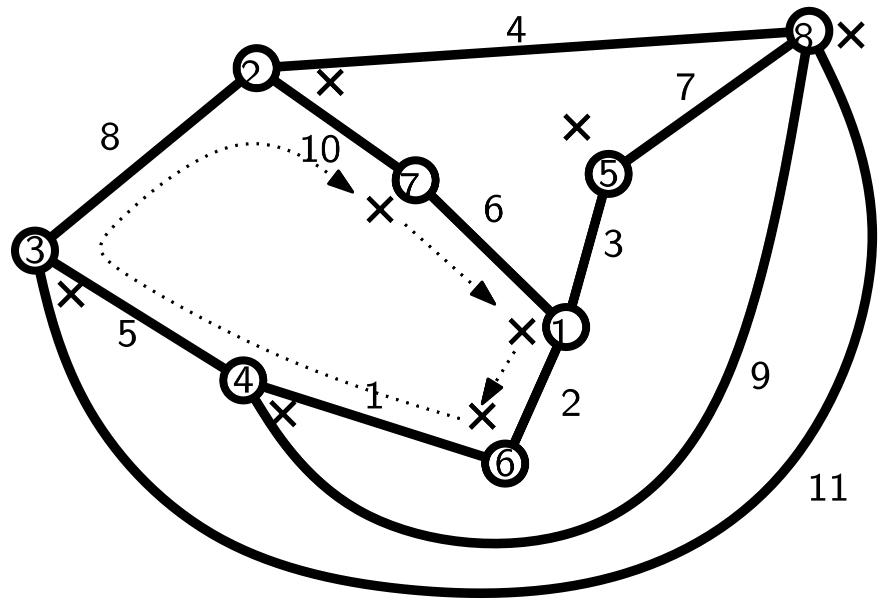

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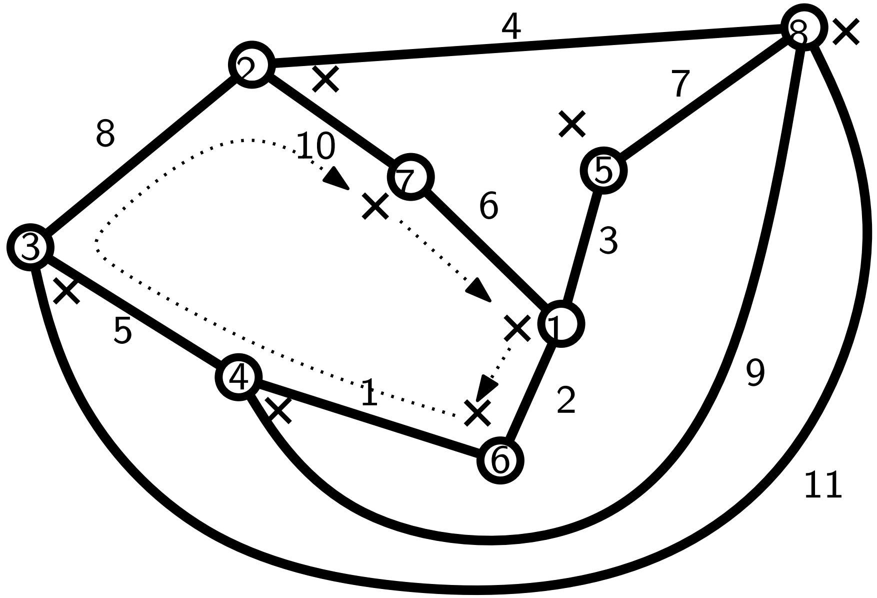

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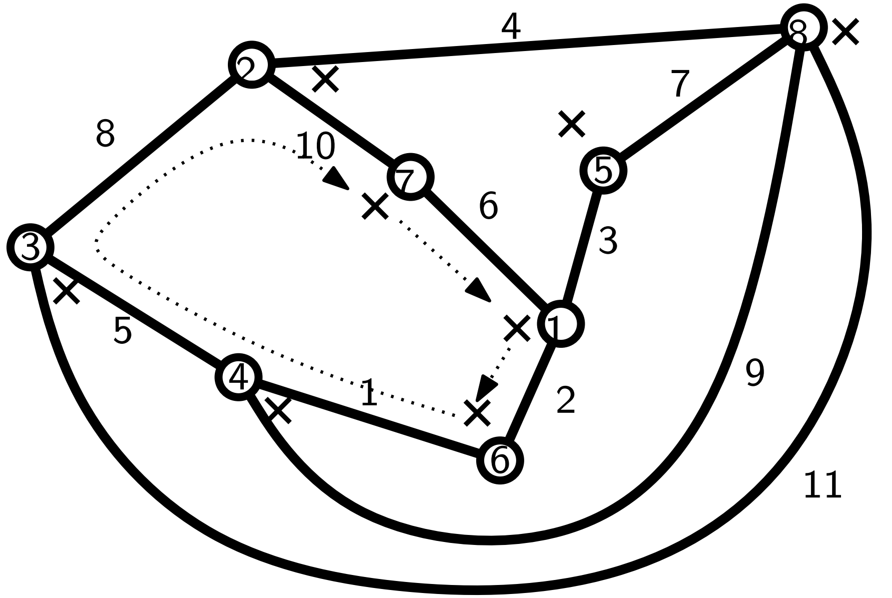

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Non-minimal ($m = n + \ell - 2 + 2g$) $\Leftrightarrow$ increasing map of genus g

Origin of the problem...

and a common setup for all permutations/maps relations?

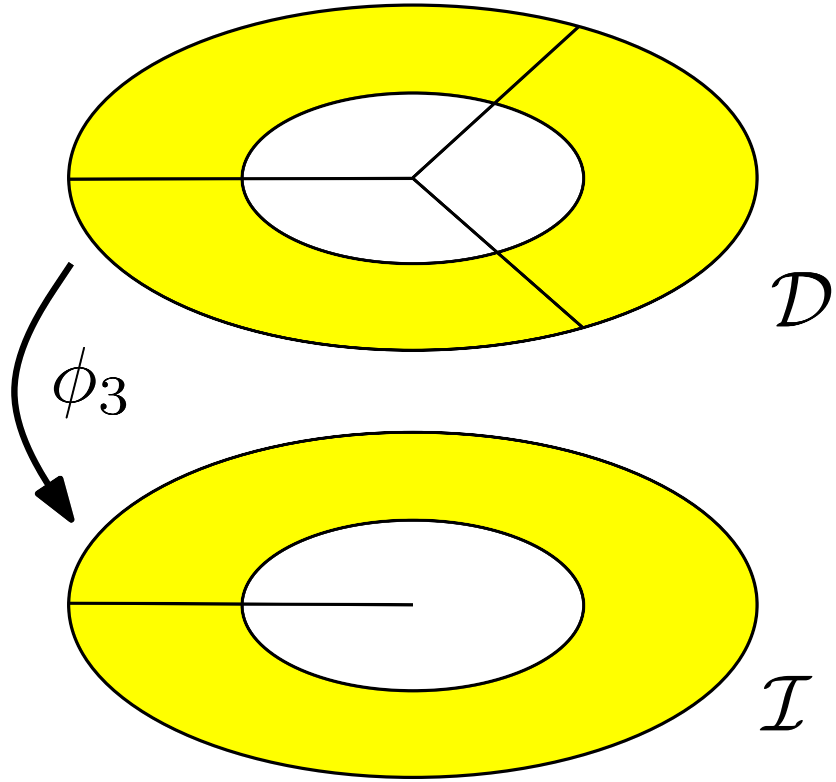
Ramified coverings of the sphere by itself

See book Lando-Zvonkin for more details.

Ramified coverings of the sphere by itself

Let $D = \{z \mid |z| < 1\} \subset \mathbb{C}$ be the unit open disc, and let $\sim$ denote equivalence up to homeomorphisms (bijective, bicontinuous mappings).

A mapping $\phi : \mathcal{D} \to \mathcal{I}$ is a covering if, for all x in $\mathcal{I}$ there exists $n \geq 1$ and a neighborhood V of x such that $\phi^{-1}(V) \sim D \times \{1, \ldots, n\}$, and the restriction of ϕ to each sheet D_i (connected component of the preimage) is an homeomorphism $\phi_{|D_i} : D_i \xrightarrow{\sim} D$.



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Consider $\phi_k : A_r \to A_{r^k}$ with $\phi_k(z) = z^k$.

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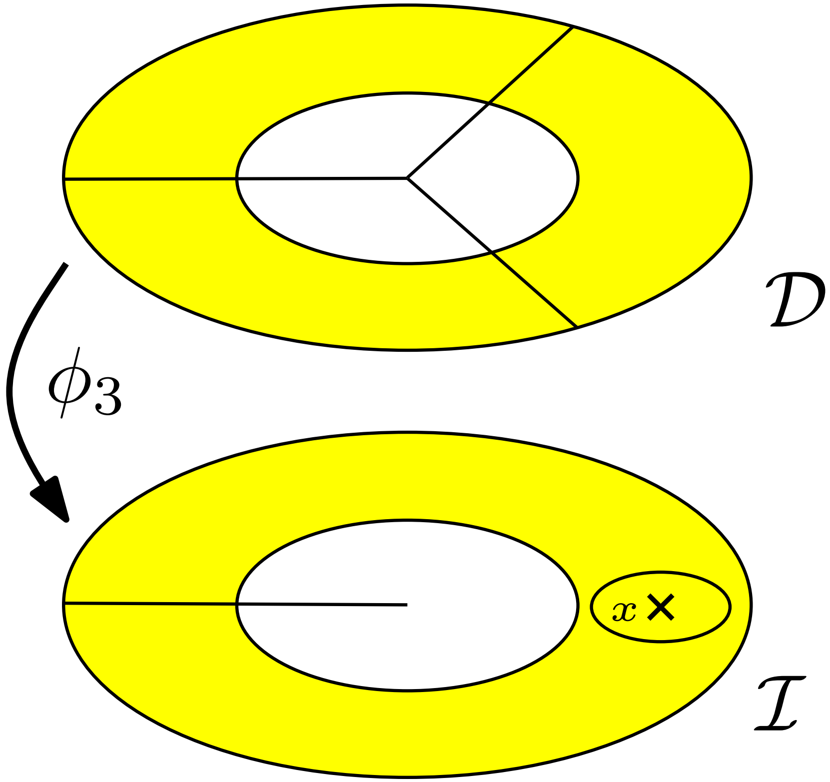

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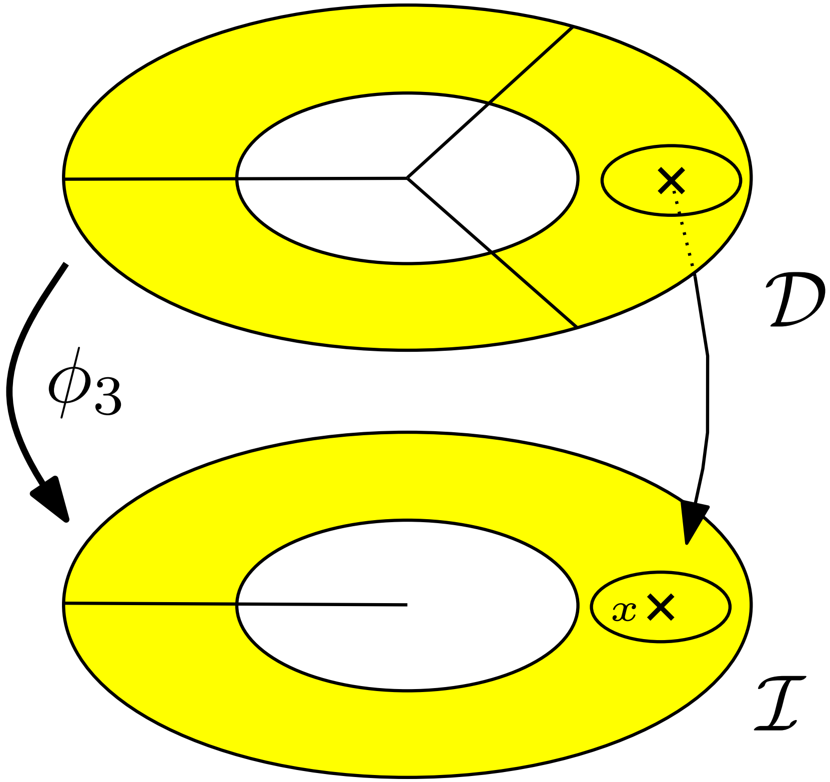

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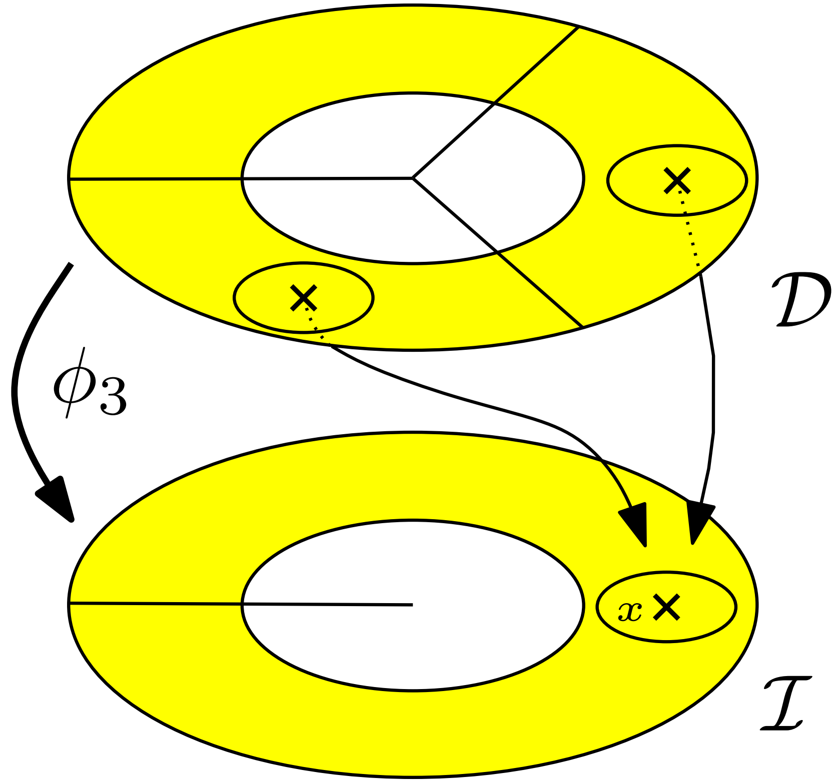

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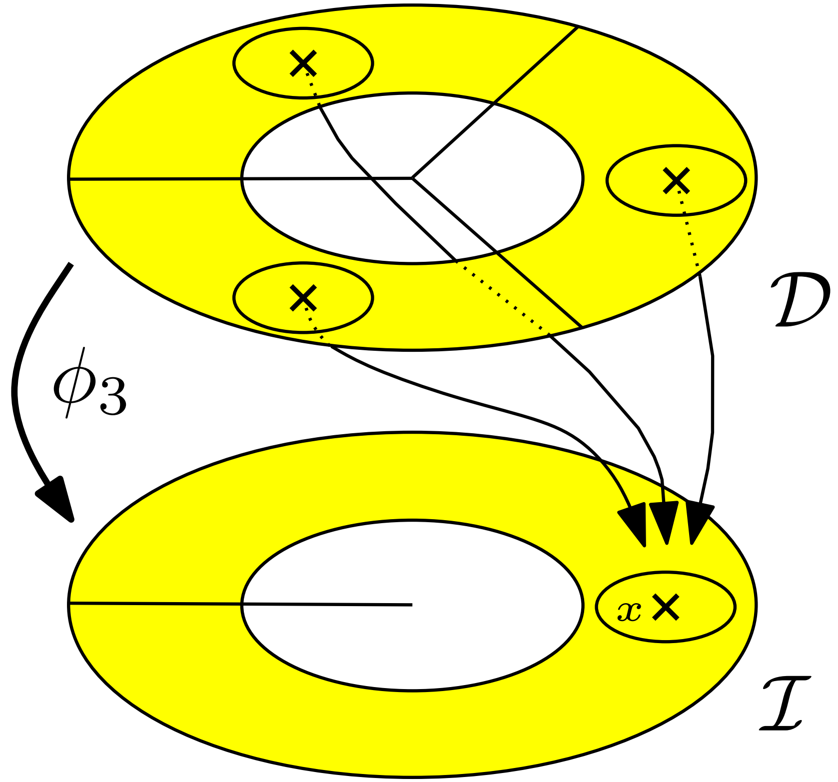

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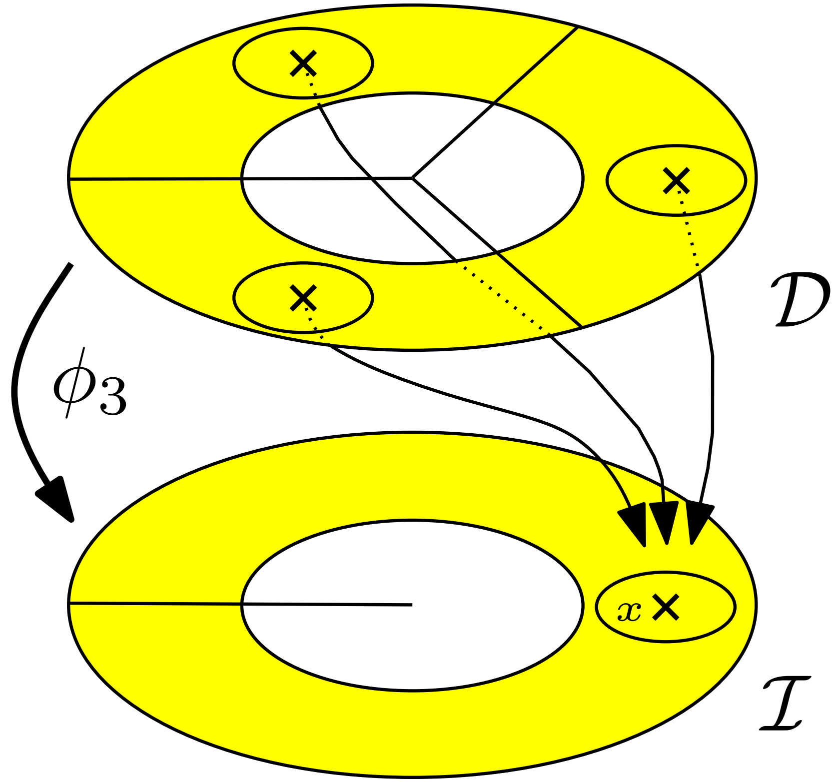

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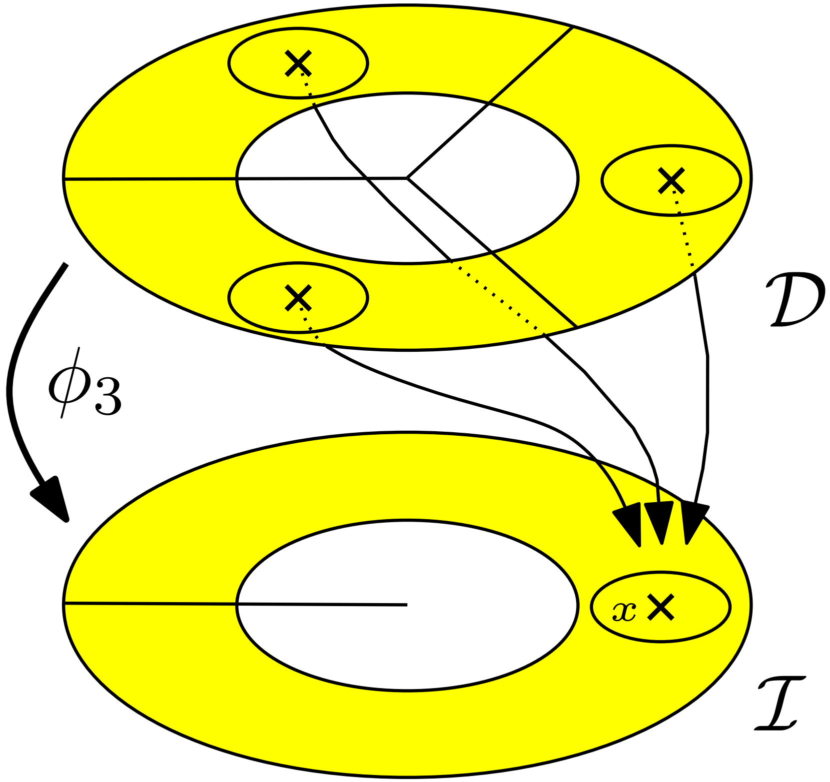

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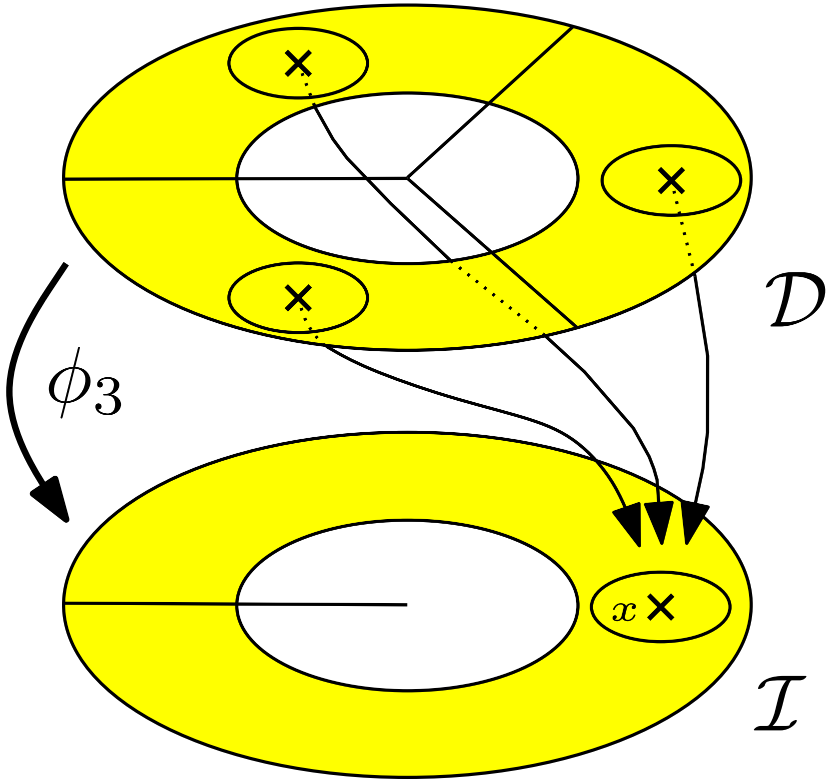

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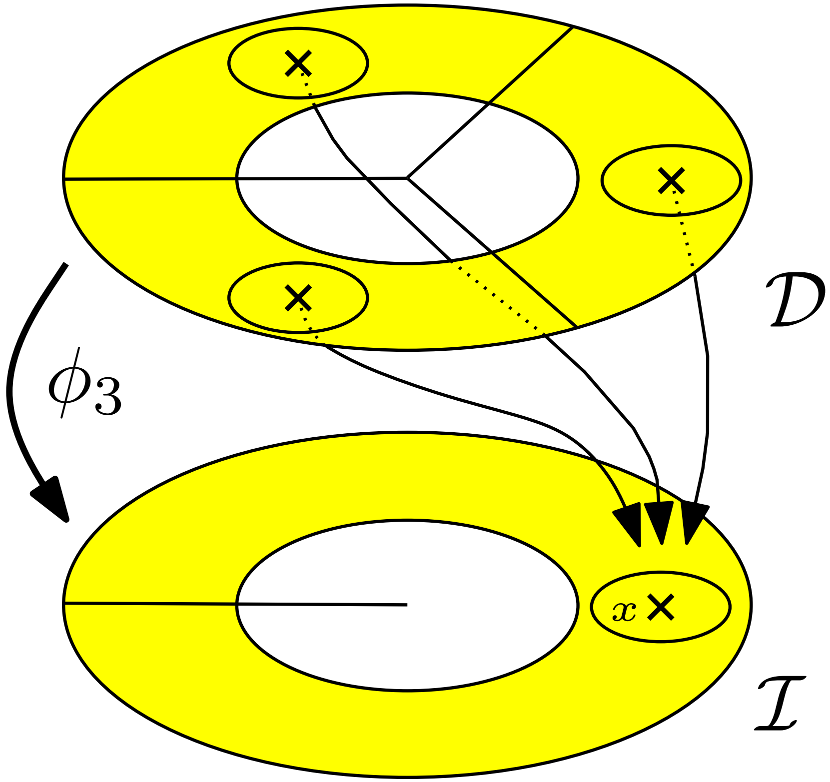

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What is we try to extend from A_r to D?

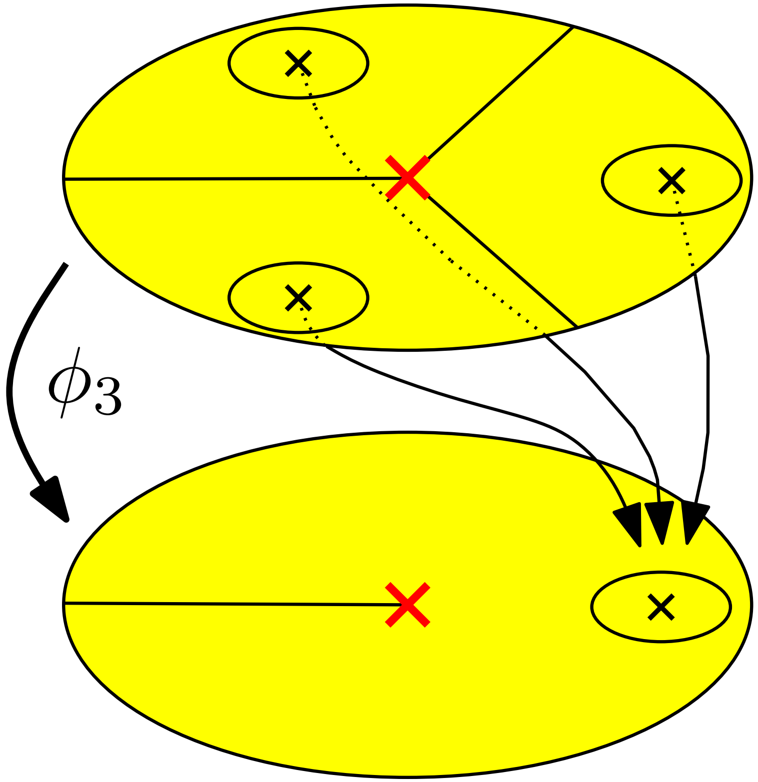
Ramified coverings of the sphere by itself

Recall $\phi_k : A_r \to A_{r^k}$ with $\phi_k(z) = z^k$.

Extend from A_r to D?

The mapping $\phi_k : D^* \to D^*$ is a covering.

but not $\phi_k : D \to D$.



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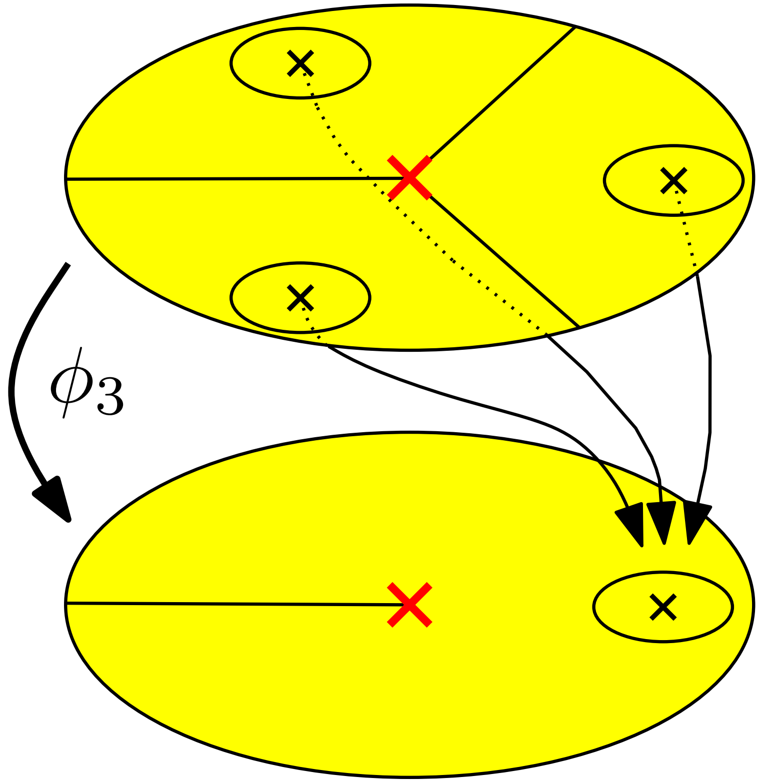

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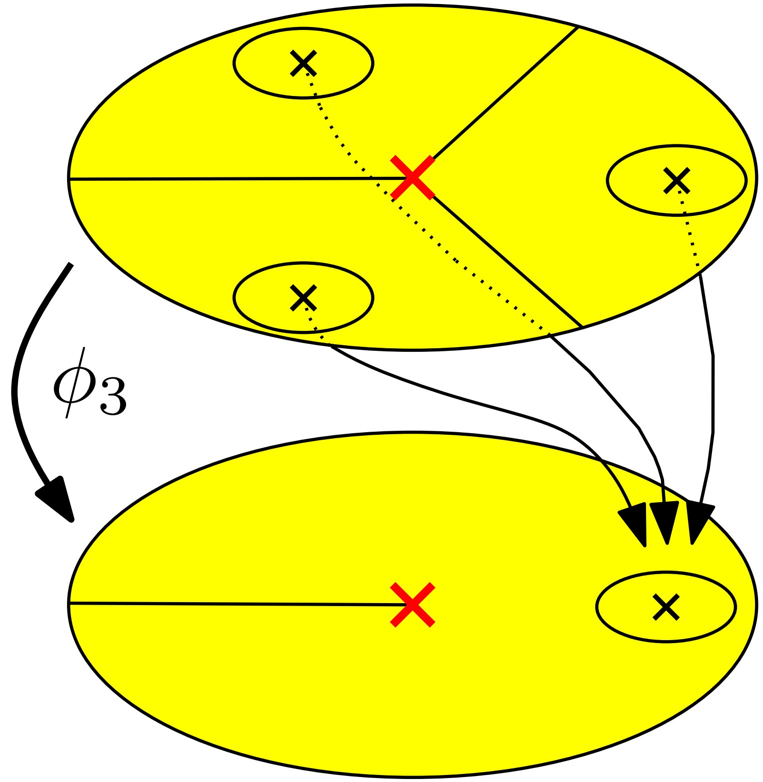

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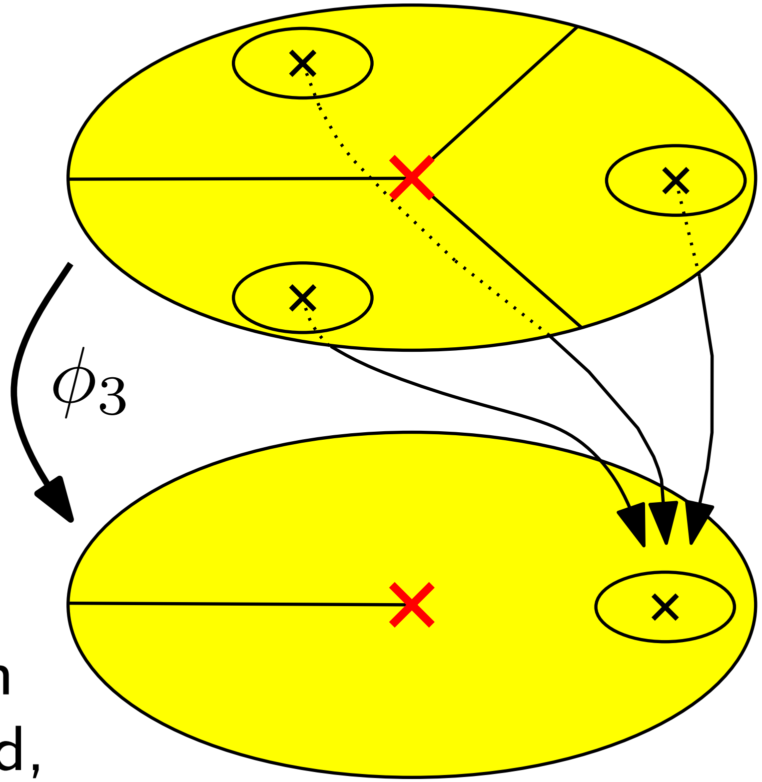

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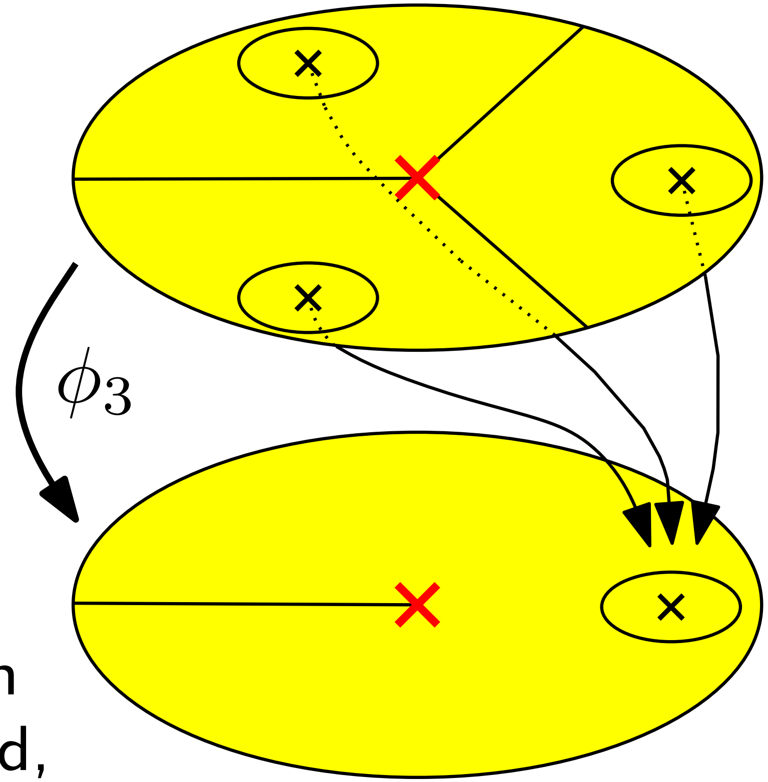

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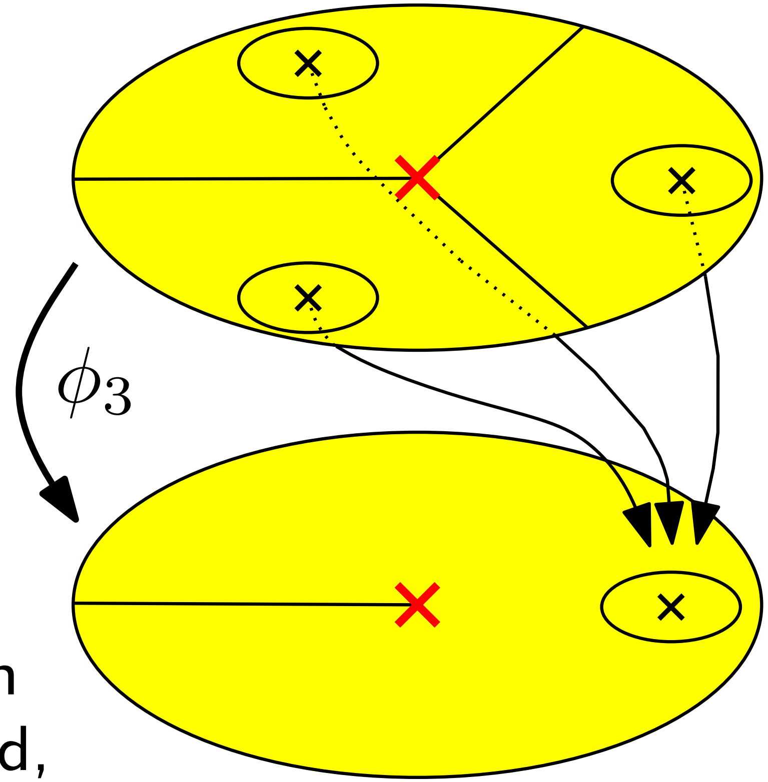

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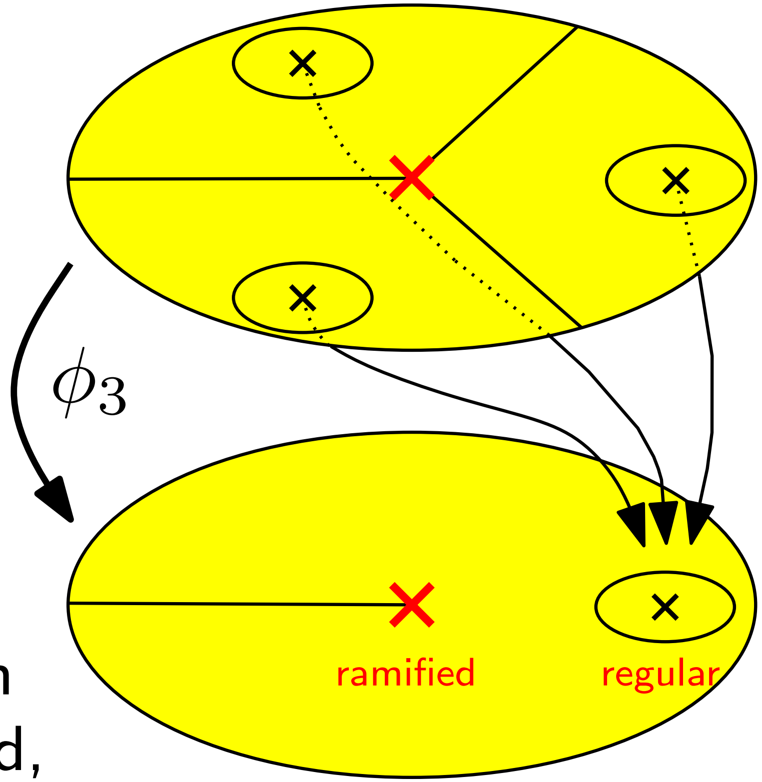

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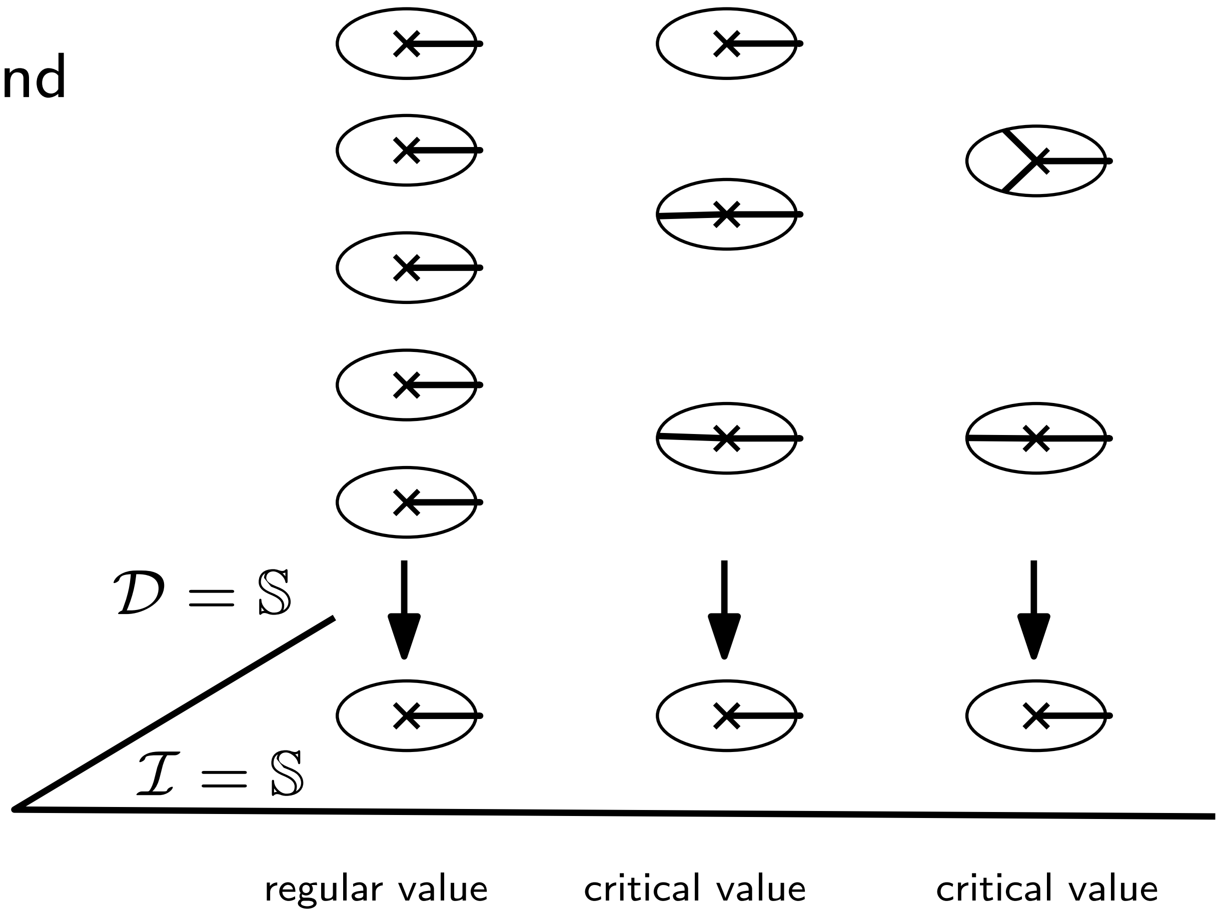
- $\phi_{\mathbb{S}\setminus\phi^{-1}(X)}$ is a covering, and
- ϕ is ramified over each x_i



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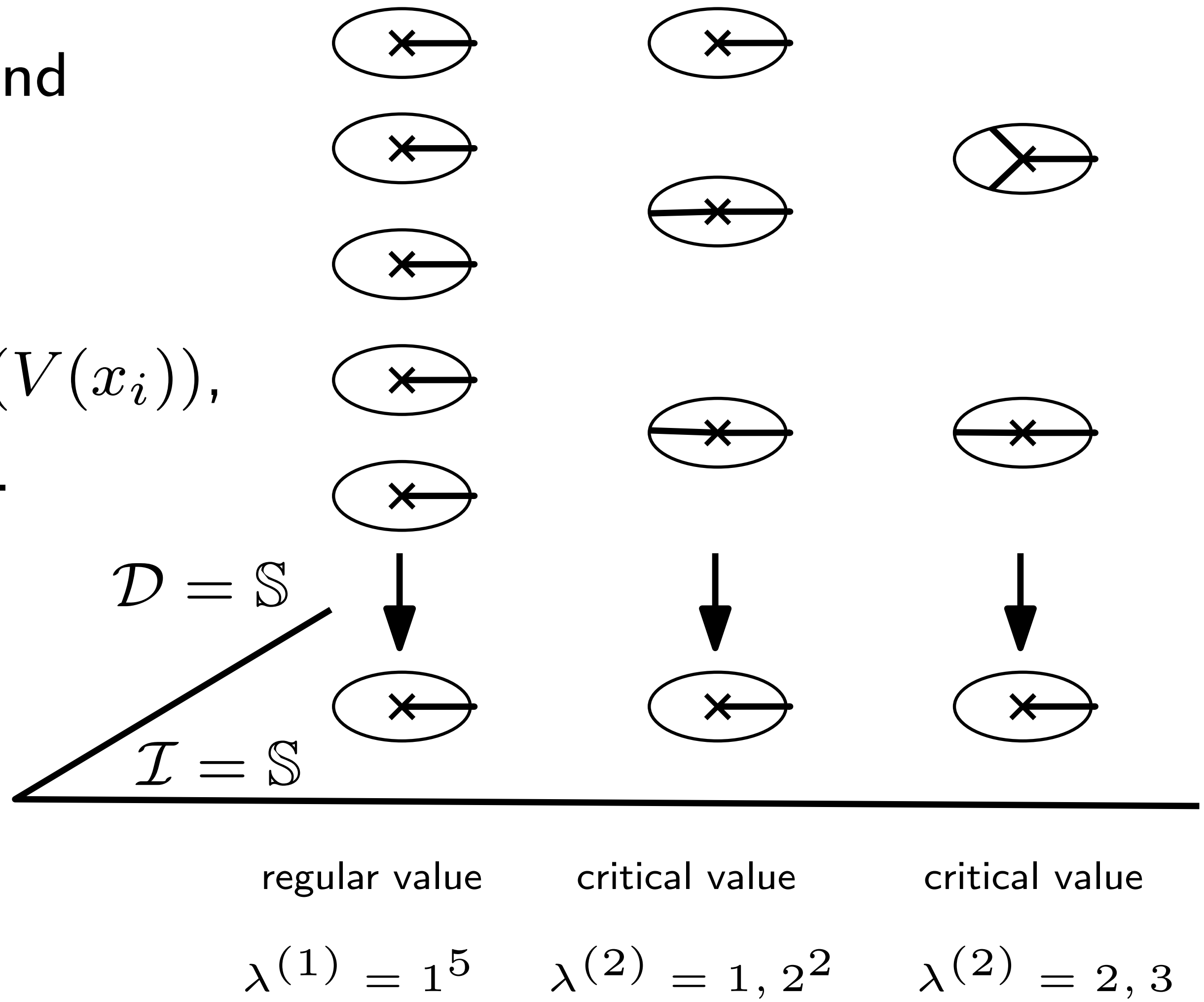
- ϕ is ramified over each x_i

On each component V_j of $\phi^{-1}(V(x_i))$, $\phi \sim \phi_{\lambda_j^{(i)}}$ for some integer $\lambda_j^{(i)}$.



Ramified coverings of the sphere by itself (Cont'd)

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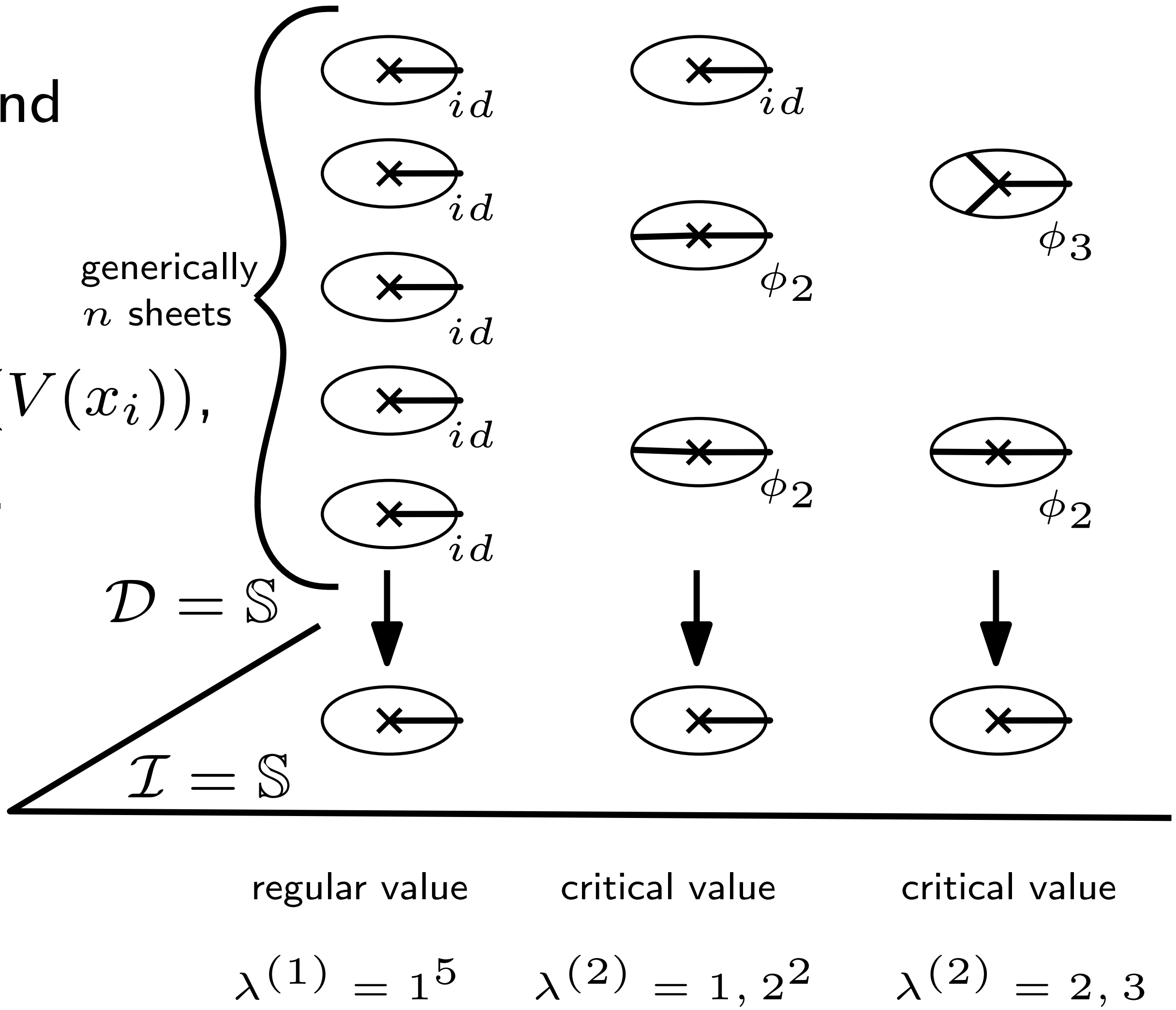

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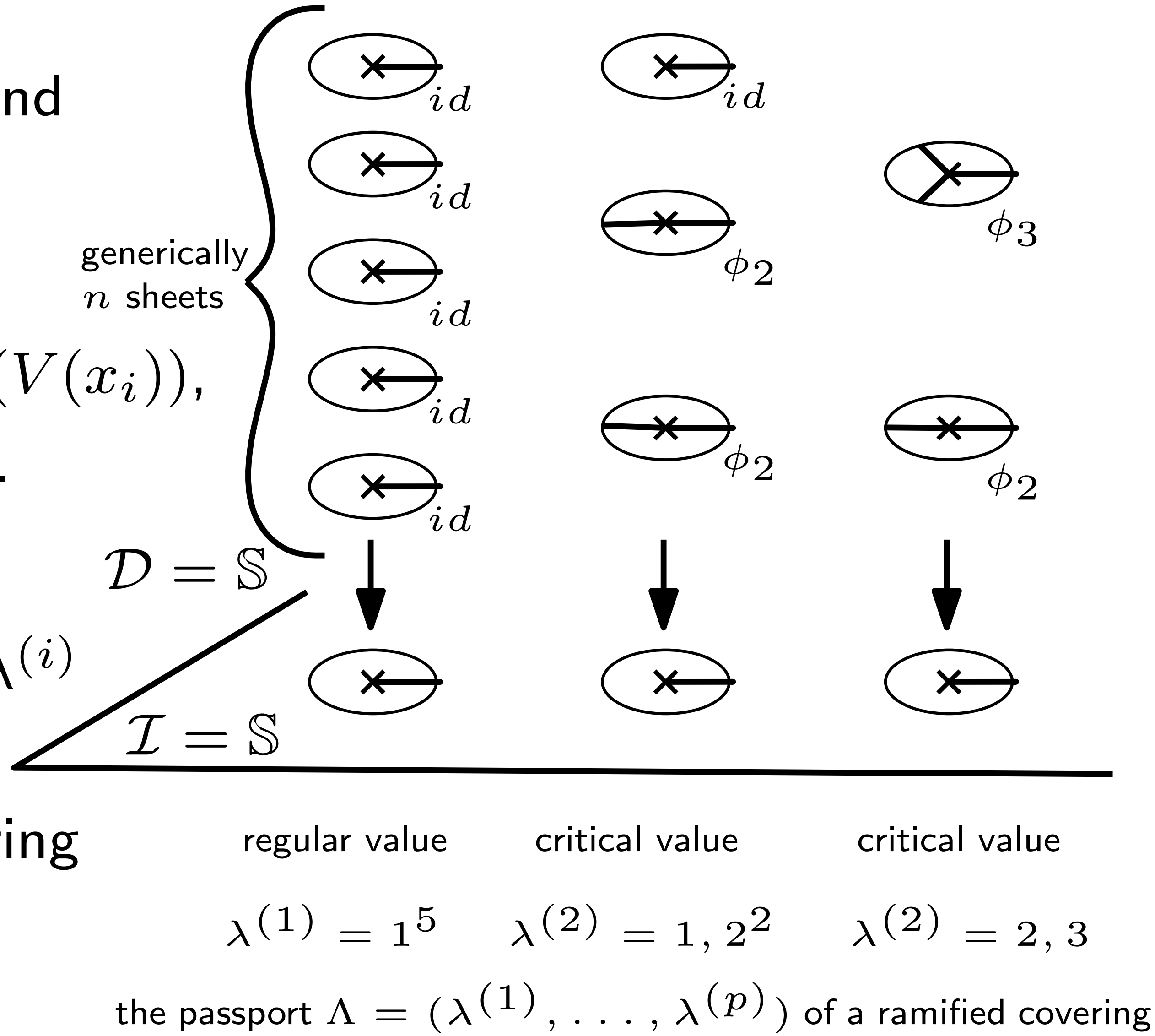
The ramification type over a critical value x_i is the partition $\lambda^{(i)}$

The passport of a ramified covering is the list $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$



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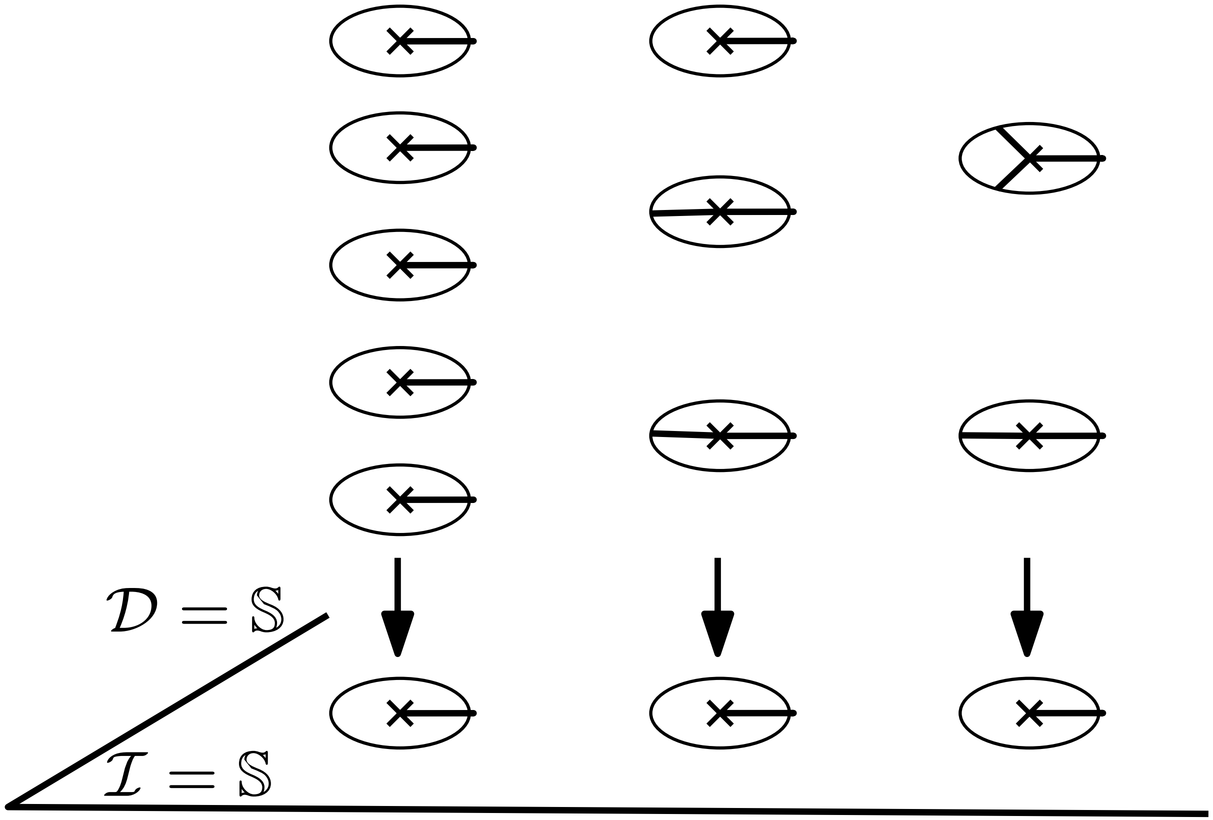

regular value | critical value | critical value

$\lambda^{(1)} = 1^5$ | $\lambda^{(2)} = 1, 2^2$ | $\lambda^{(2)} = 2, 3$

the passport $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of a ramified covering

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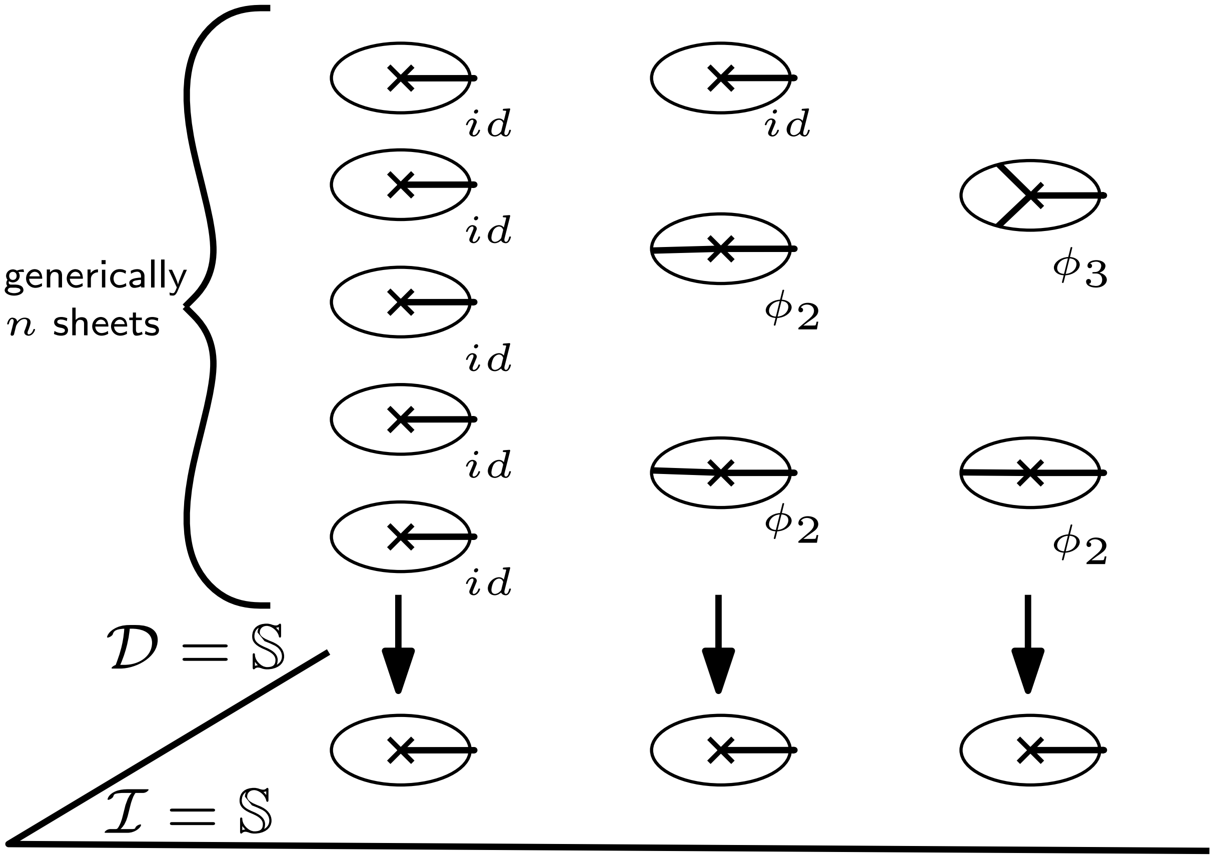

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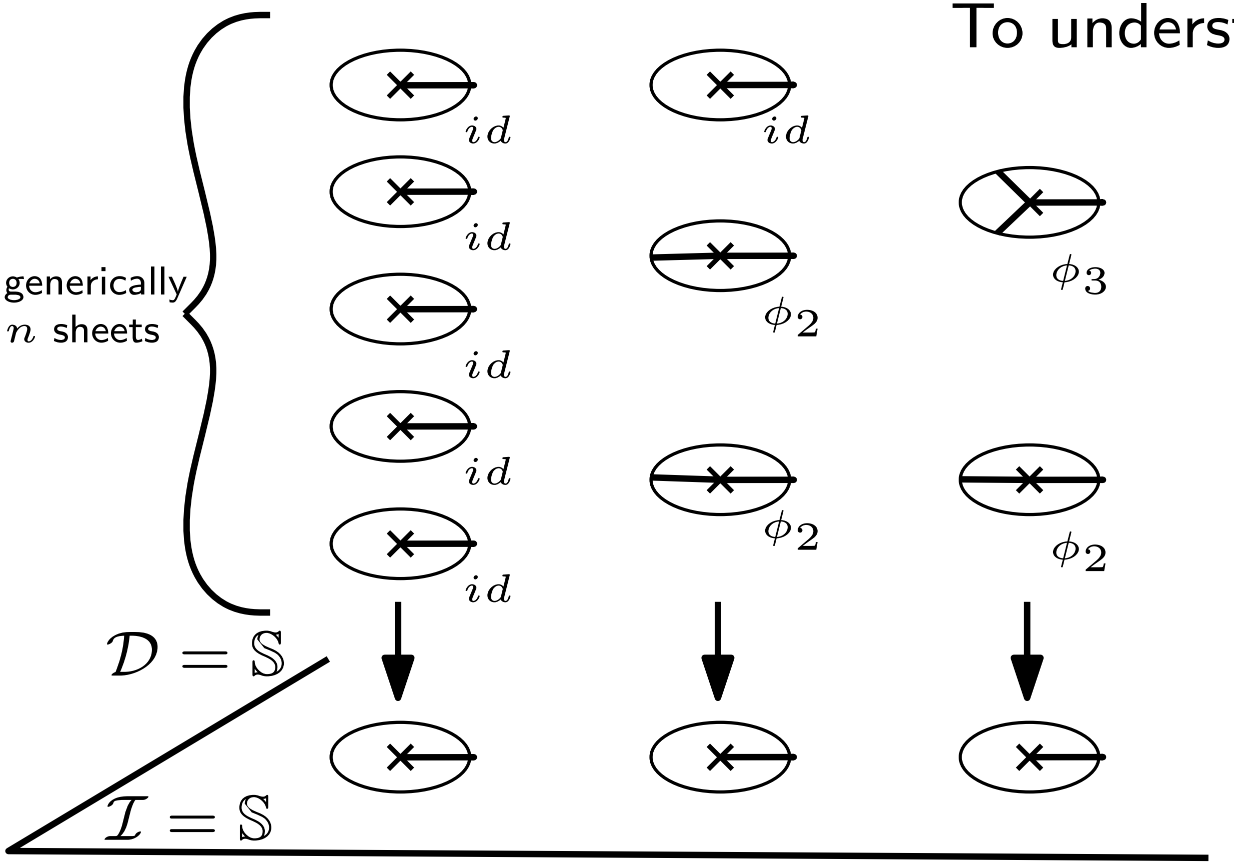
To understand the "shape" of the covering, draw paths on $\mathcal{I}$ and study its preimages.



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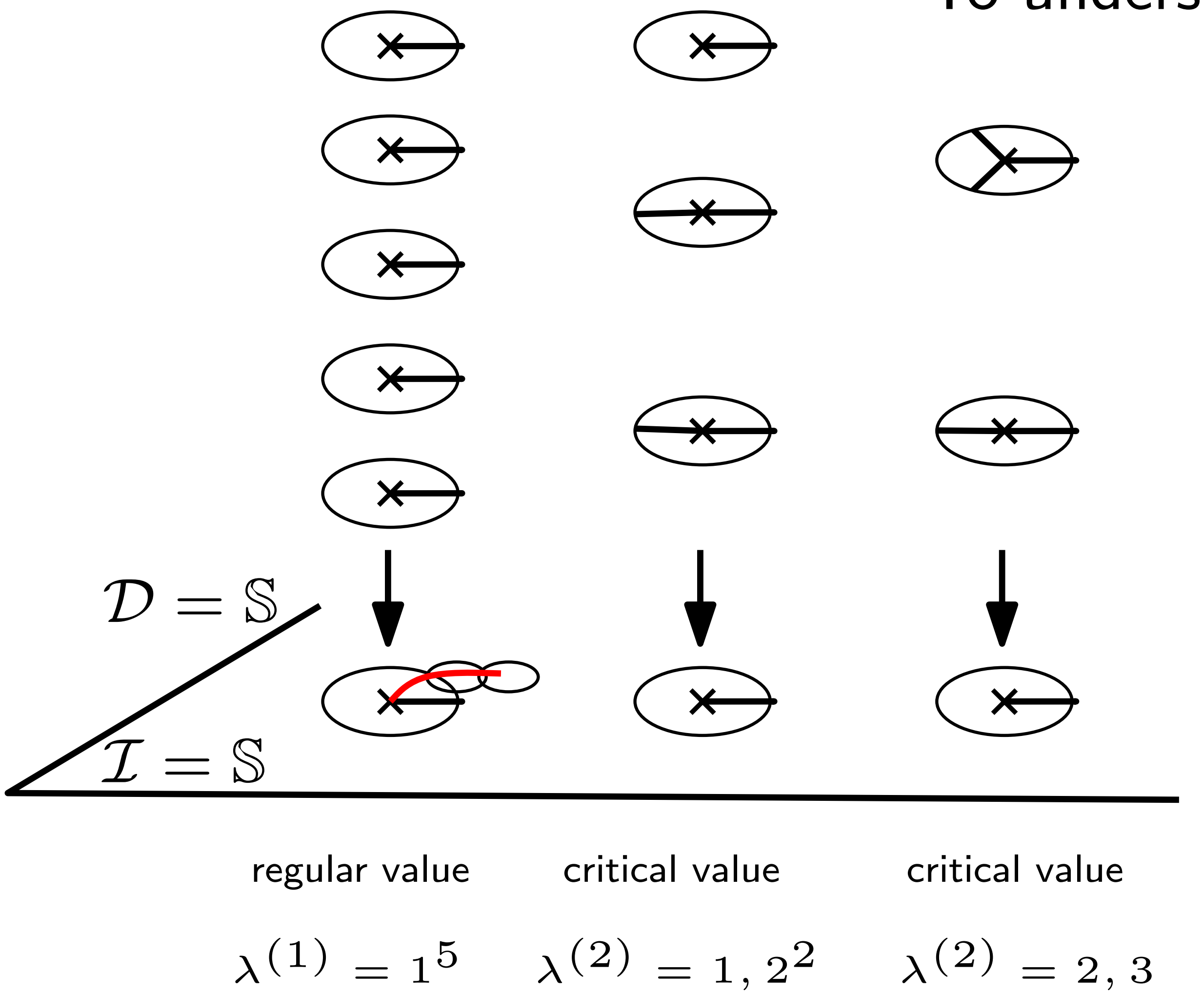

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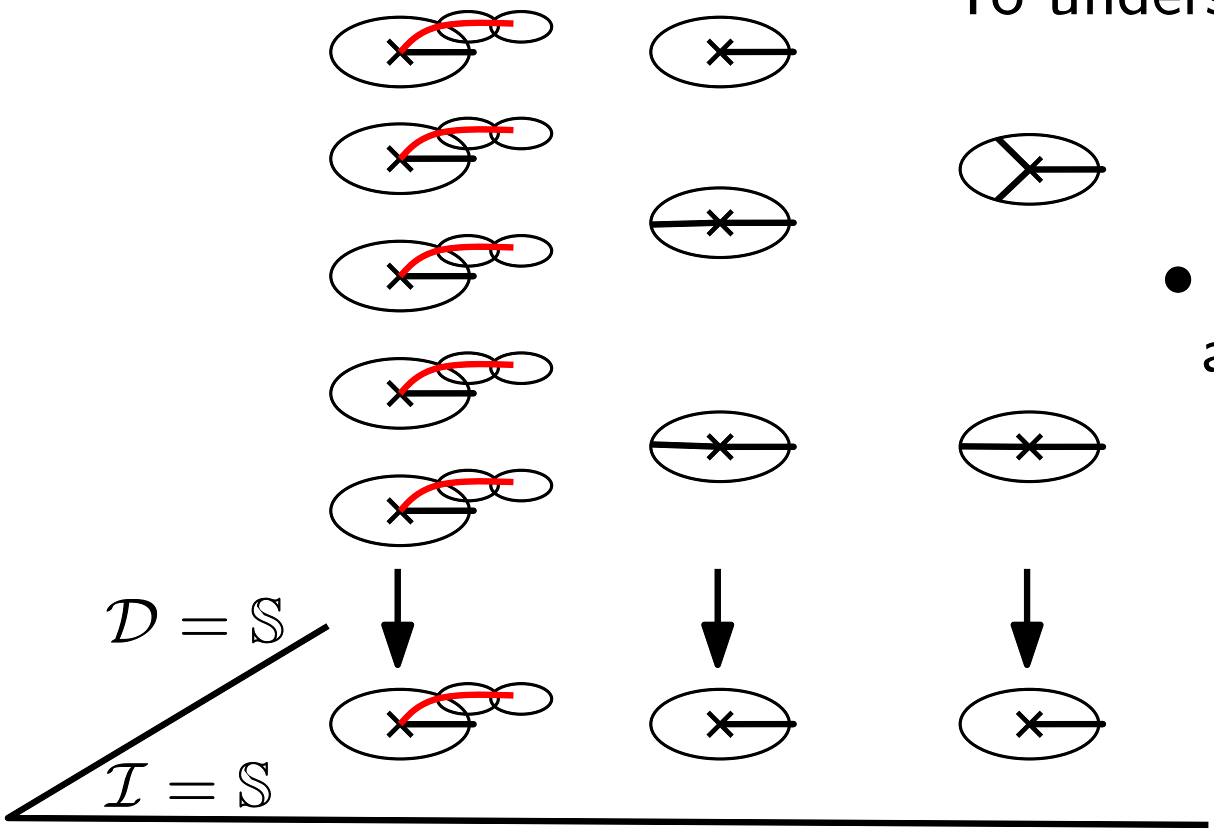

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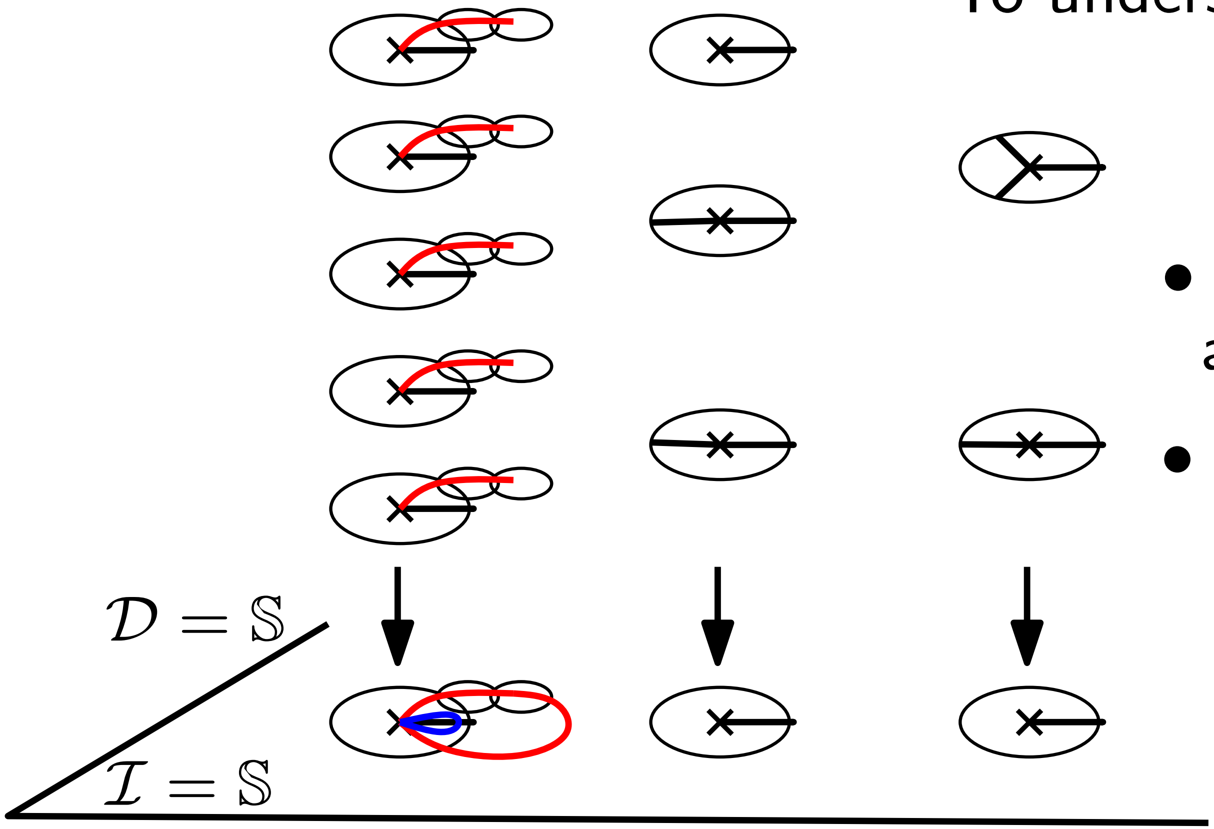

To understand the "shape" of the covering,

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- a contractible loop on $\mathcal{I}$

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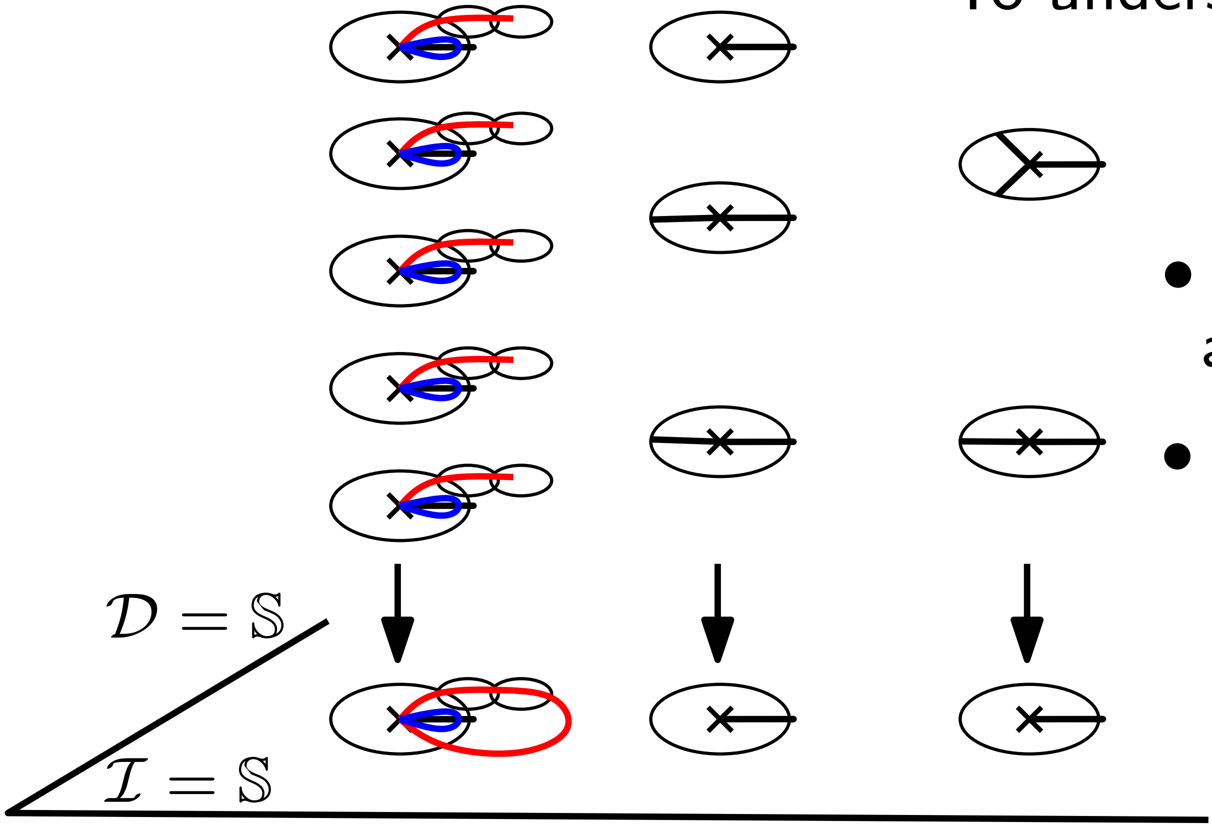

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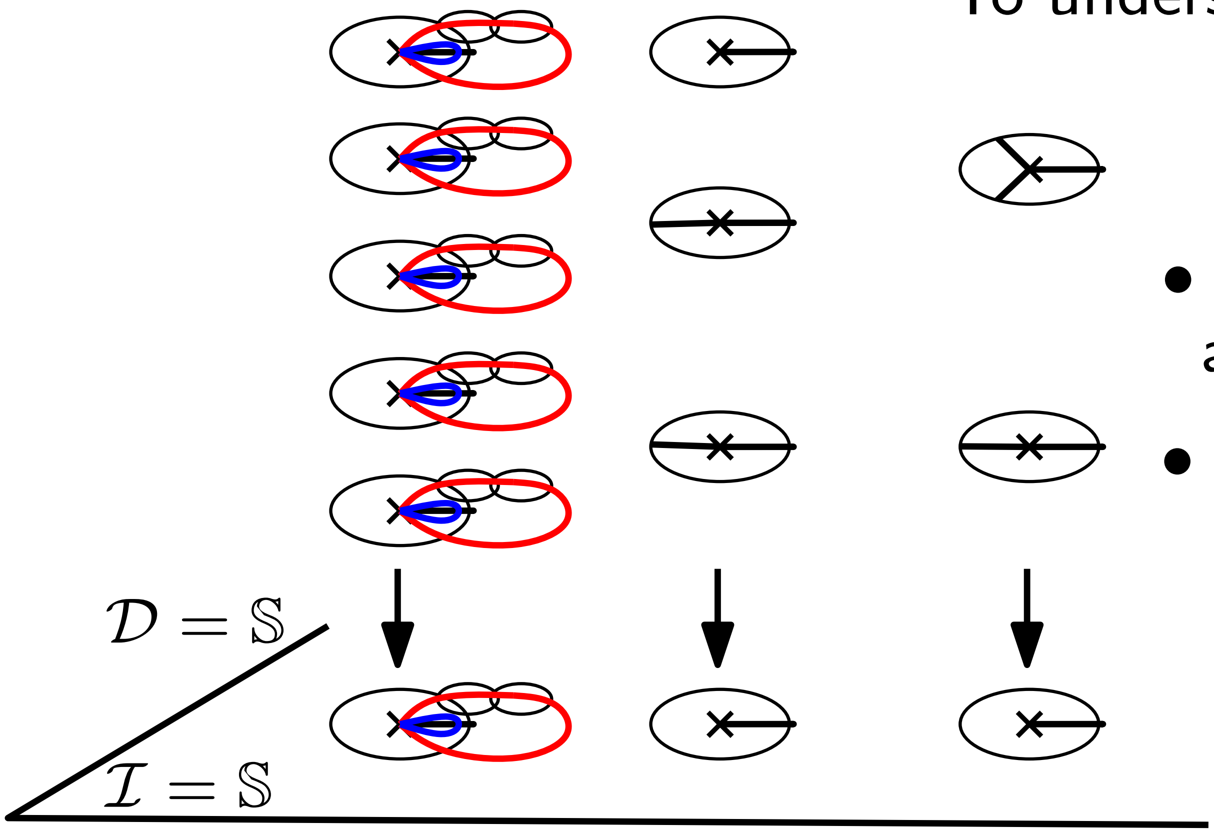
Ramified coverings of the sphere by itself (Cont'd)



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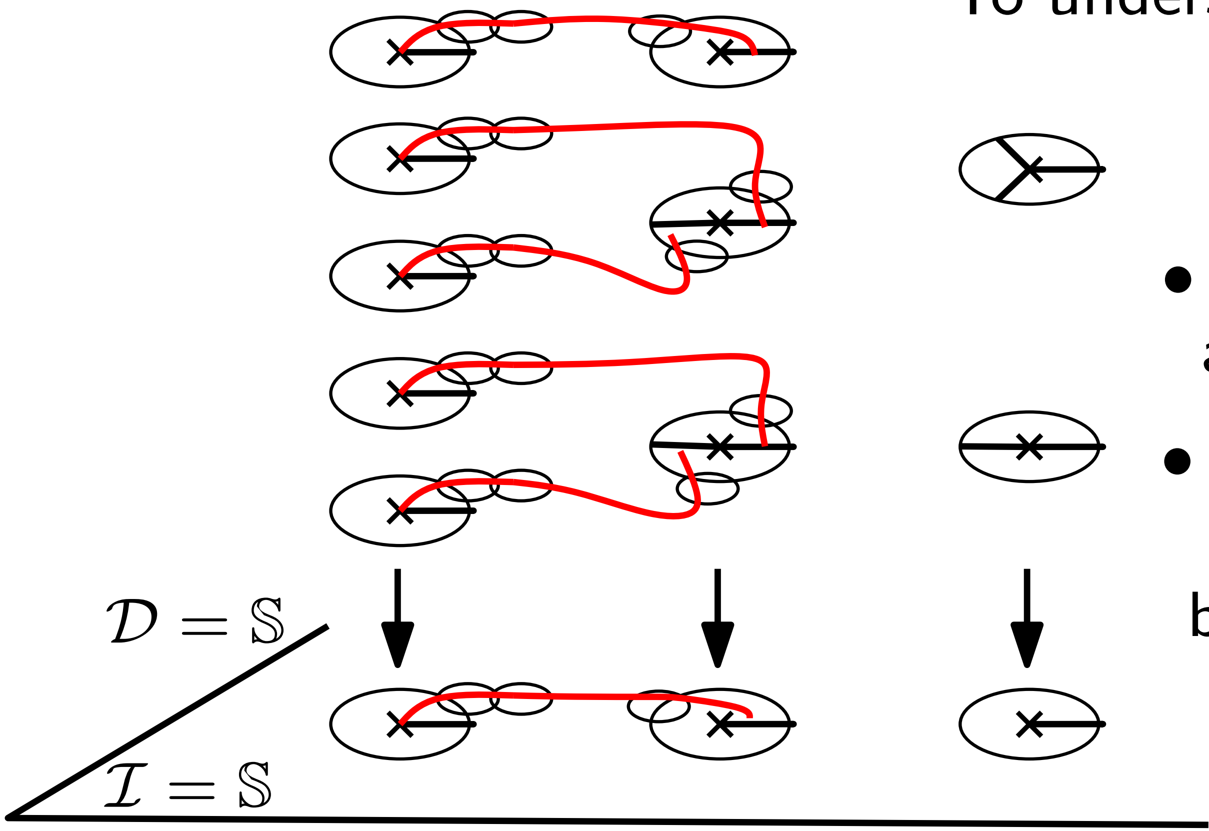

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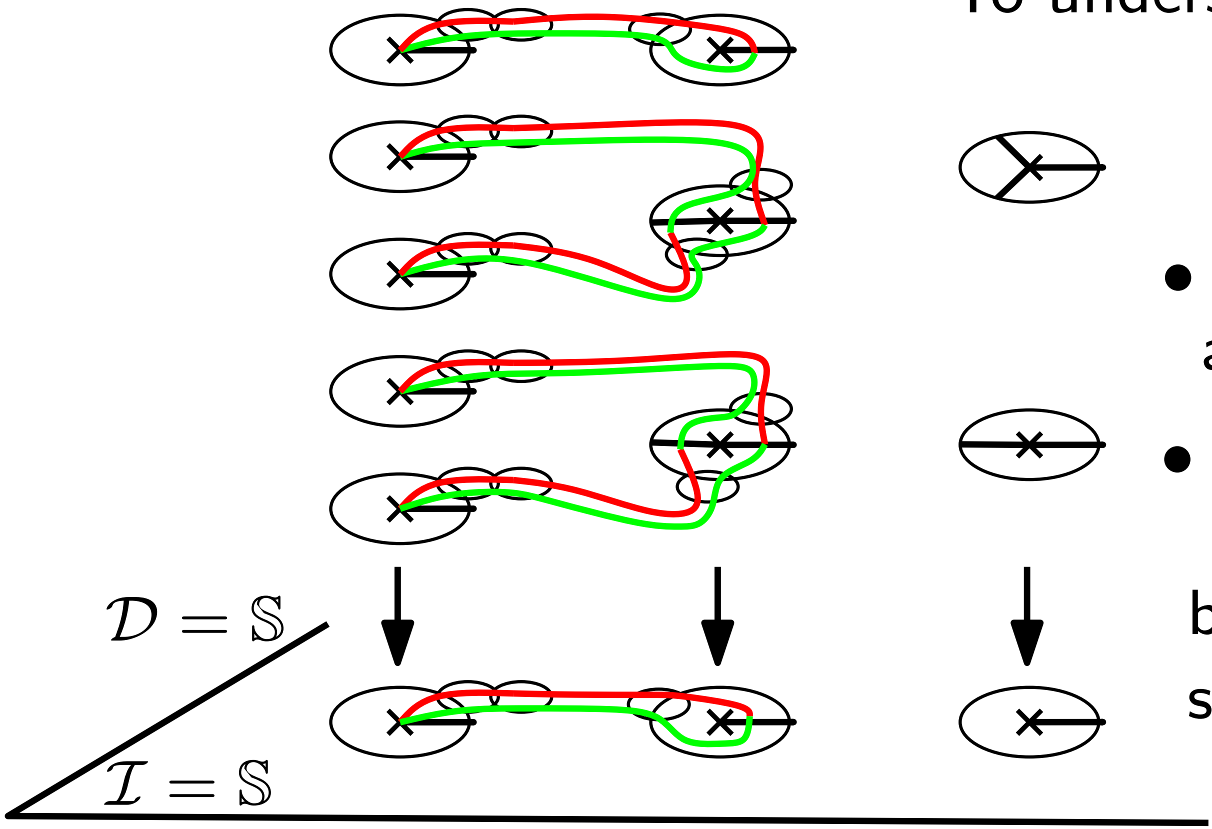

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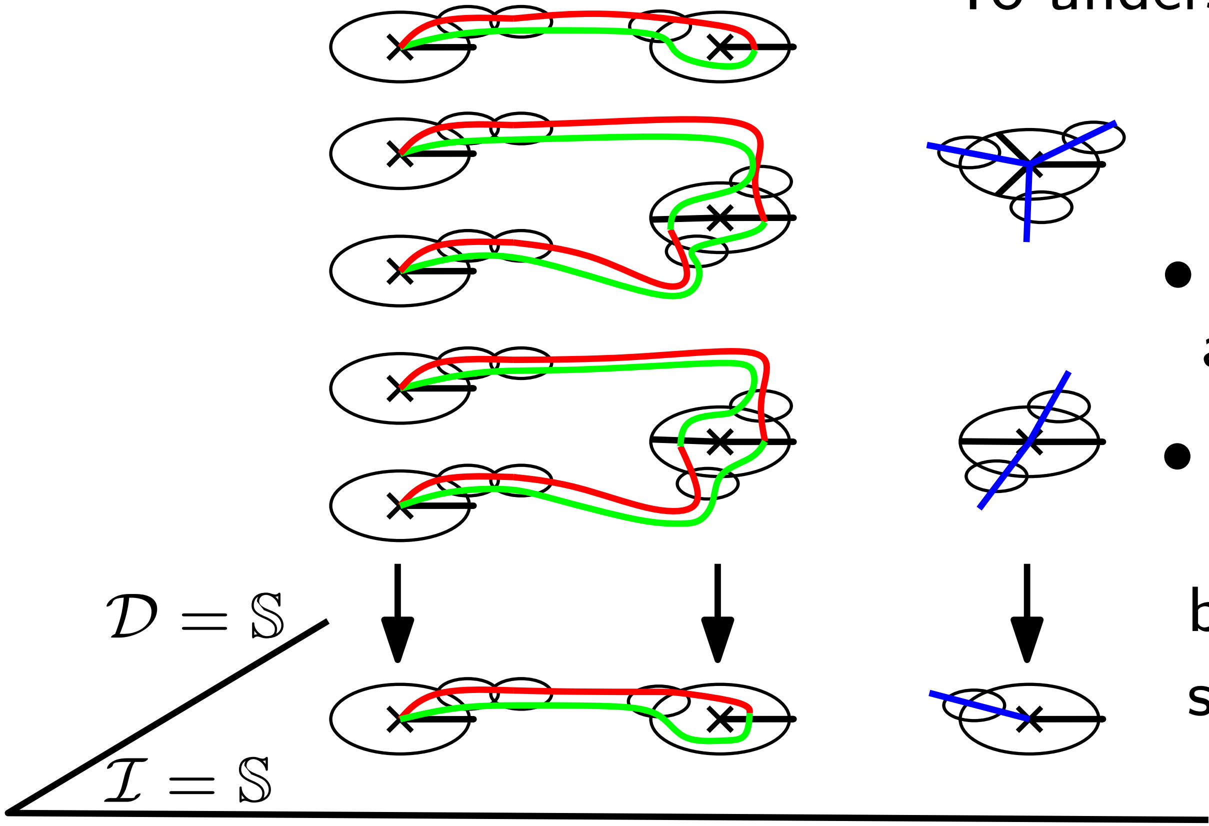

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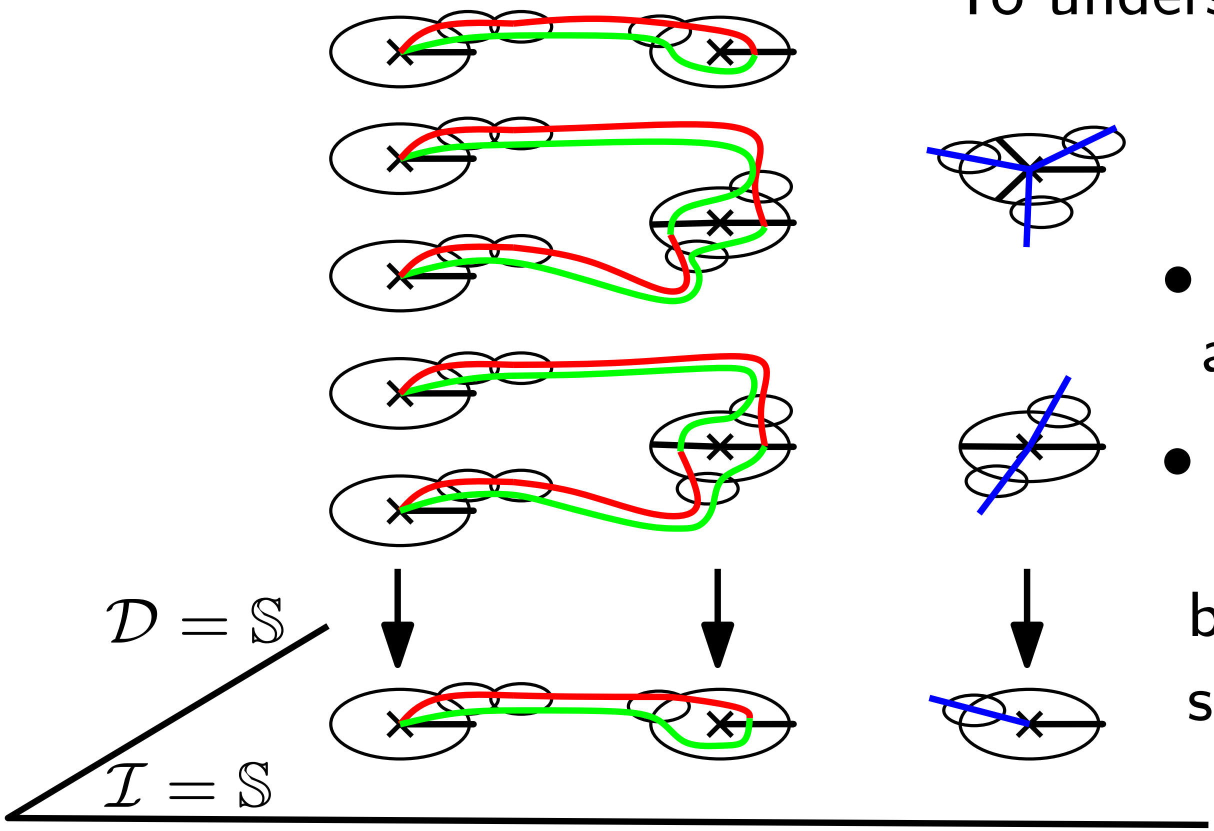

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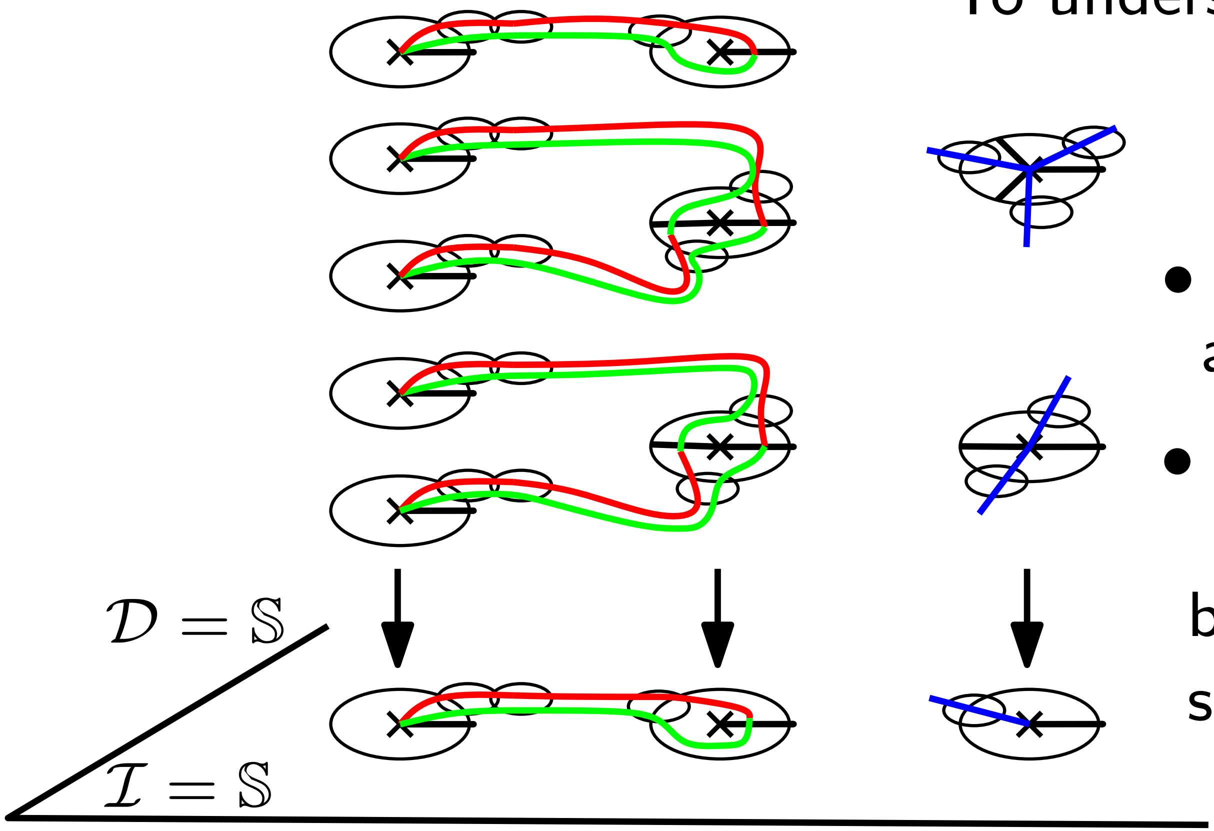

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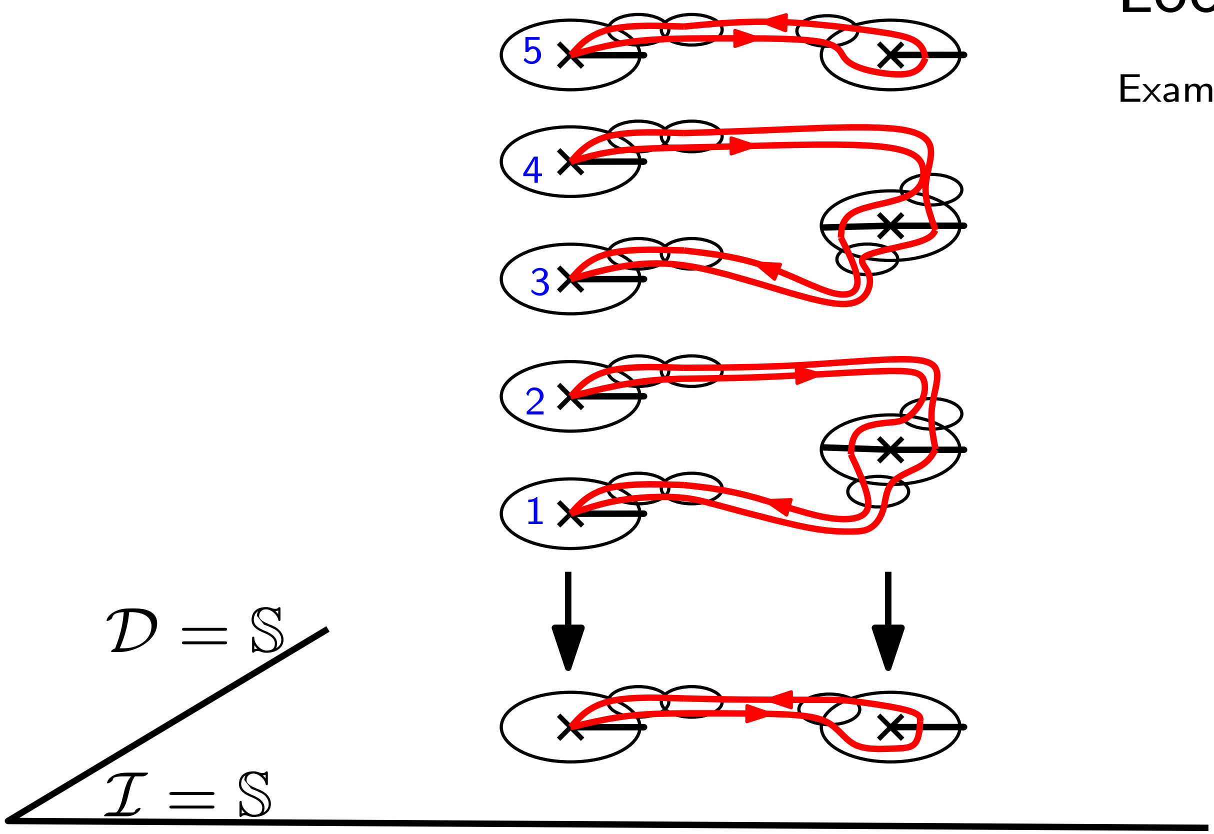
$\Rightarrow$ The partitions $\lambda^{(i)}$ are partitions of n, degree of the covering.

Monodromy, and permutations

Let us label $\{1, \ldots, n\}$ the preimages of a regular point.



Loop $\Rightarrow$ permutation of sheet labels

Example: $(1,2)(3,4)(5)$ in cyclic notation

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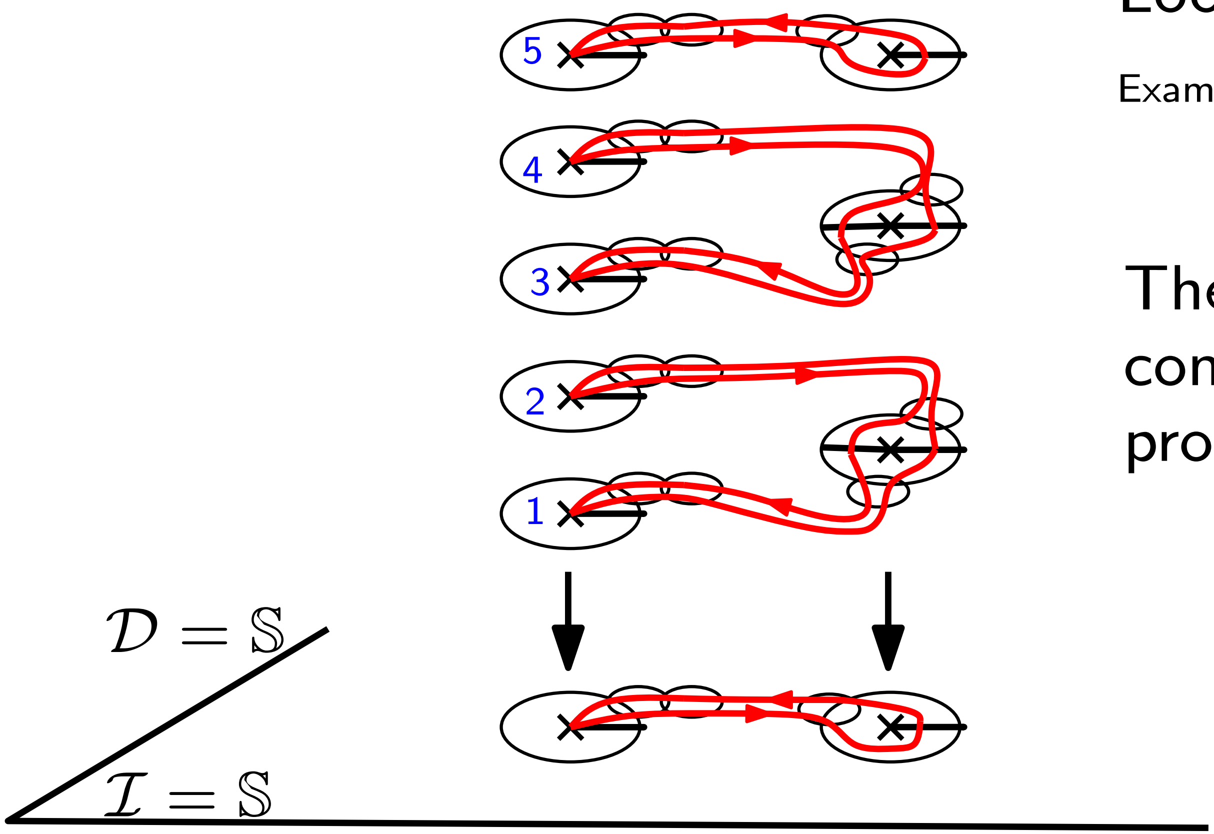

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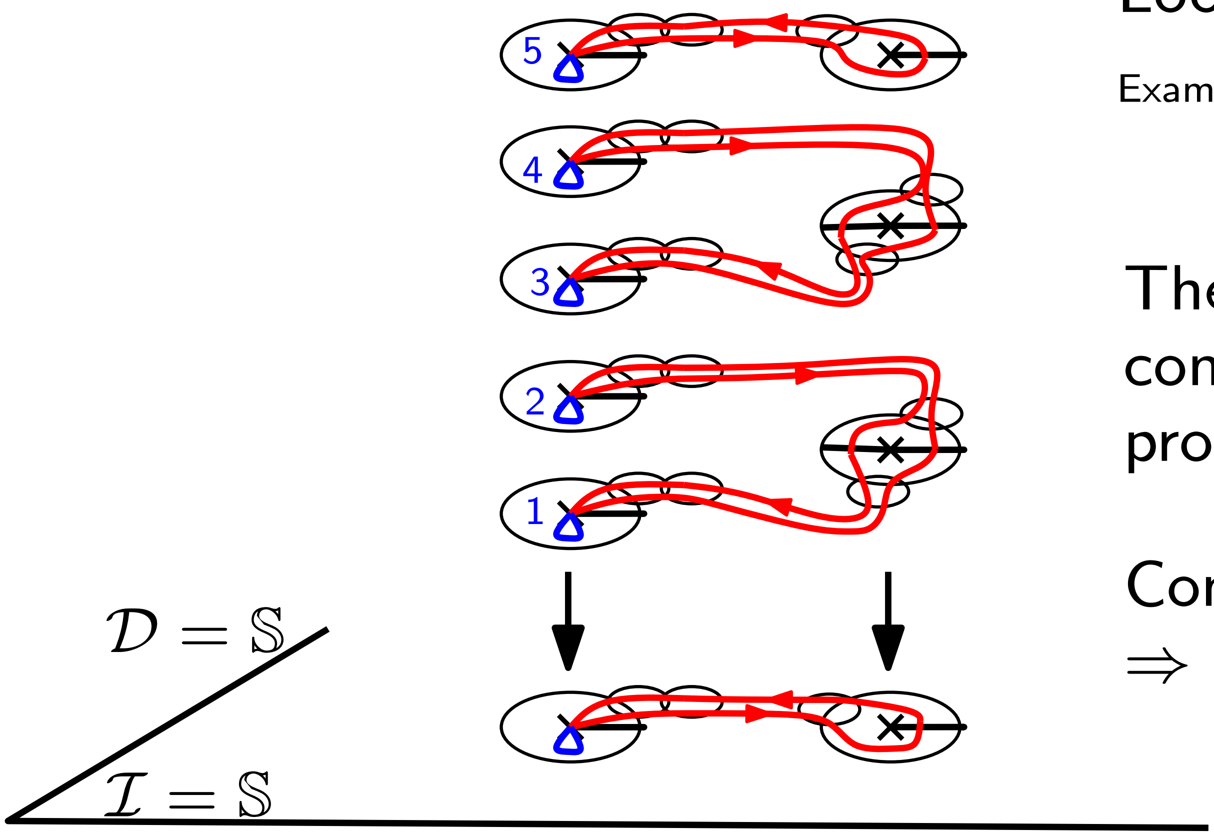

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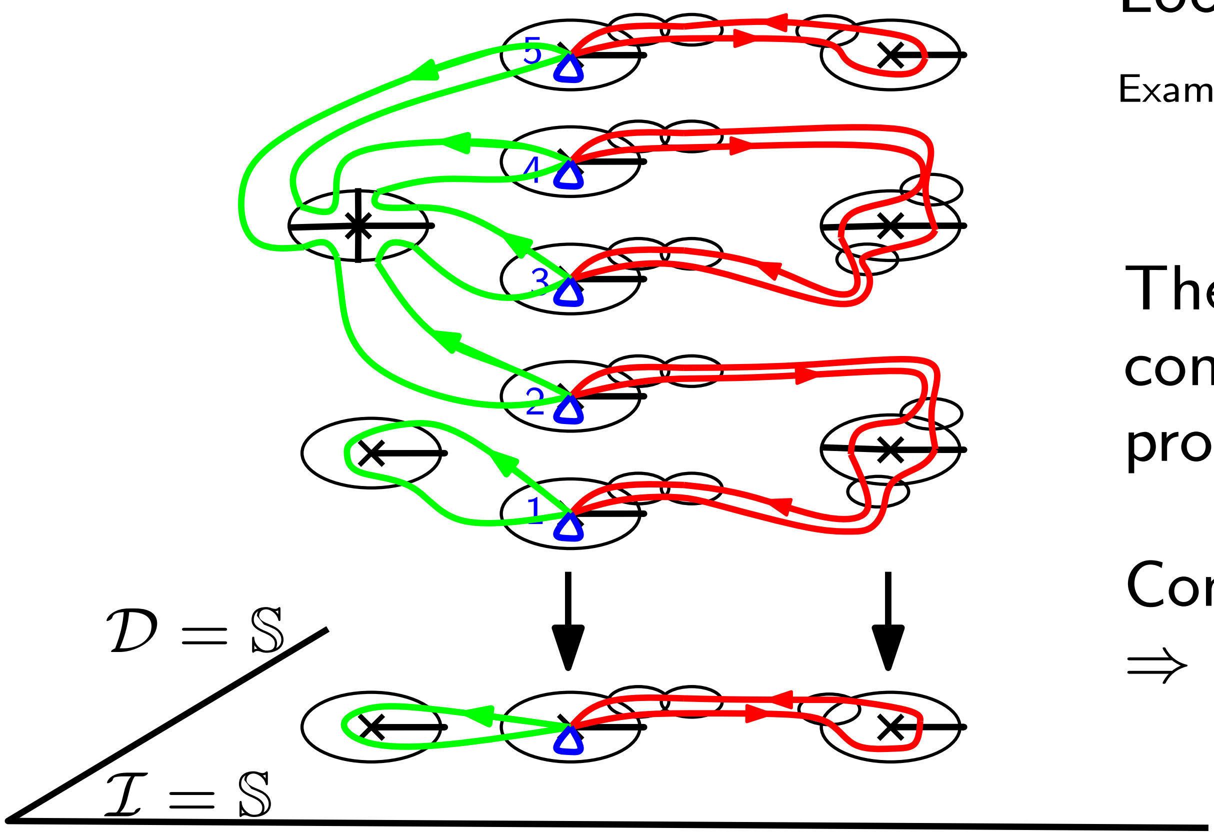

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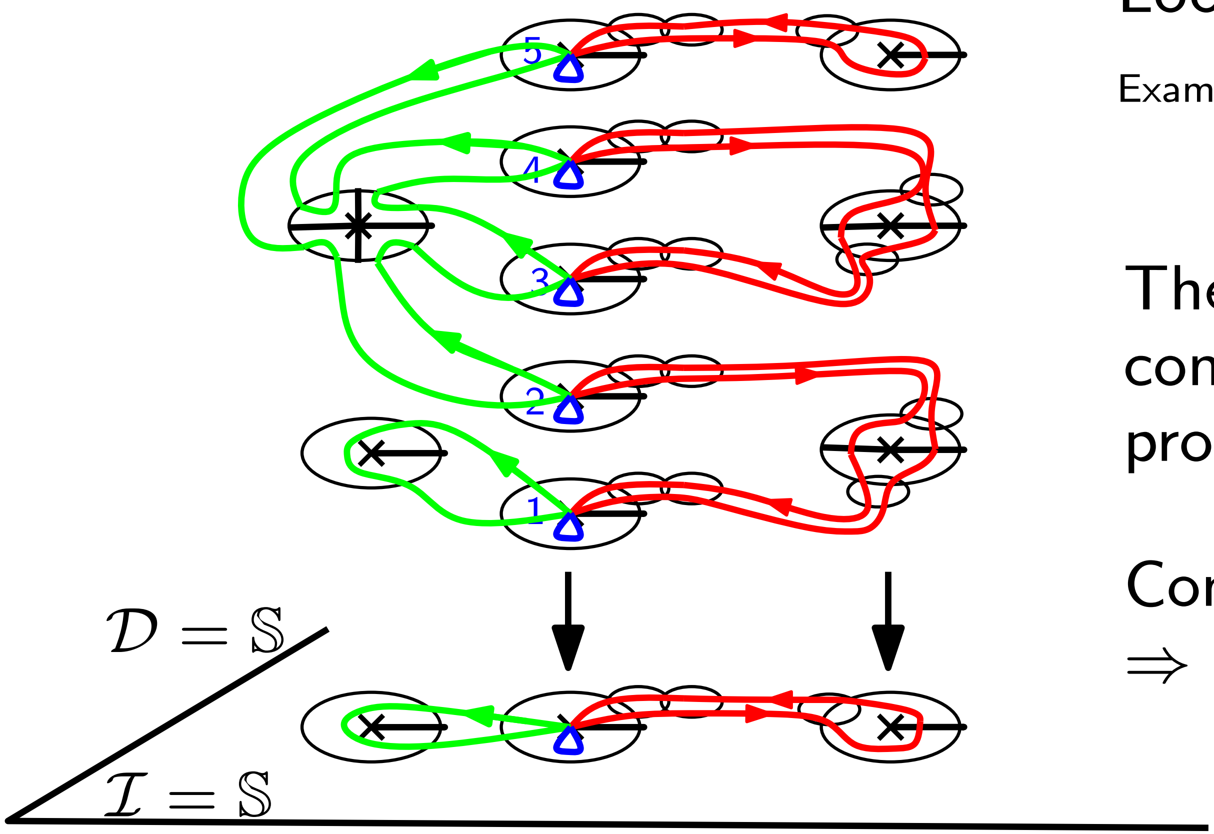

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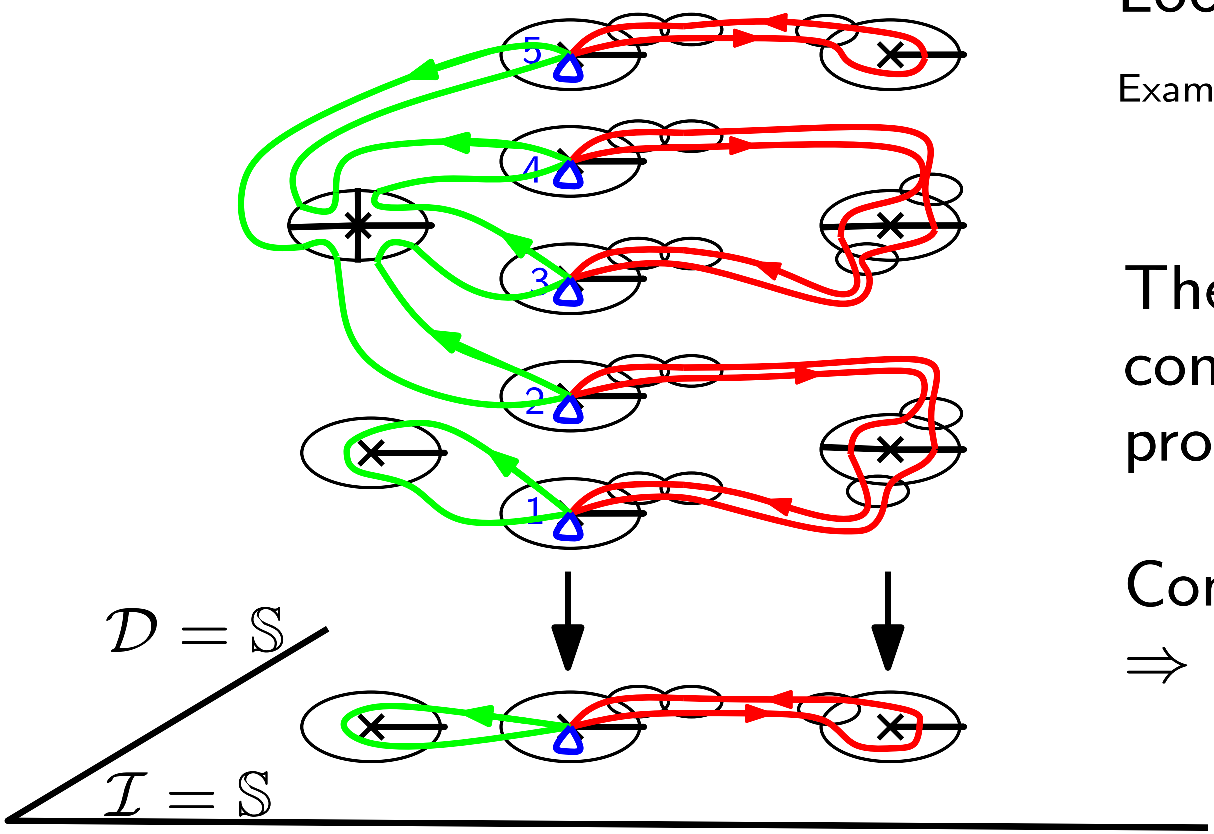

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coverings with 3 critical values and bipartite maps

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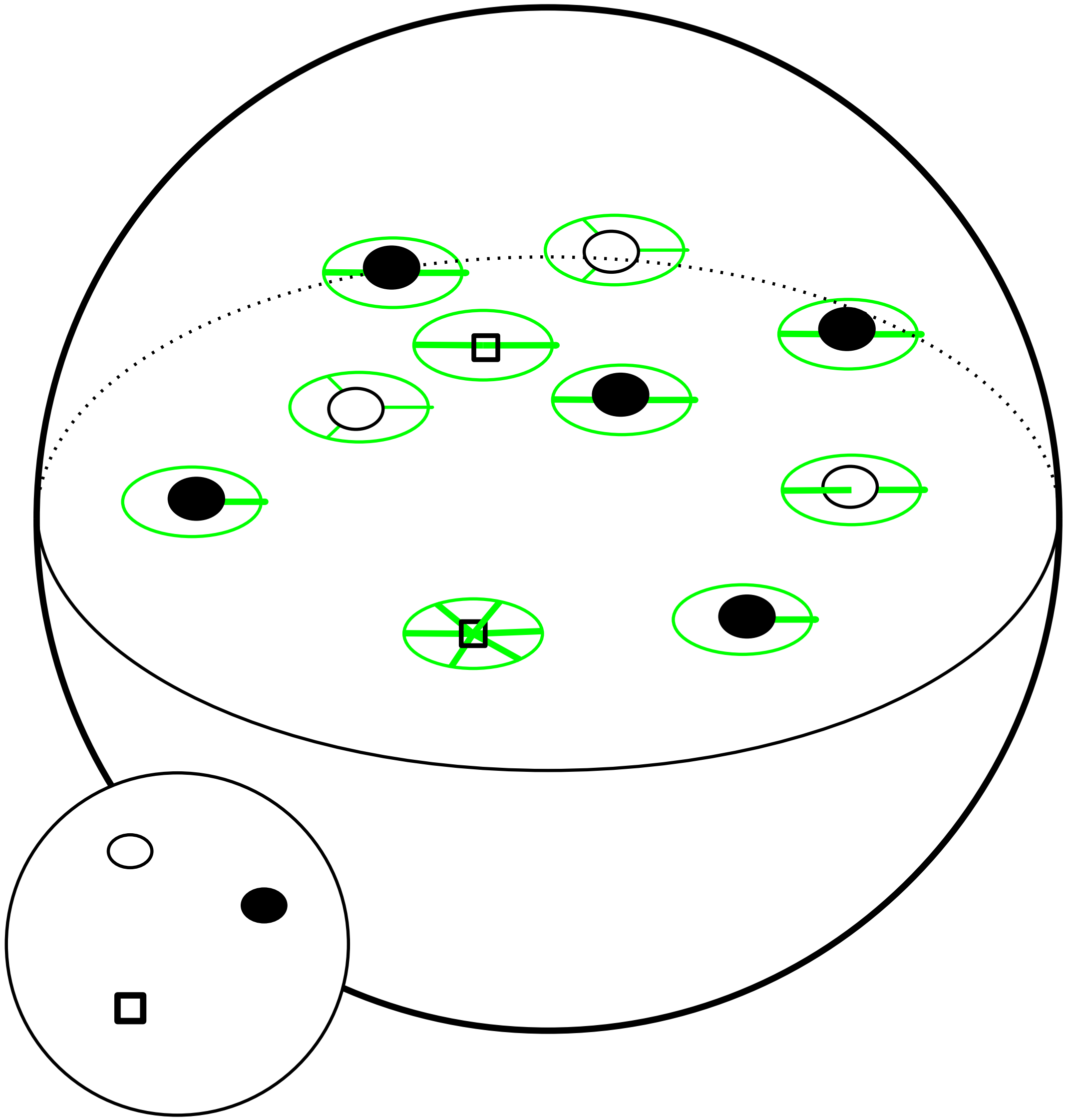

$\mathcal{I} = \mathbb{S}$

3 critical values $\quad \lambda^{\bullet} = 2^3 1^2 \quad \lambda^{\circ} = 3^2 2 \quad \lambda^{\square} = 62$

coverings with 3 critical values and bipartite maps

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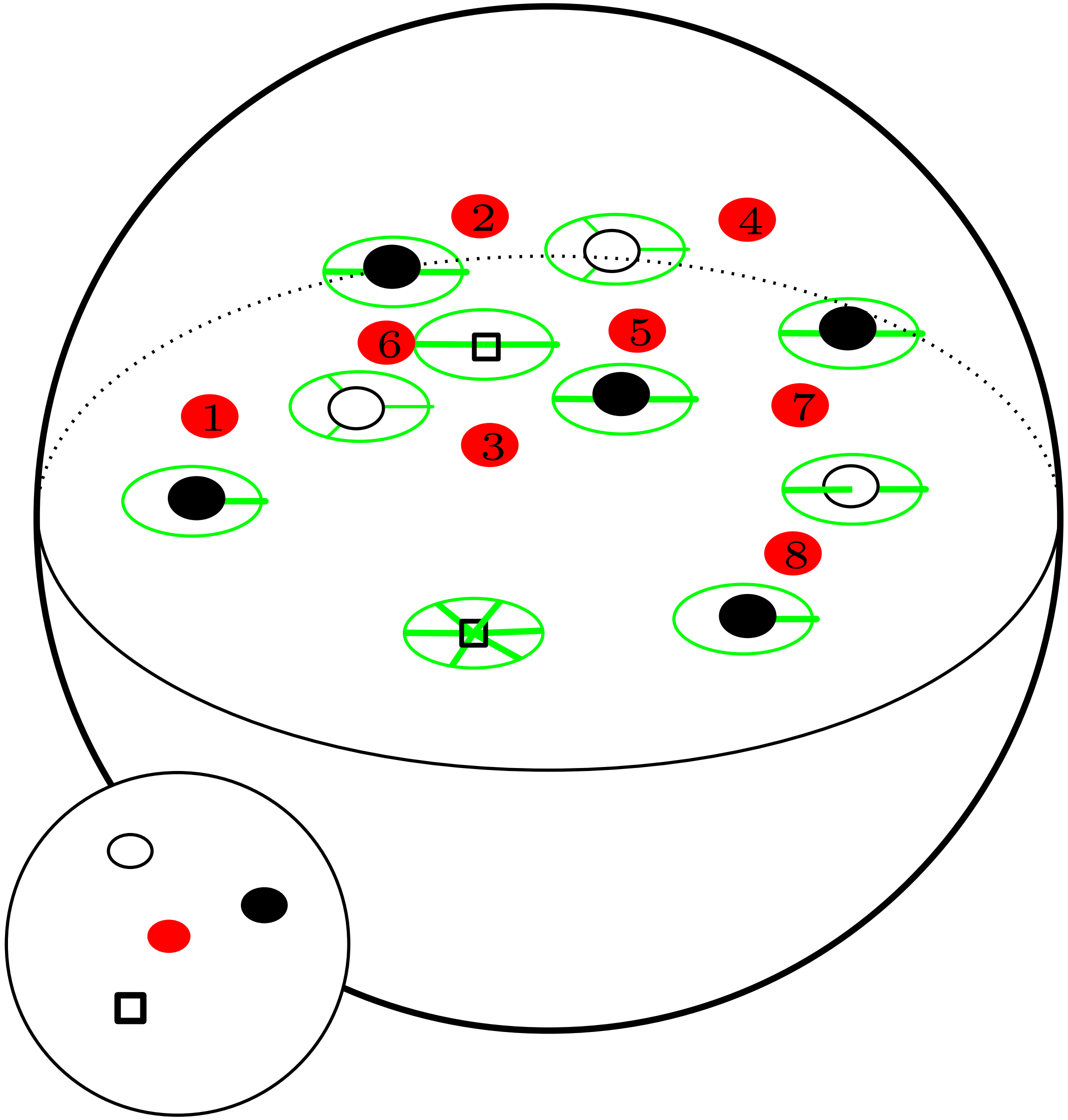

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1 regular value with labeled preimages

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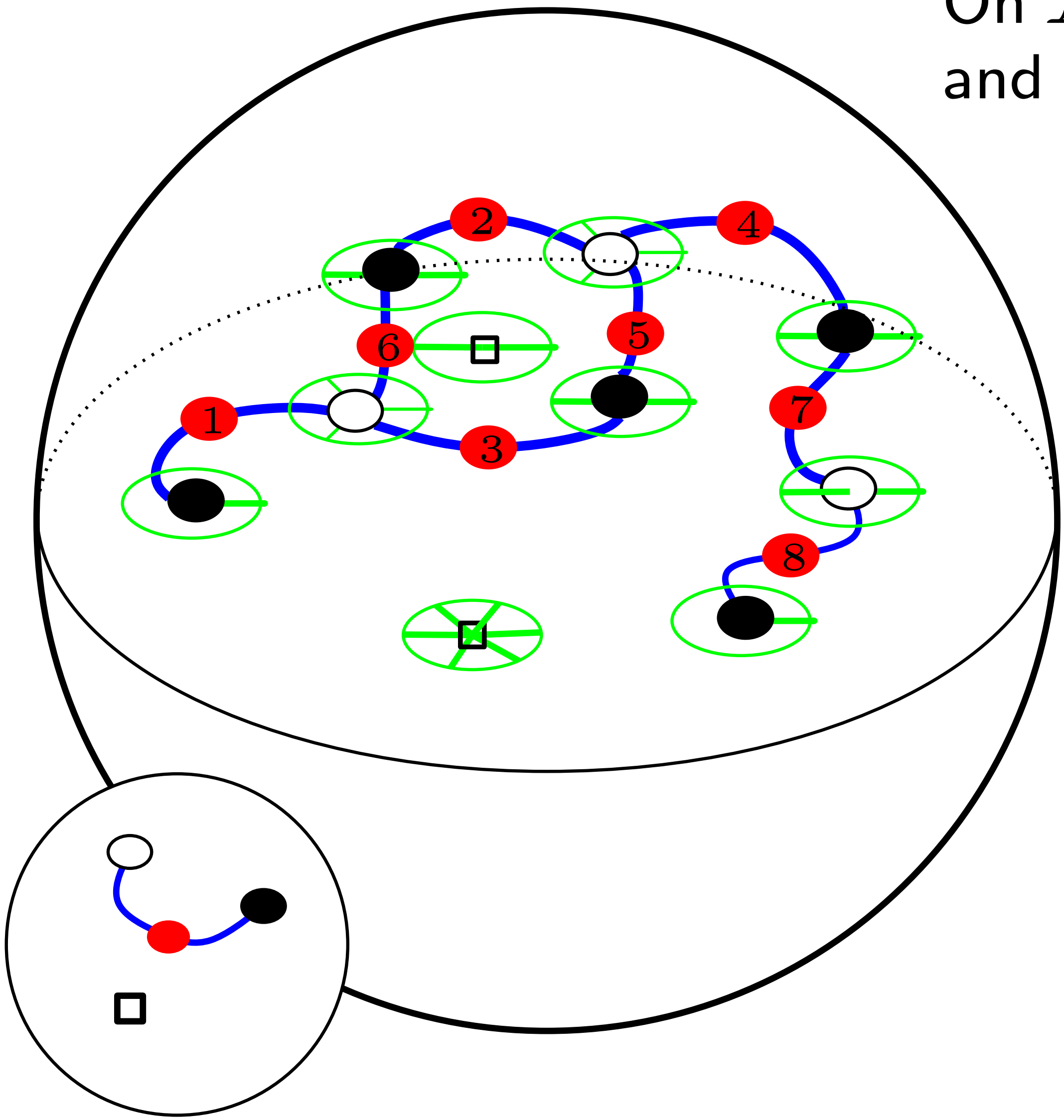

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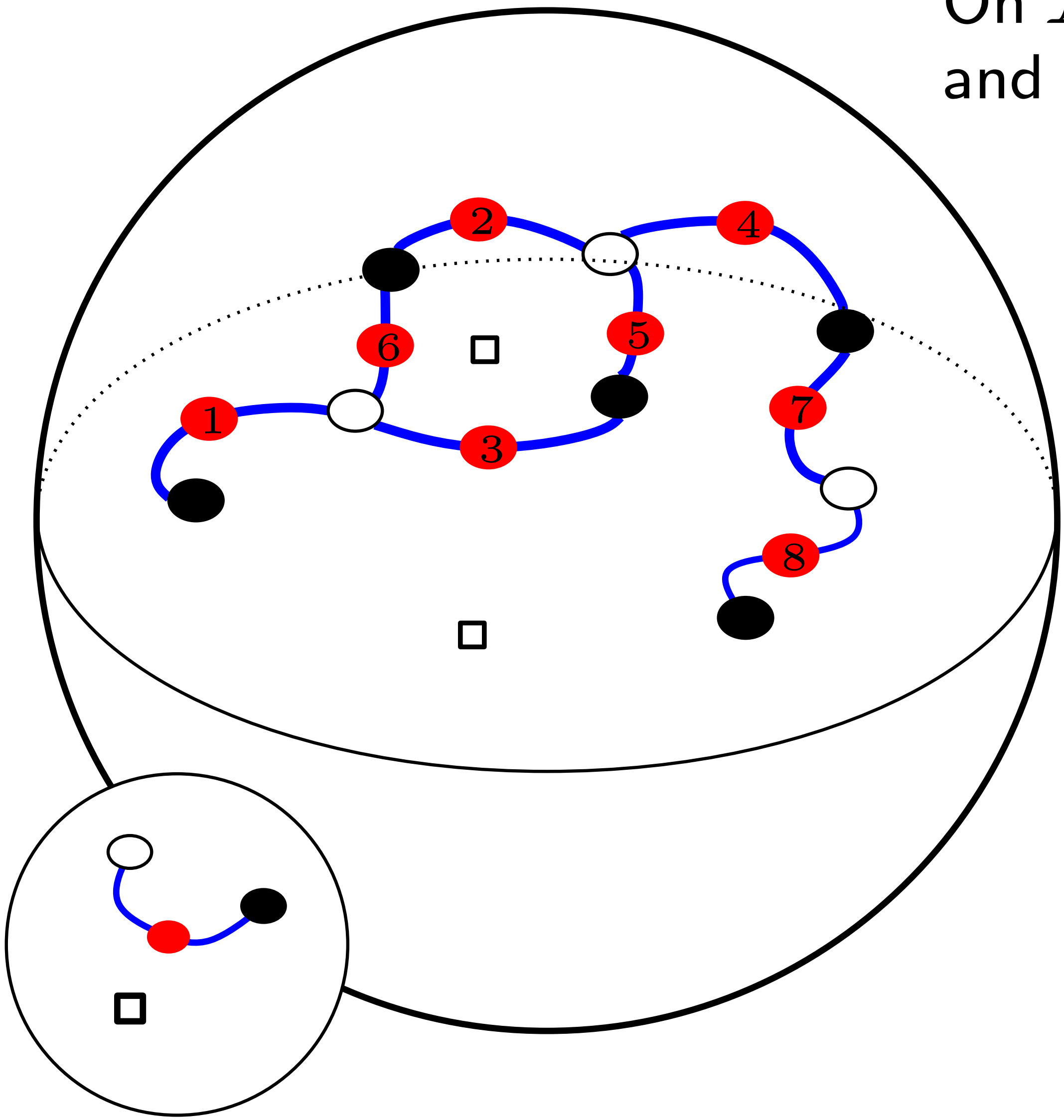

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that is, a graph embedded on the sphere with simply connected faces

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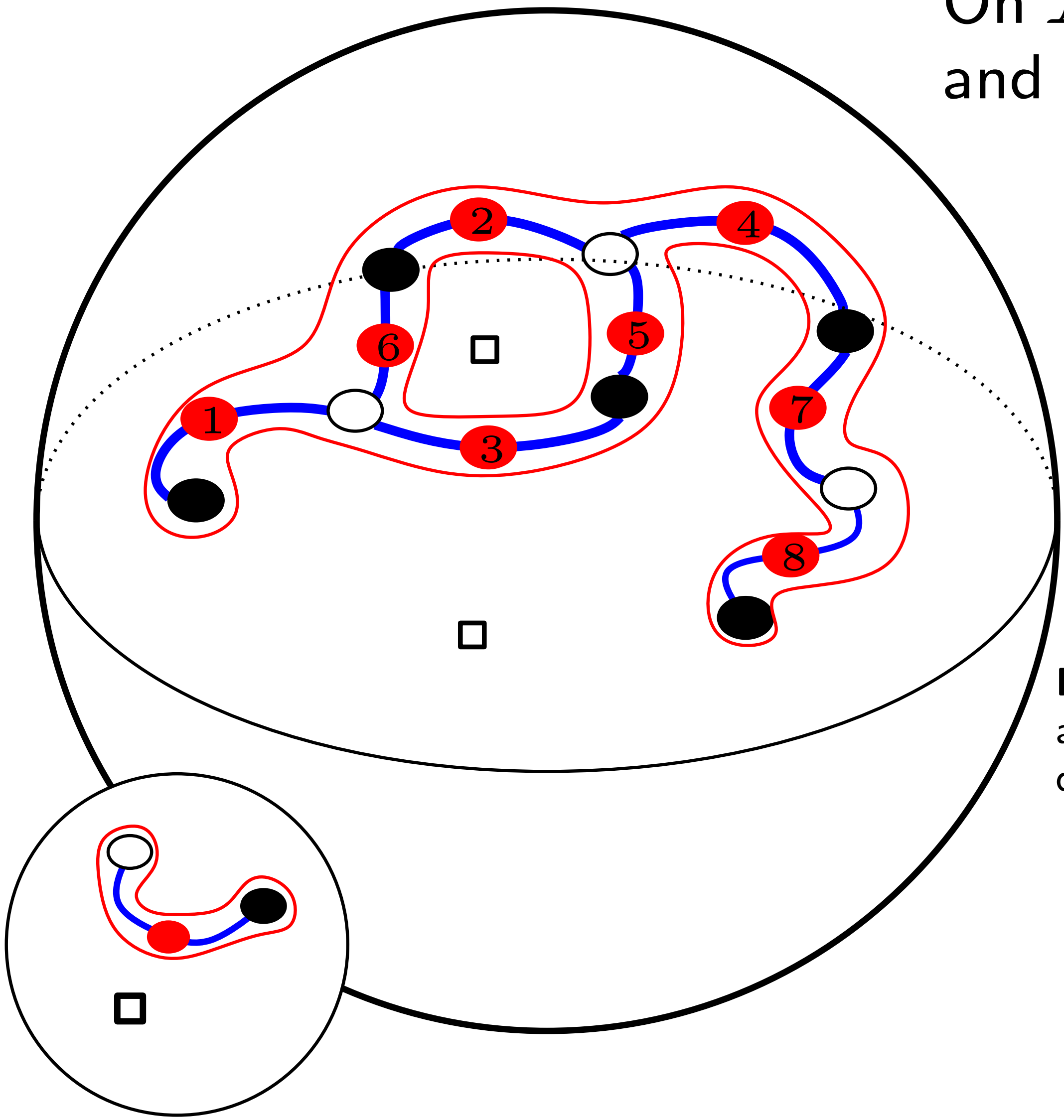

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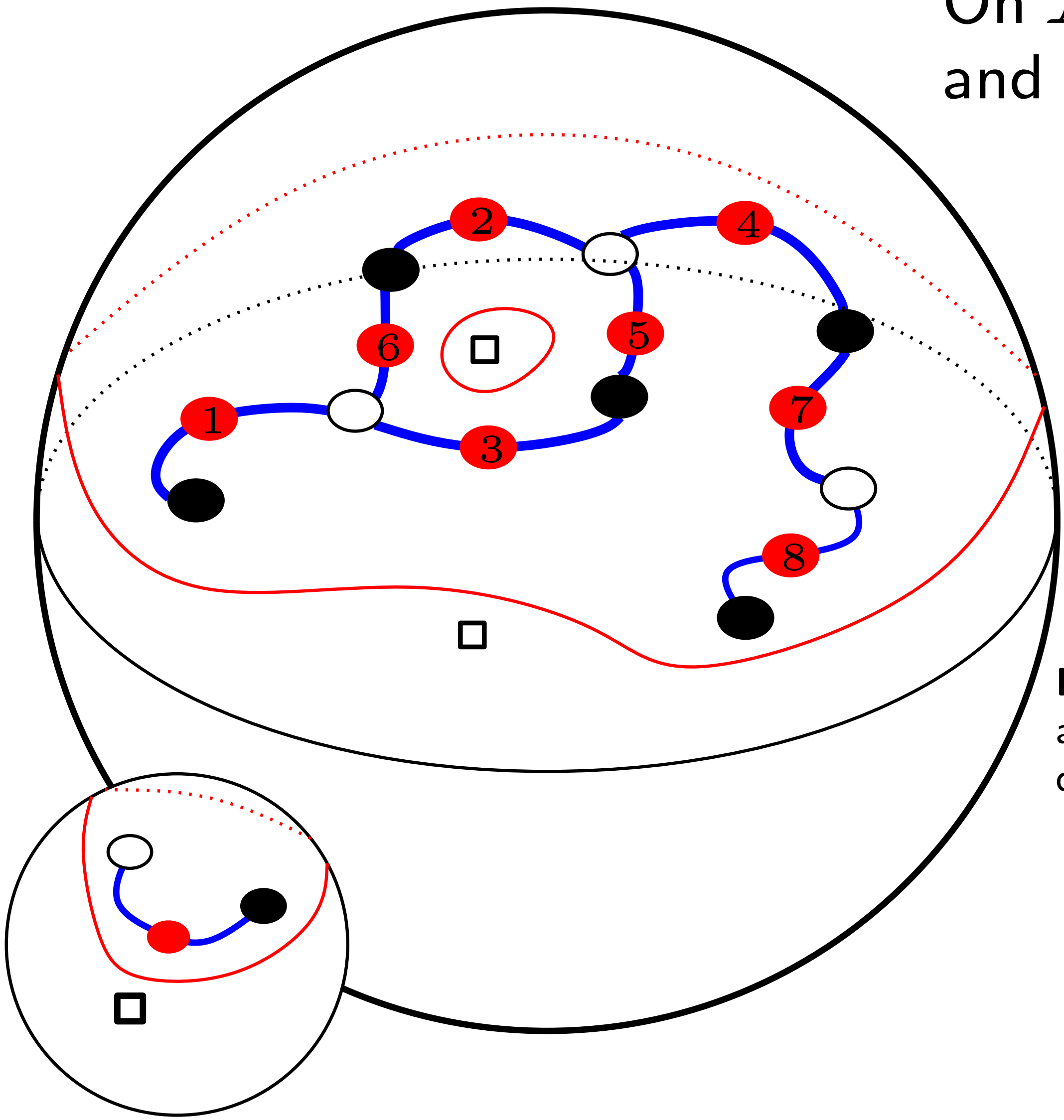

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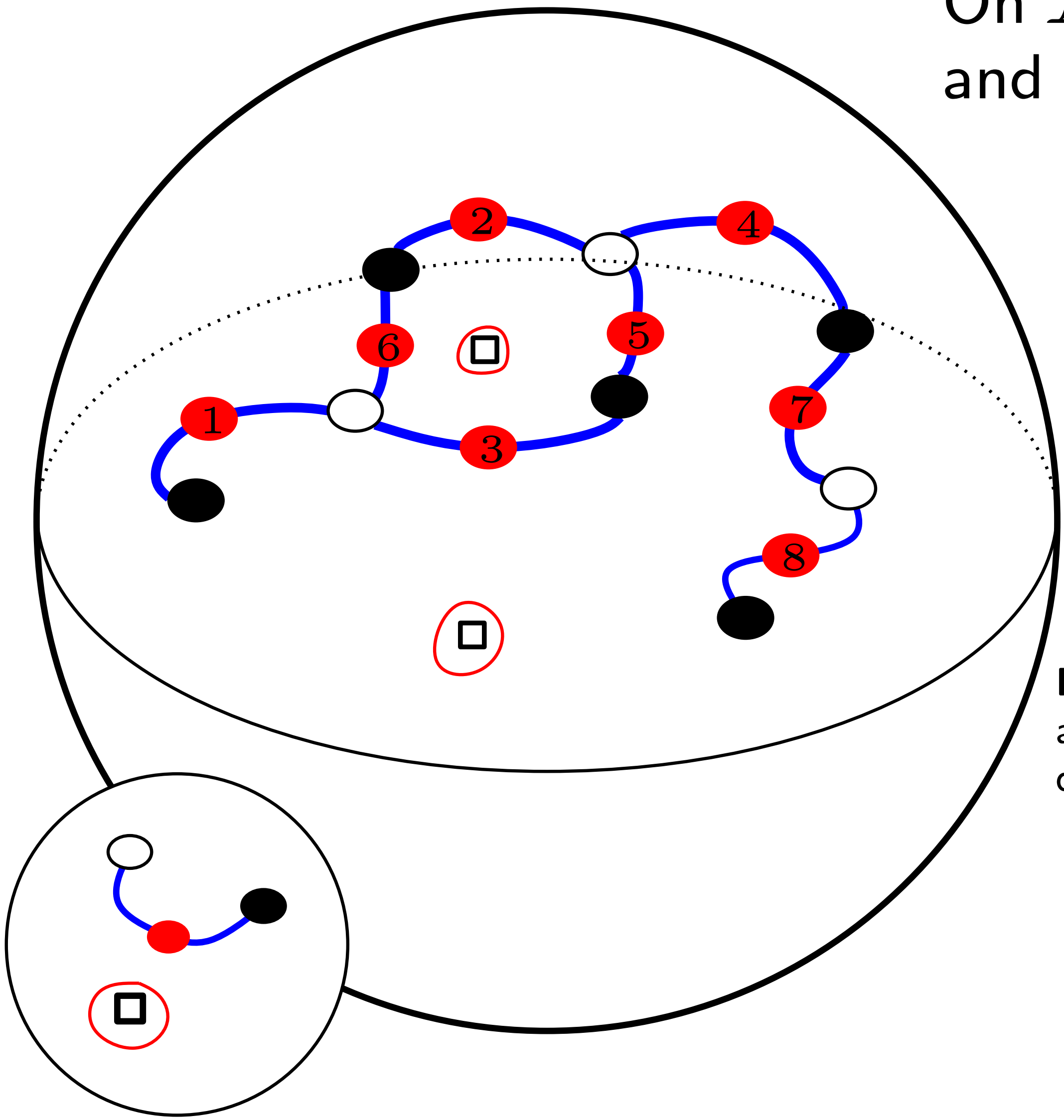

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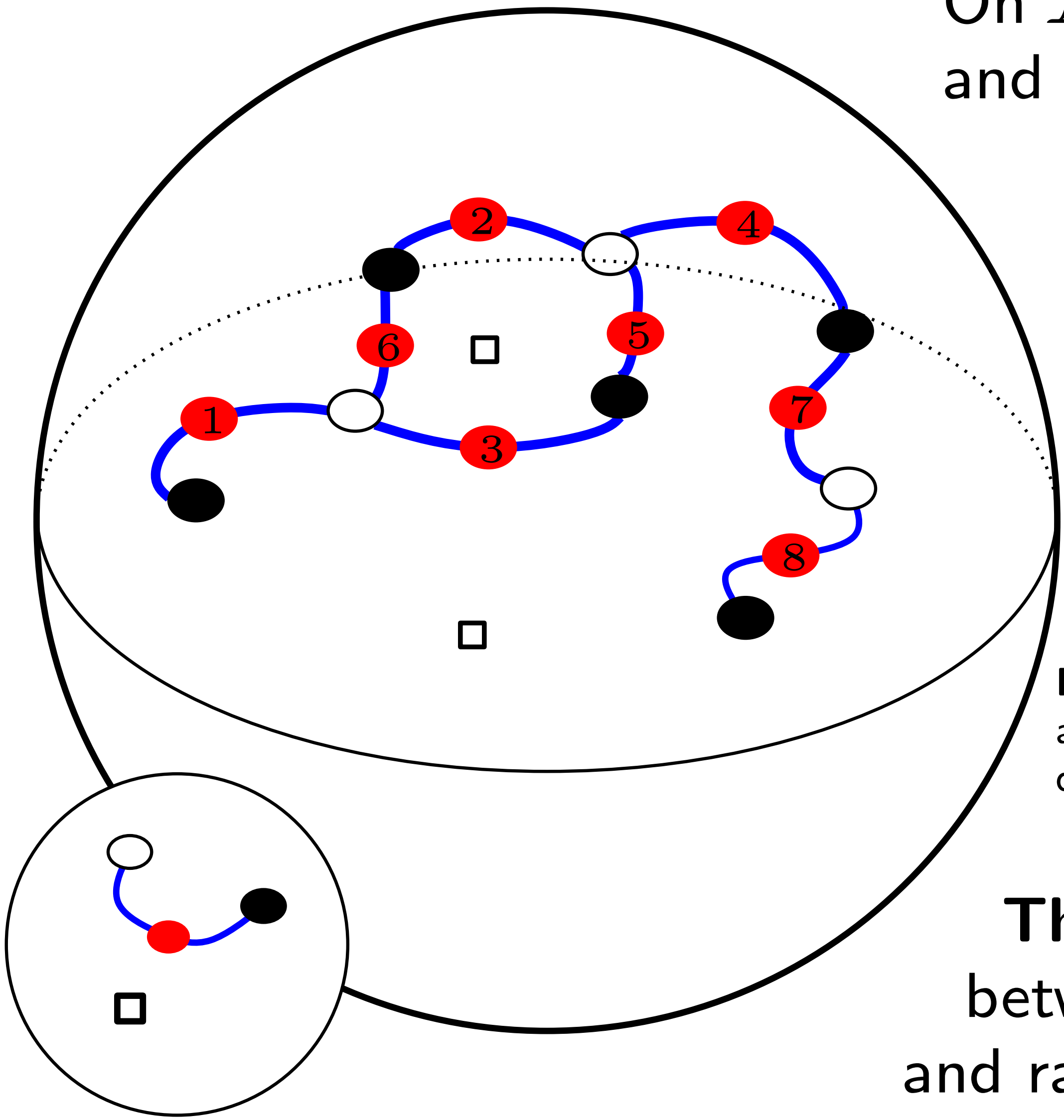

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Theorem. This is a bijection between bipartite planar maps and ramified coverings of $\mathbb{S}$ by $\mathbb{S}$ with 3 critical values.

3 critical values, bipartite maps and permutations

A loop around a critical value yields a permutation

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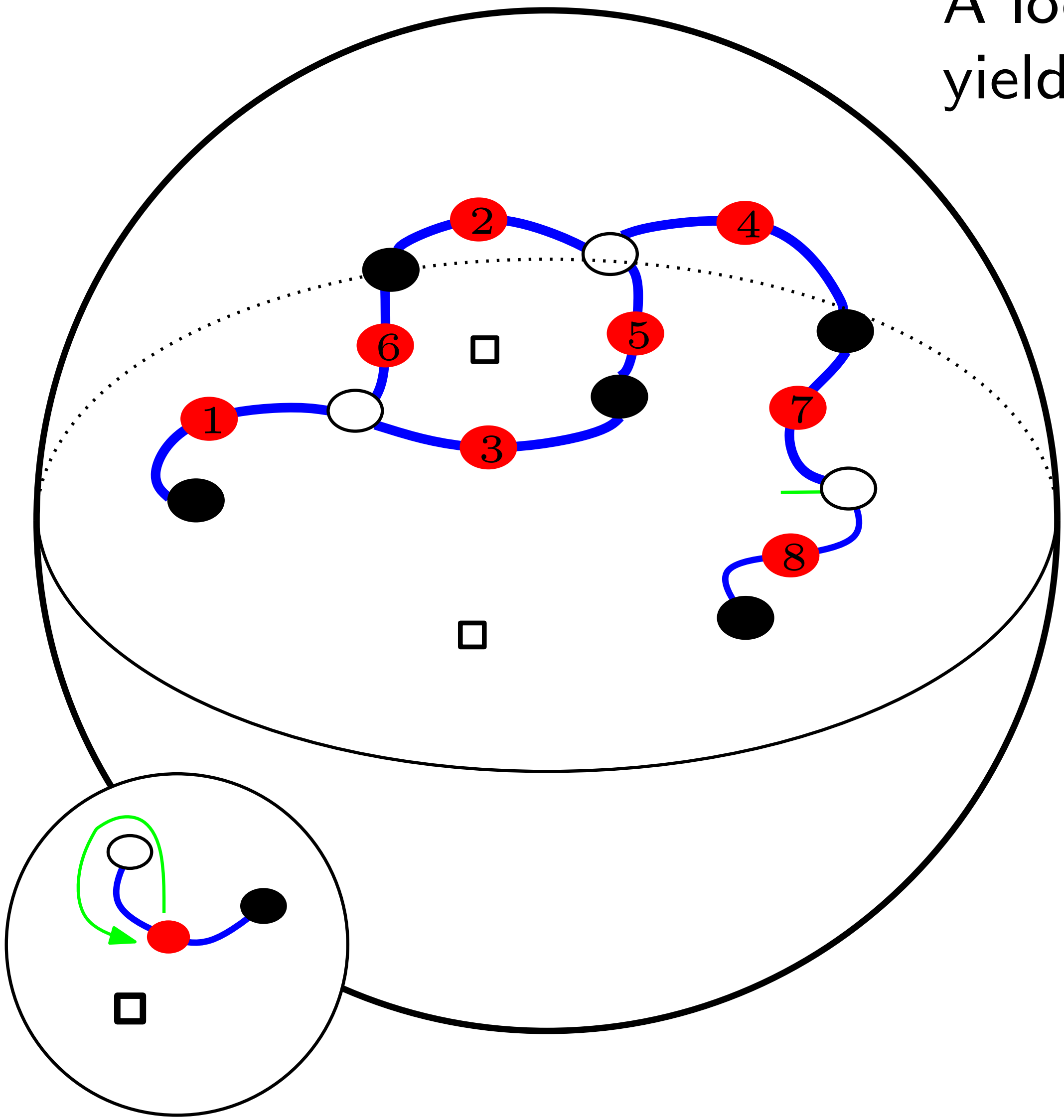

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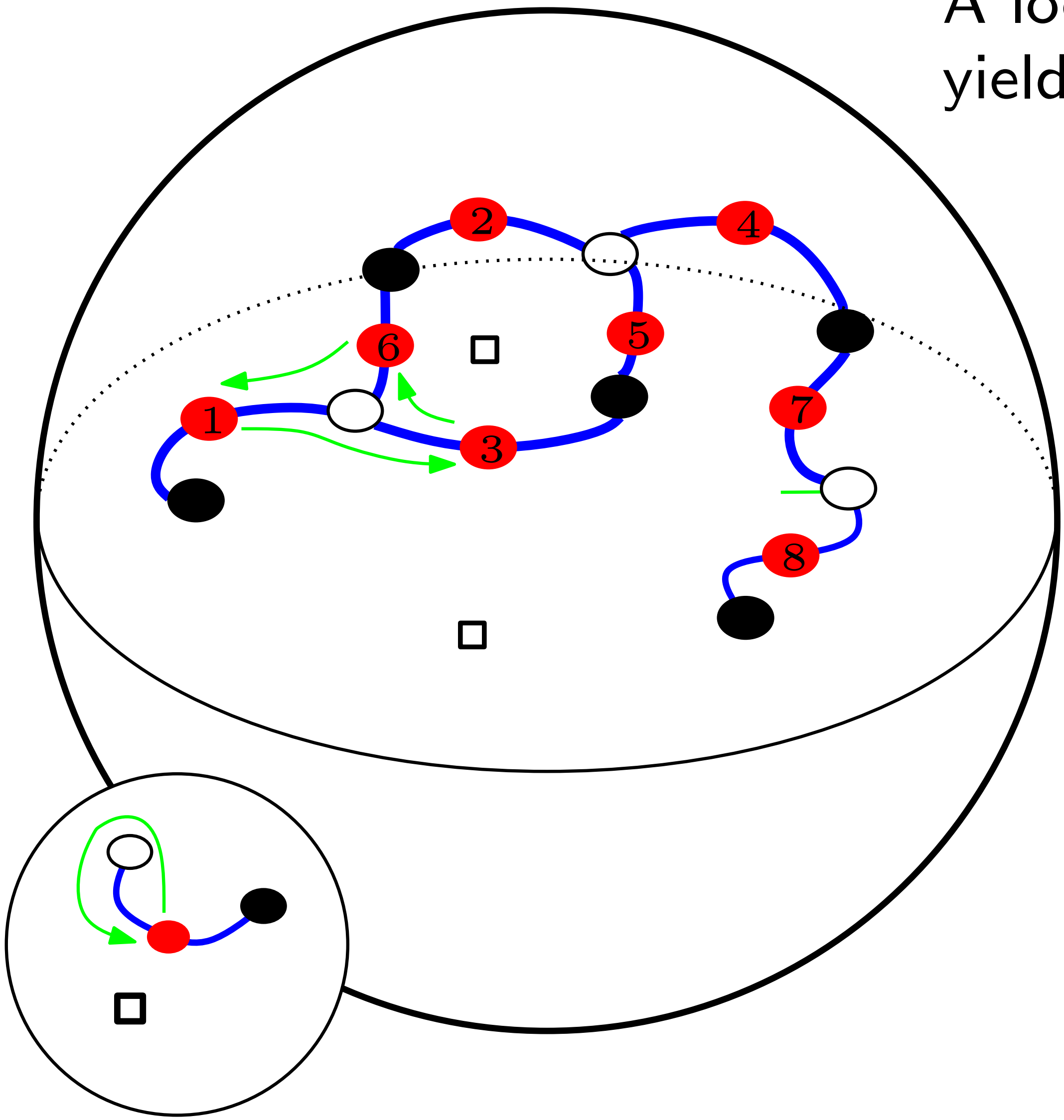
$$\sigma_\circ = (1,3,6)(2,5,4)(7,8)$$

with cyclic type λ°



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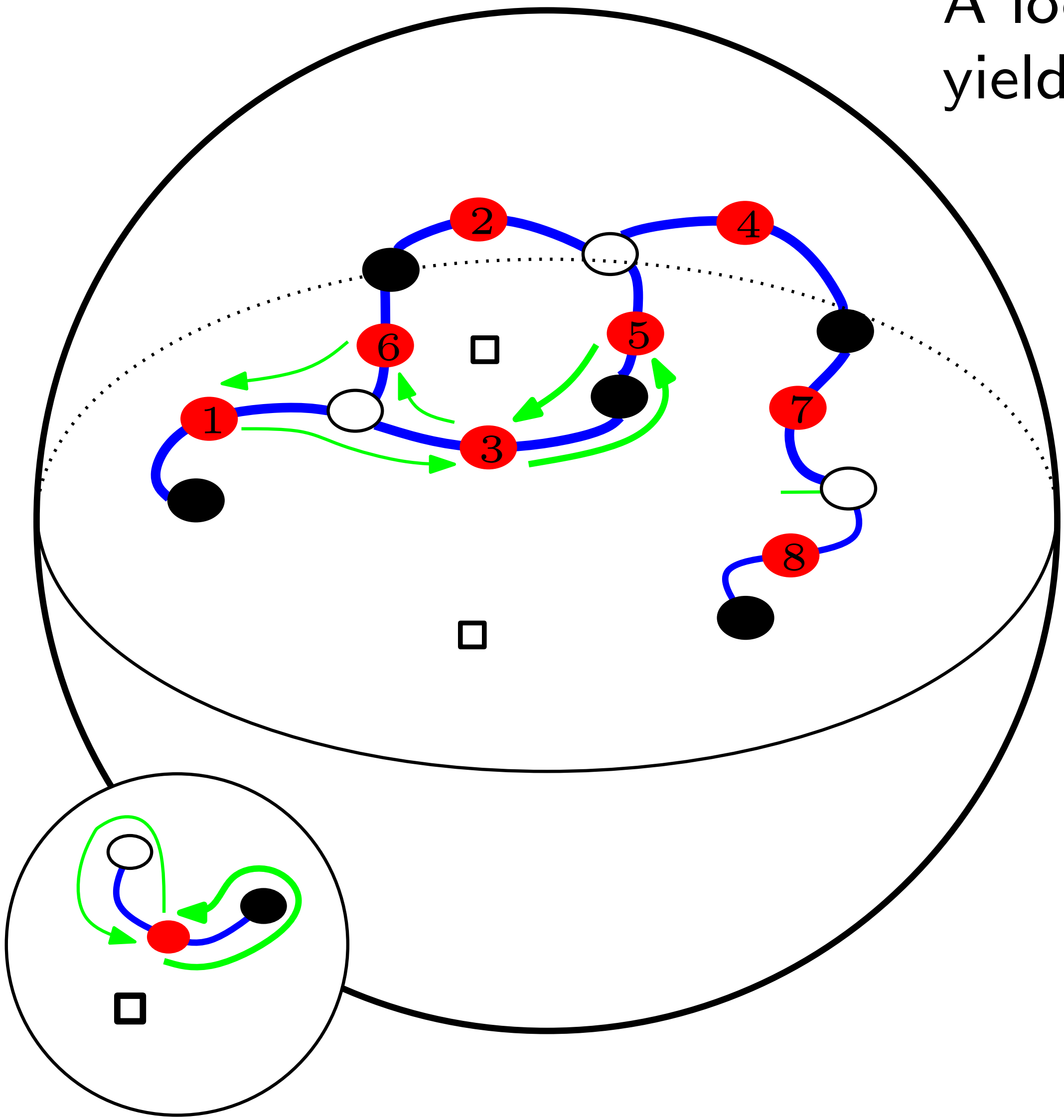

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$\sigma_{\bullet} = (1)(2,6)(3,5)(4,7)(8)$
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Cycle types $\Leftrightarrow$ degree distributions

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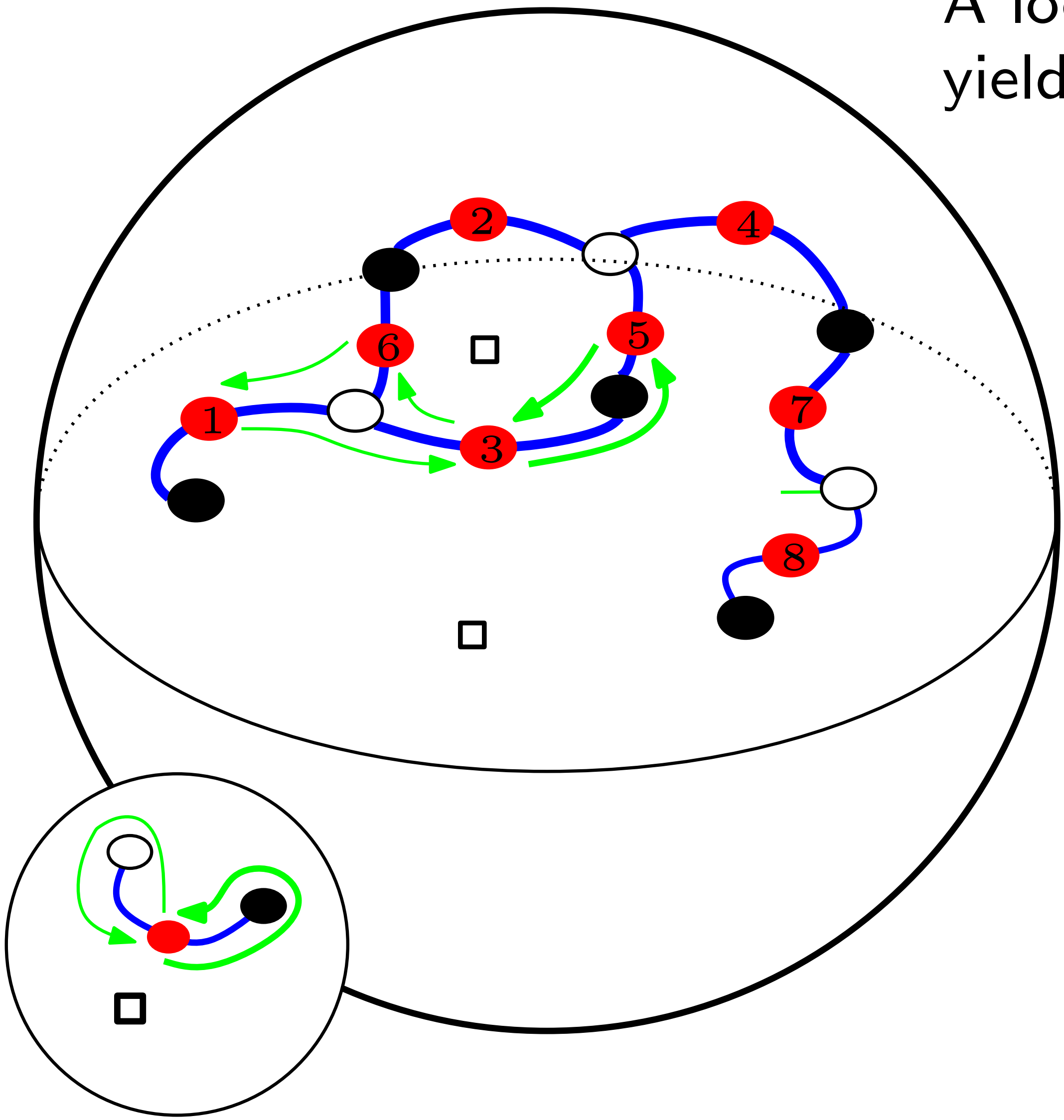

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Cycle types ⇔ degree distributions

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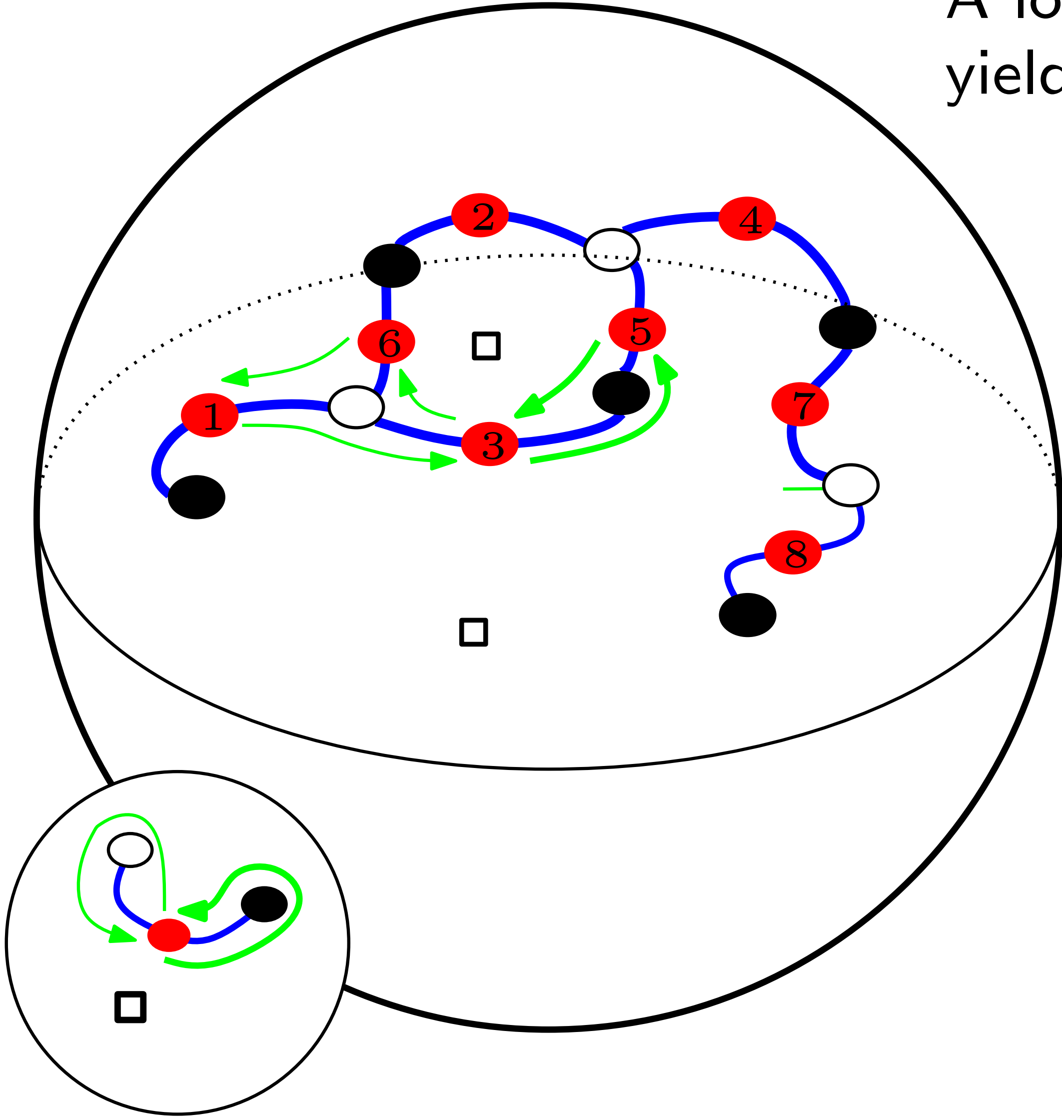

$\mathcal{I} = \mathbb{S}$

A loop around a critical value yields a permutation

$\sigma_{\circ} = (1,3,6)(2,5,4)(7,8)$
with cyclic type λ°

$\sigma_{\bullet} = (1)(2,6)(3,5)(4,7)(8)$
with cyclic type $\lambda^{\bullet}$

Cycle types $\Leftrightarrow$ degree distributions

What about $\sigma_{\square}$ and $\lambda_{\square}$?

3 critical values $\quad \lambda^{\bullet} = 2^3 1^2 \quad \lambda^{\circ} = 3^2 2 \quad \lambda^{\square} = 62$

1 regular value with labeled preimages

3 critical values, bipartite maps and permutations

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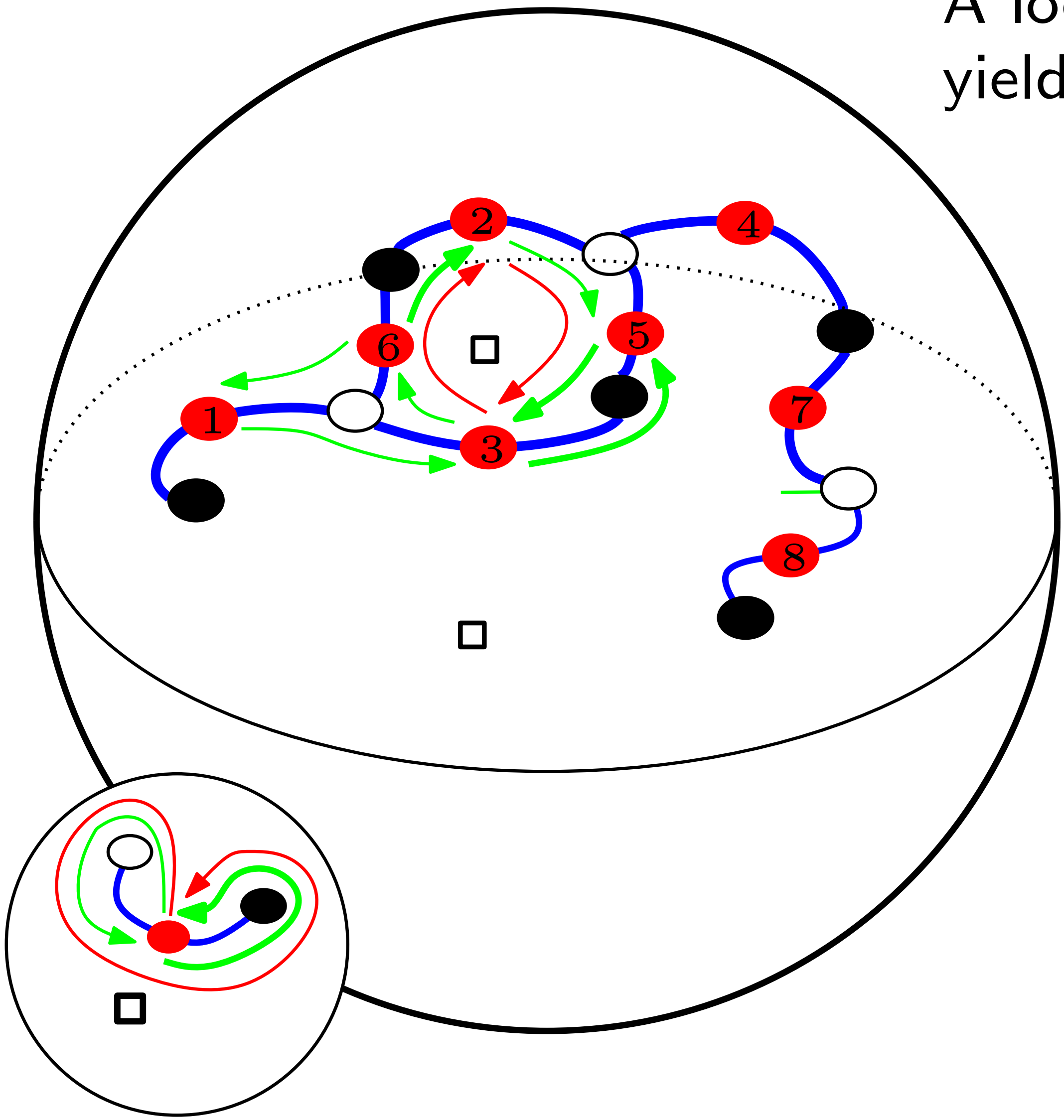

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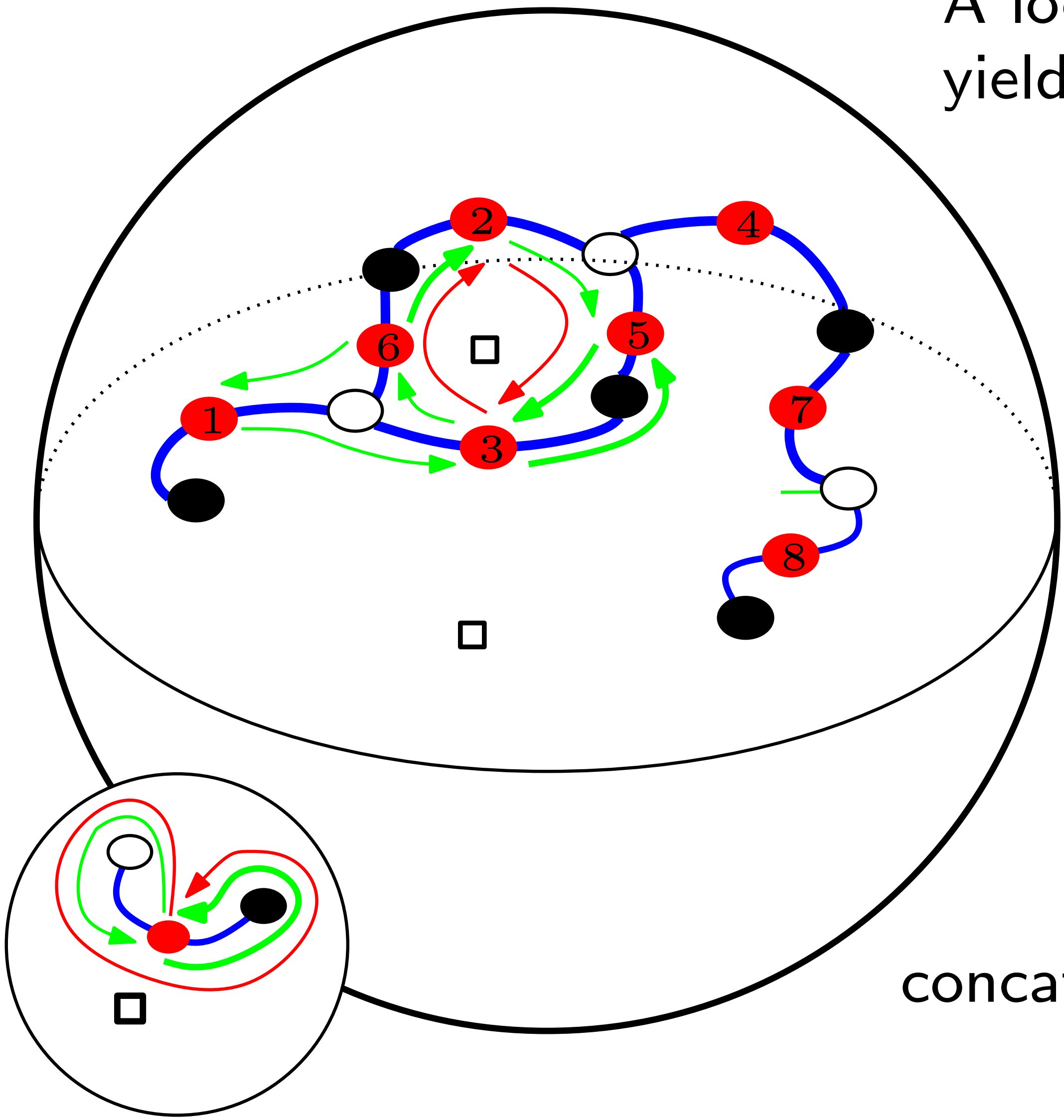

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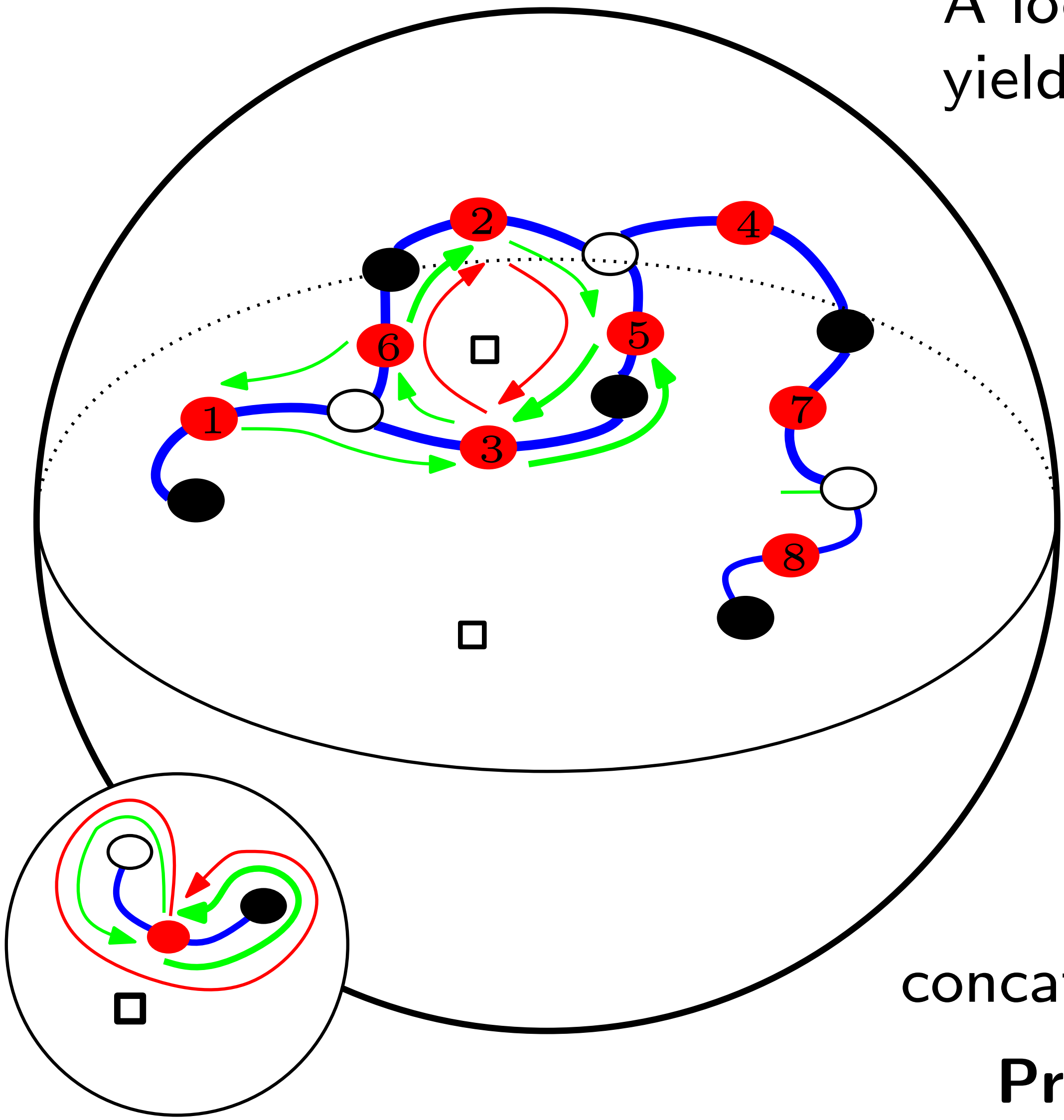

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Proposition: $\sigma_{\circ}\sigma_{\bullet} = \sigma_{\square}$.

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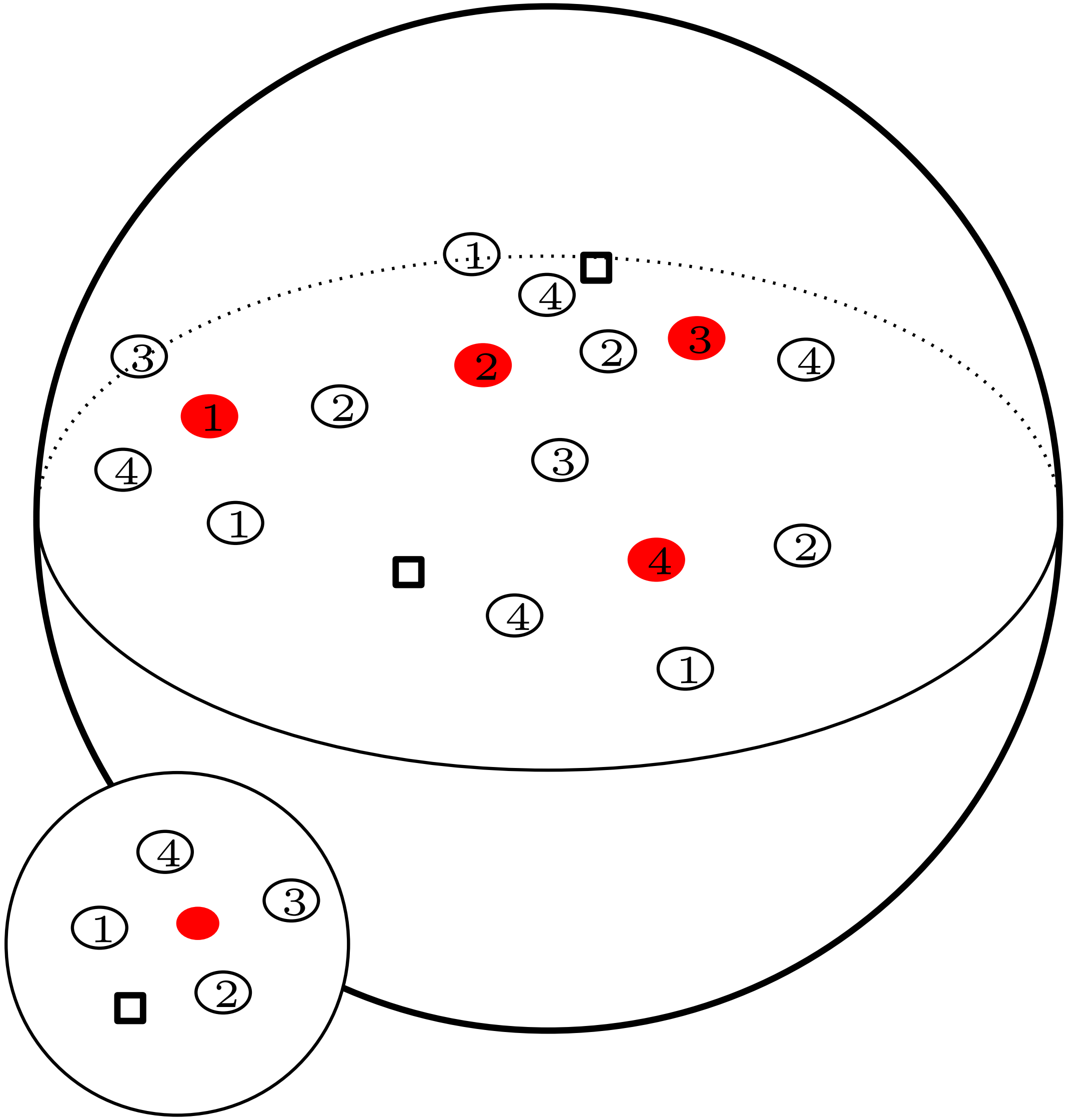
$m+1$ critical values, m-constellations, permutations



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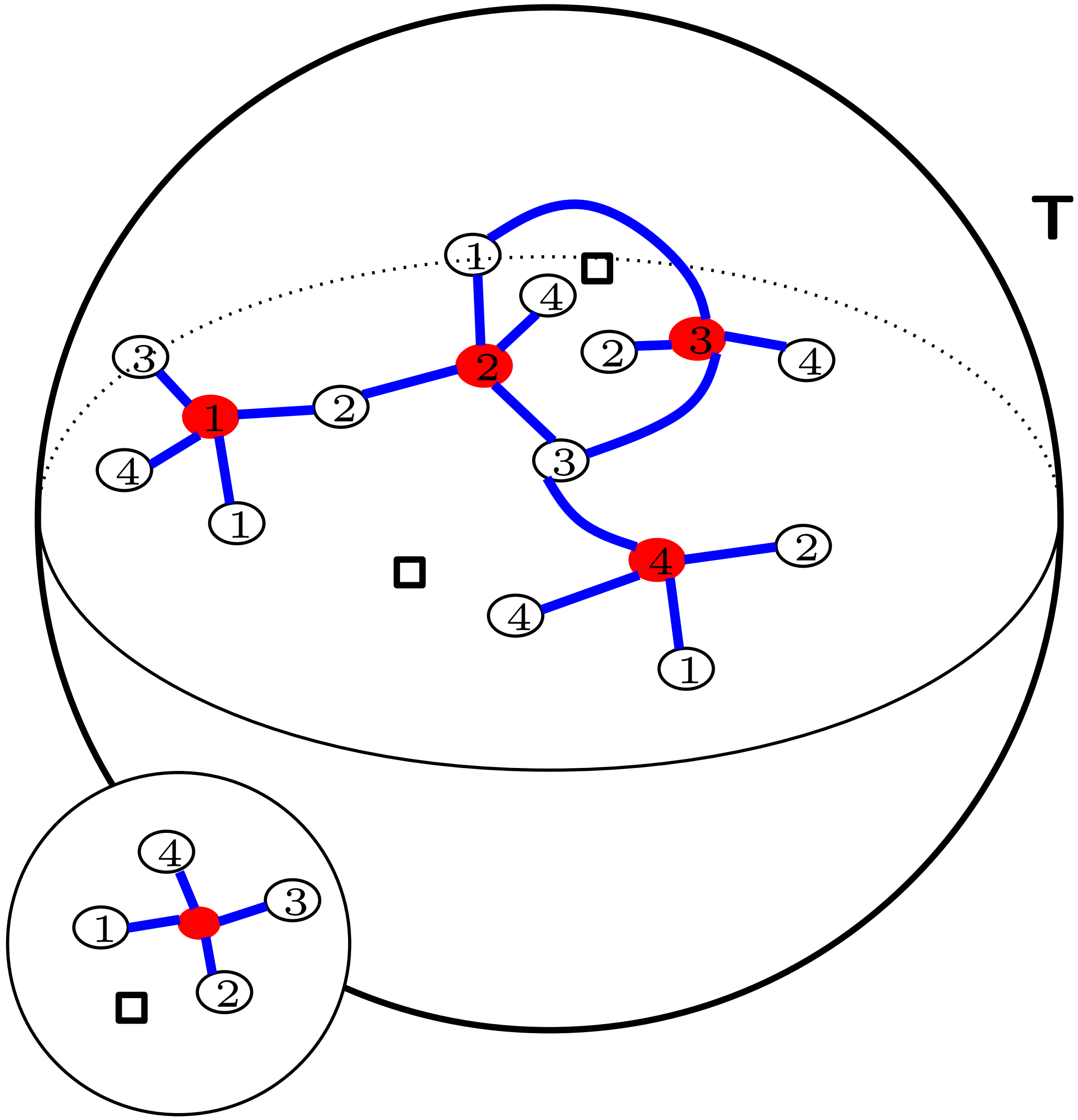

The preimage of the m-star is called a star-constellation.

Thm. Planar star-constellations with:
- n labelled m-stars,
- $\lambda_j^{\square}$ faces of degree j,
- $\lambda_j^{(i)}$ color i vertices of degree j

are in bijection with minimal transitive factorizations

$$\sigma_1 \cdots \sigma_m = \sigma_{\square}$$

with σ_i of cyclic type $\lambda^{(i)}$.

Monodromy, permutations, constellations summary

Theorem. There is a bijection between

- Labelled ramified covering of $\mathbb{S}$ of type $\Lambda = (\lambda_0, \ldots, \lambda_m)$
- Factorizations $(\sigma_1 \cdots \sigma_m = \sigma_0)$ of type Λ
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$\mathcal{D} = \mathbb{S} \Leftrightarrow$ minimality $\Leftrightarrow$ planarity.

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Specializations.

— $m = 2$: bipartite maps with n edges

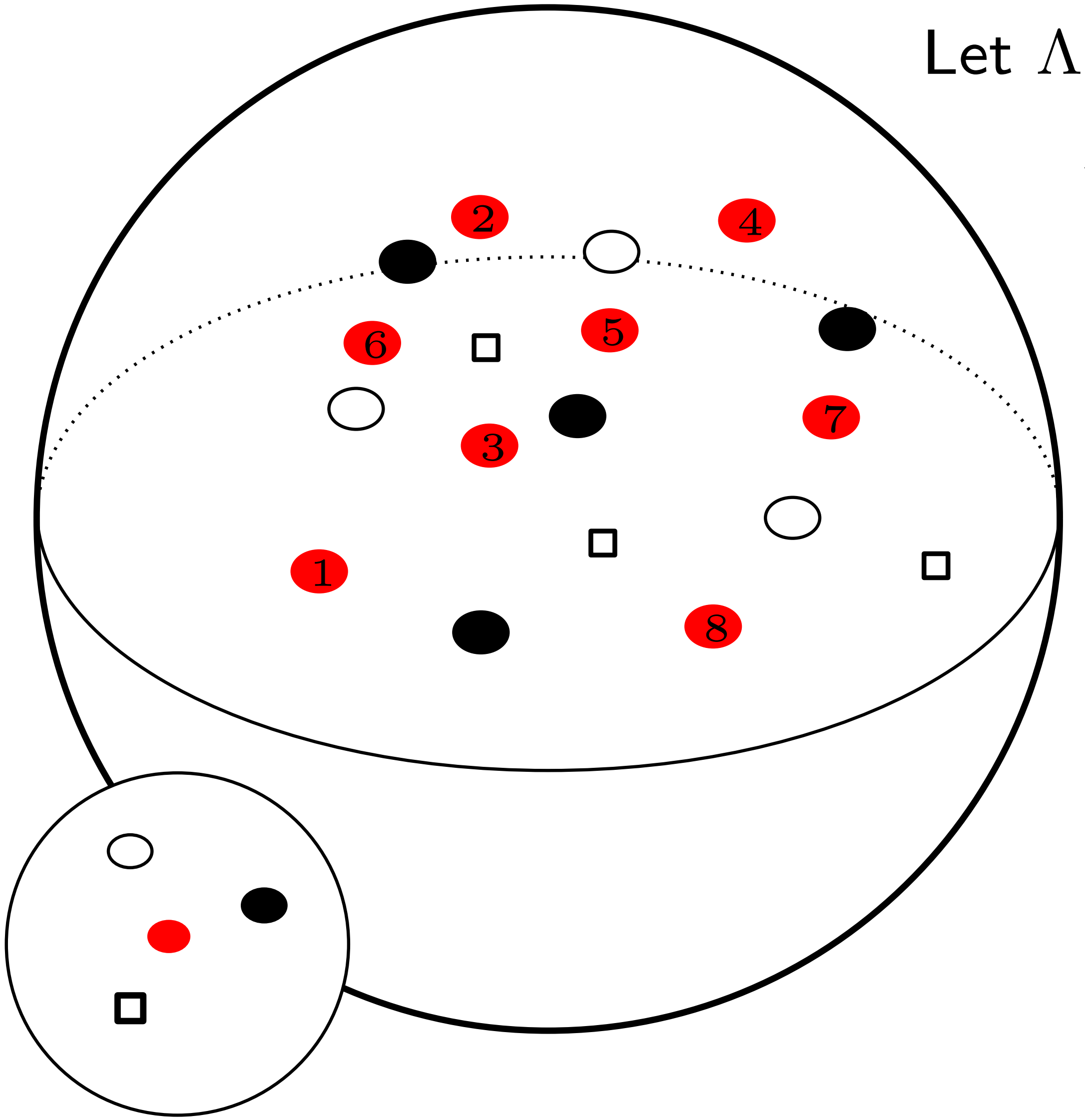
— $m = 2$ and $\lambda_\bullet = 2^{\frac{n}{2}}$: all • have deg 2 $\Leftrightarrow$ nonbipartite maps ($\frac{n}{2}$ edges)

— for all $i \geq 1$, $\lambda^{(i)} = 21^{n-2}$: factorizations in transpositions.
coverings with almost only simple branch points;   increasing maps

Ramified coverings and ”trivial” bijections:

combinatorial data structures

Application to the design of trivial bijections for maps



Let $\Lambda = (\lambda_{\square}, \lambda_{\bullet}, \lambda_{\circ})$

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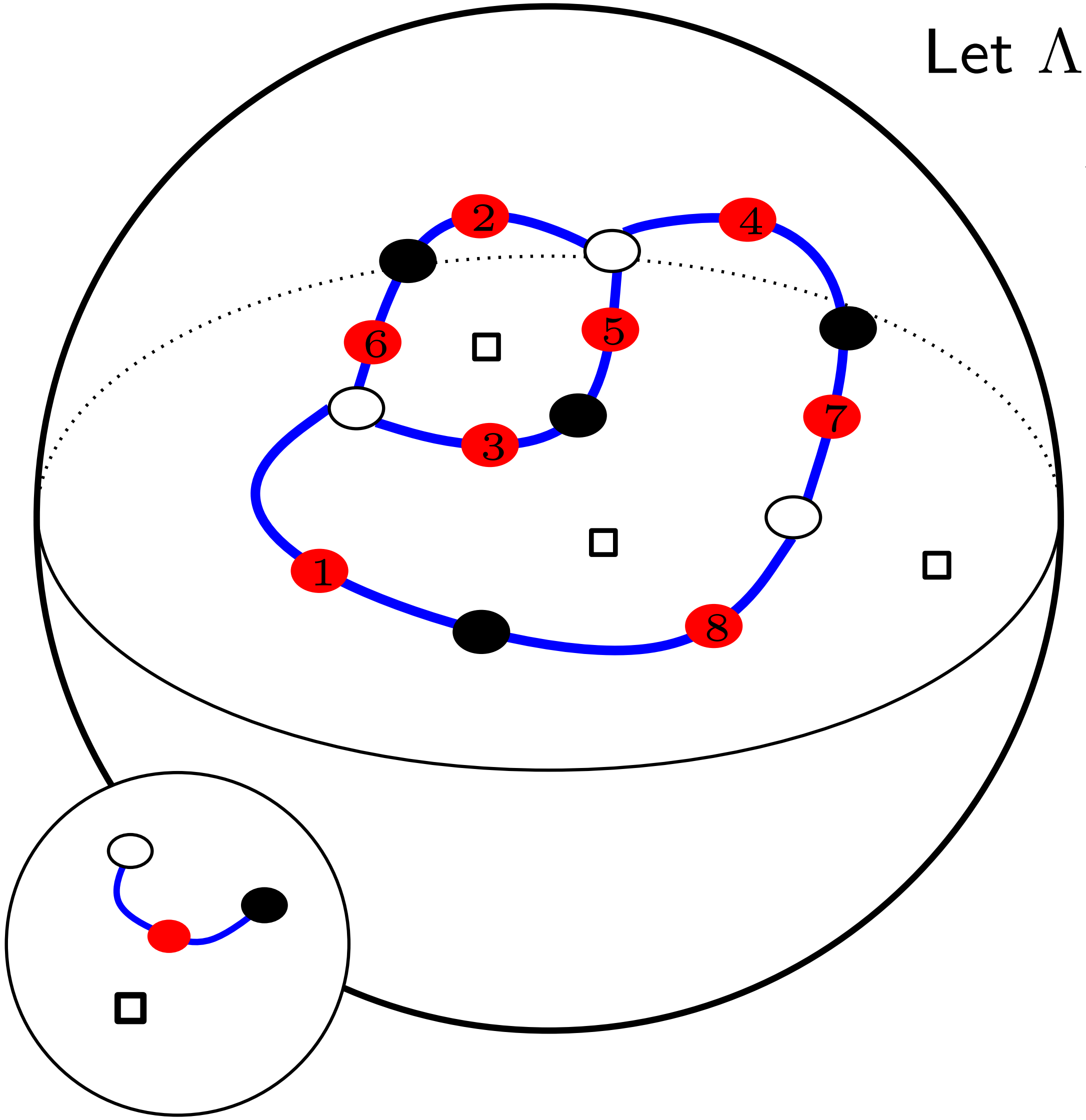

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$\circ$—$\bullet$—$\bullet$ $\Rightarrow$ bipartite map

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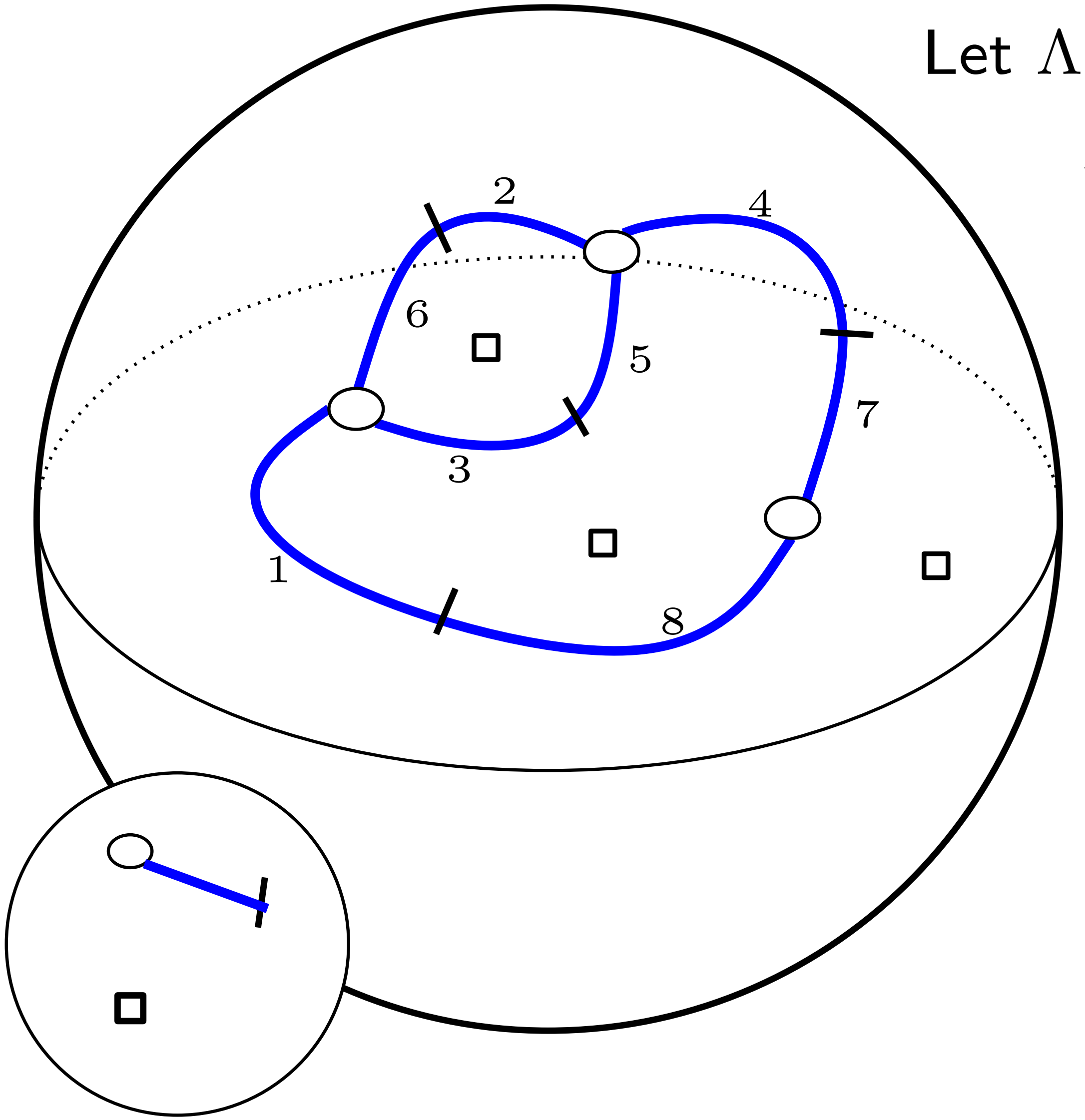

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with $\lambda_\bullet = 2^{\frac{n}{2}}$

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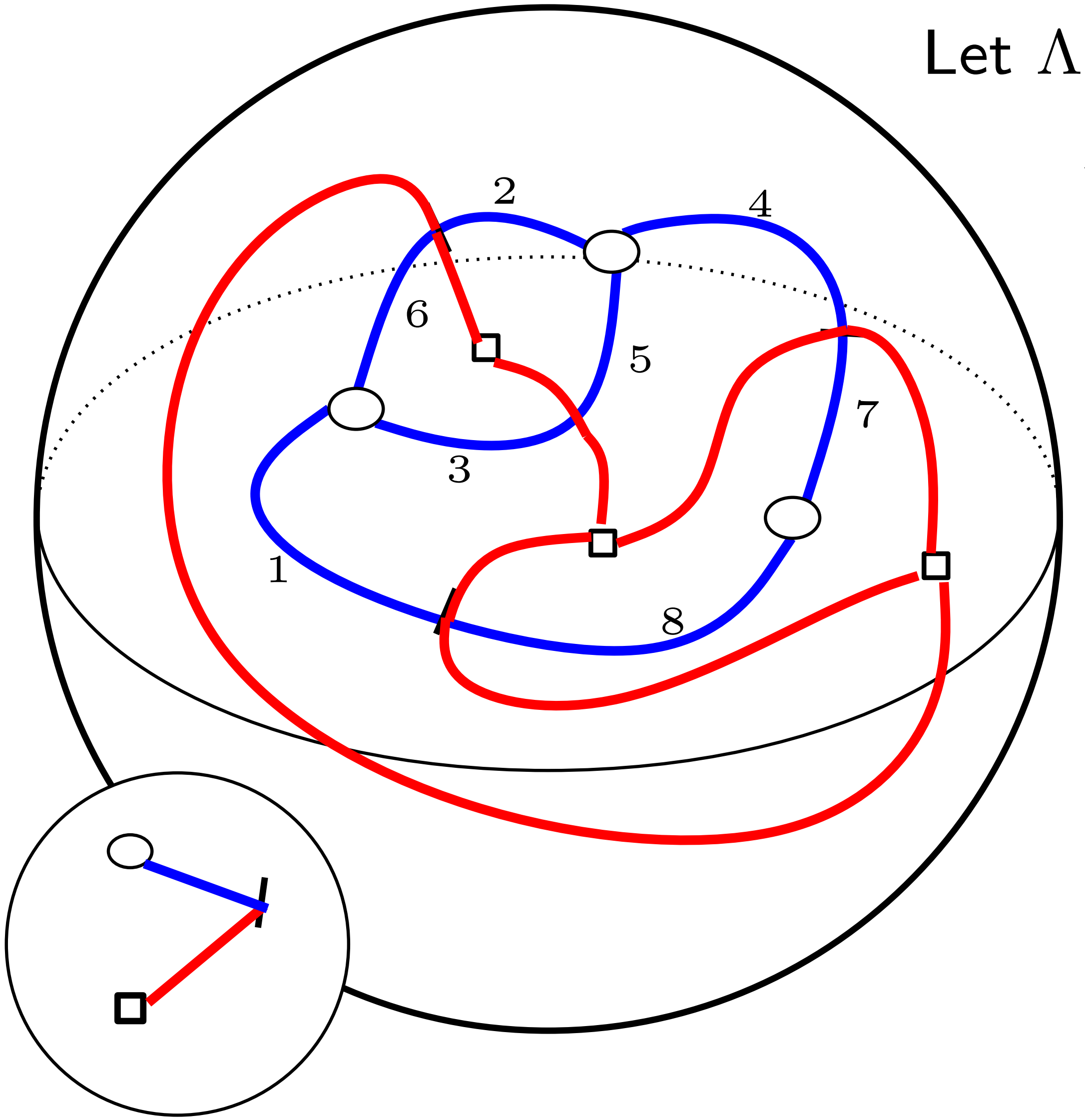

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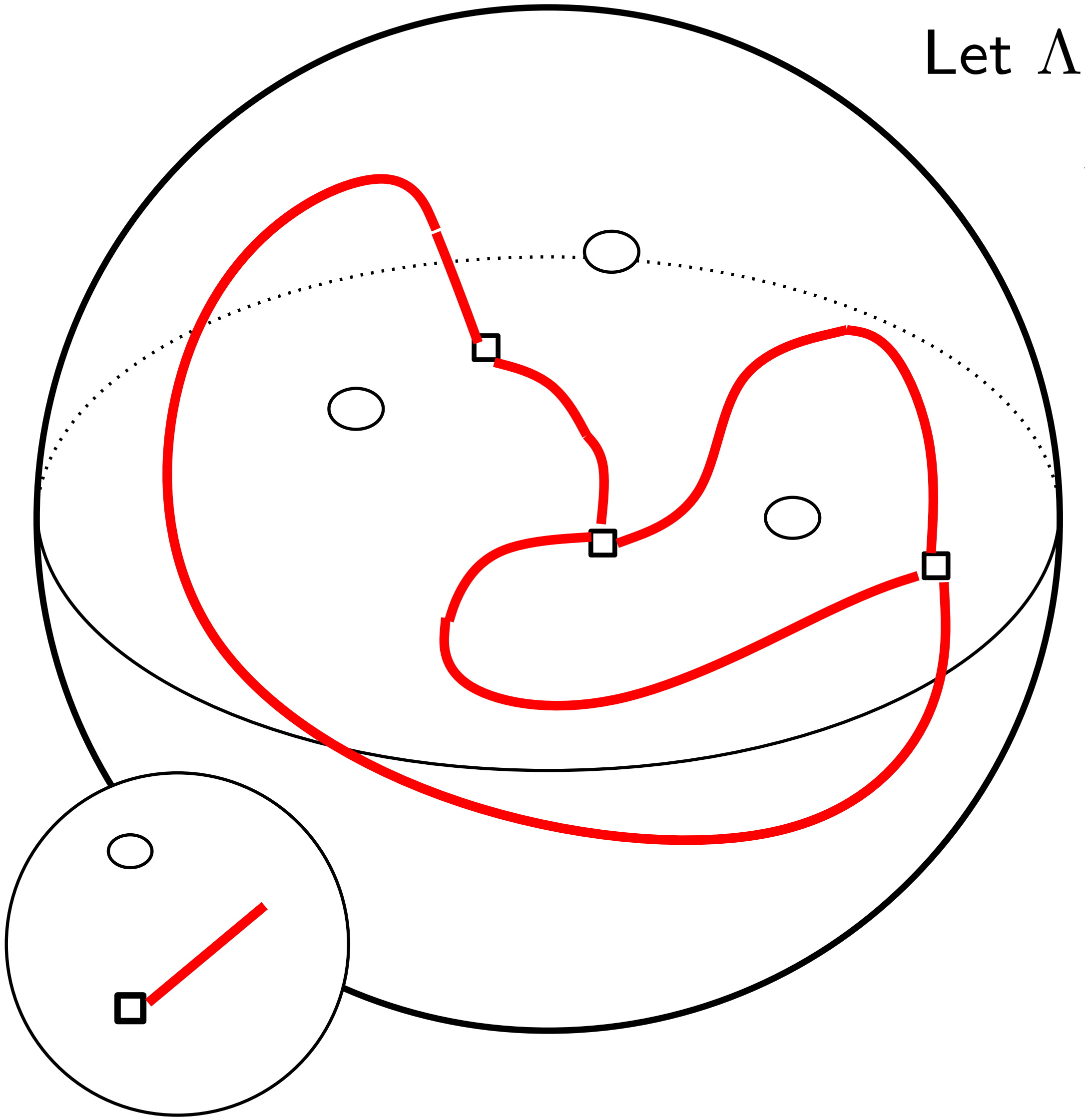
$\circ\!-\!\!\bullet\!\!-\!\bullet \Rightarrow$ bipartite map

with $\lambda_\bullet = 2^{\frac{n}{2}}$

$\circ\!-\!| \Rightarrow$ nonbipartite map

$\square\!-\!| \Rightarrow$ dual map

Application to the design of trivial bijections for maps



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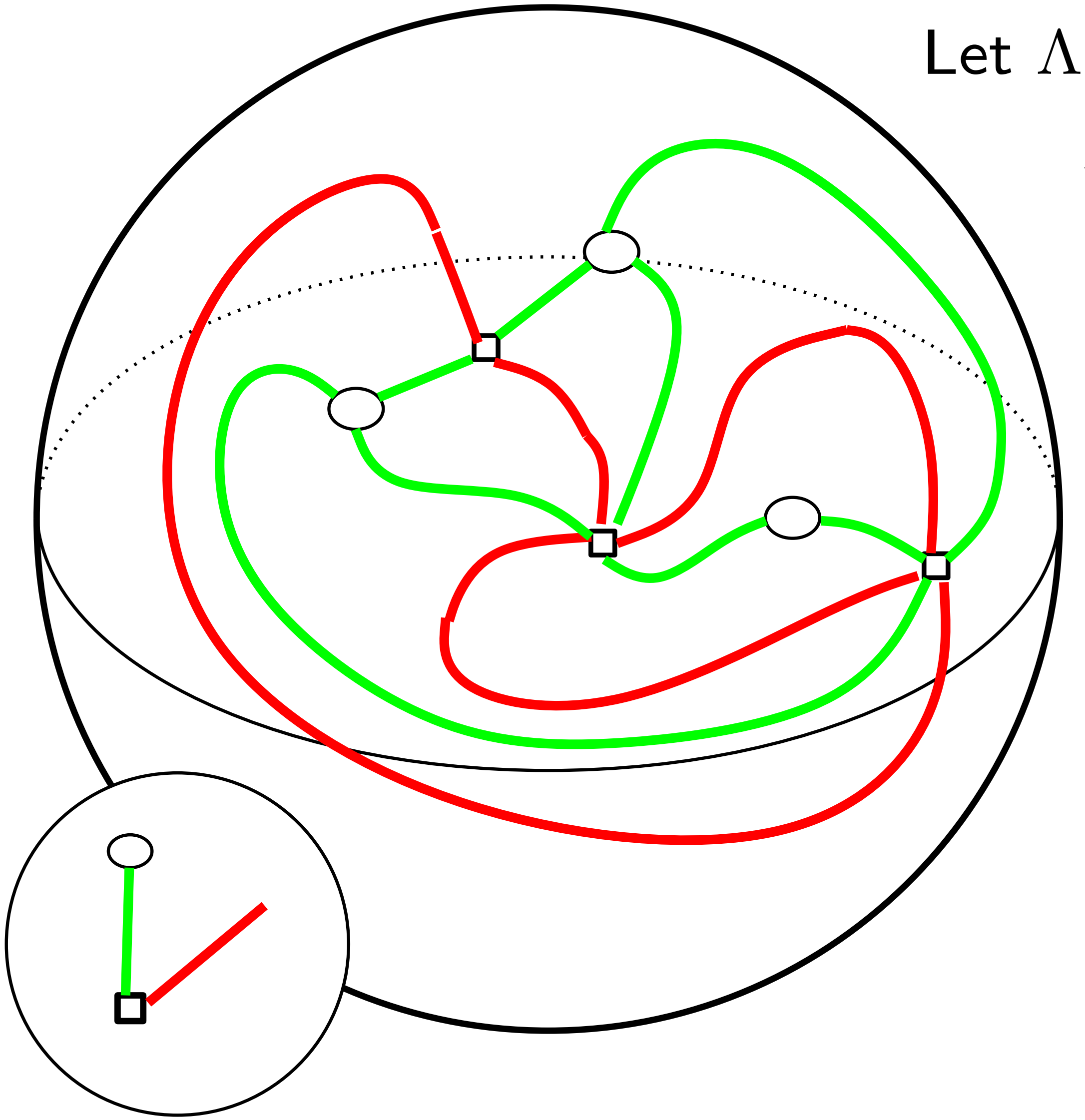

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□—○ $\Rightarrow$ quadrangulation

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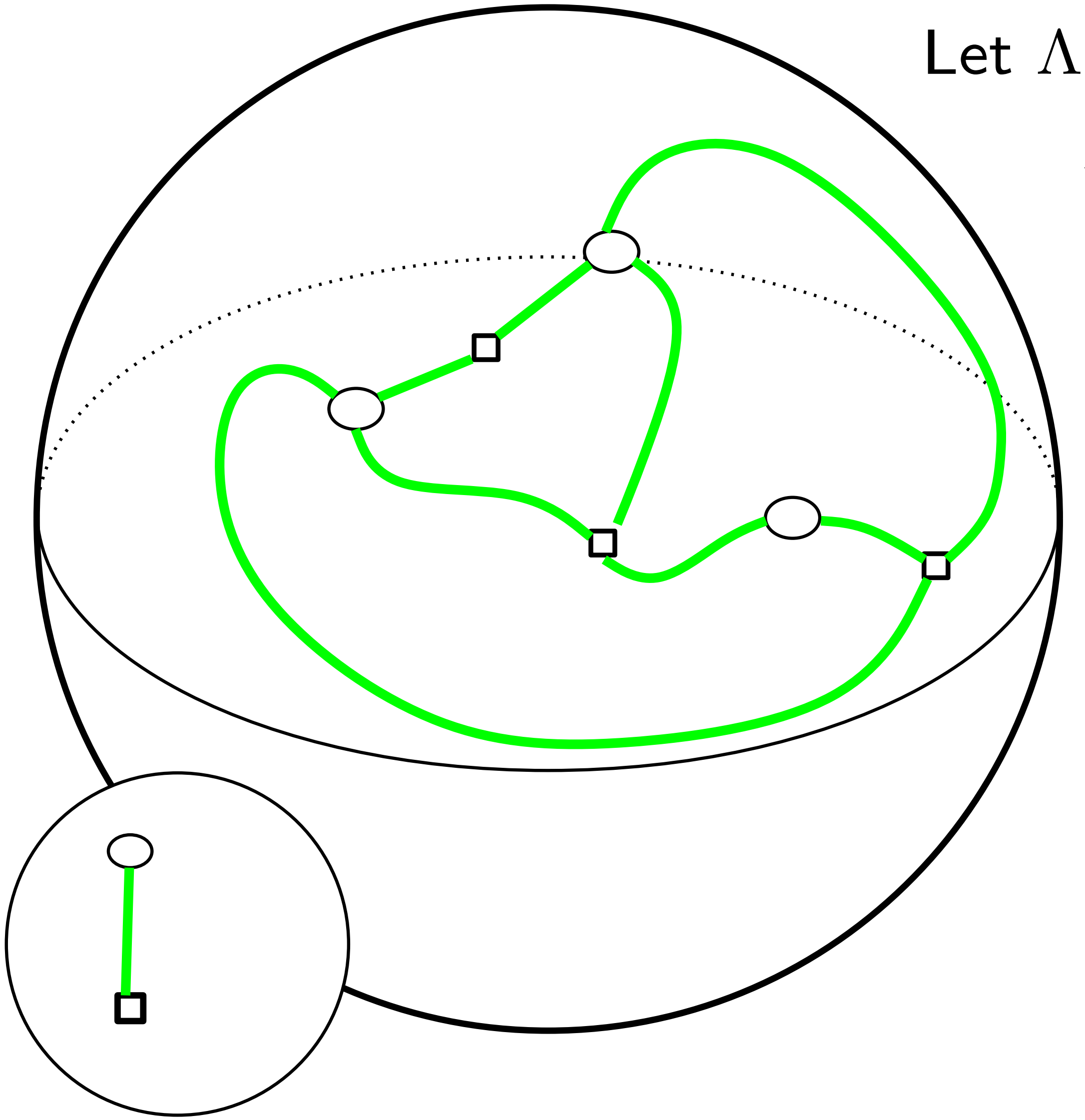

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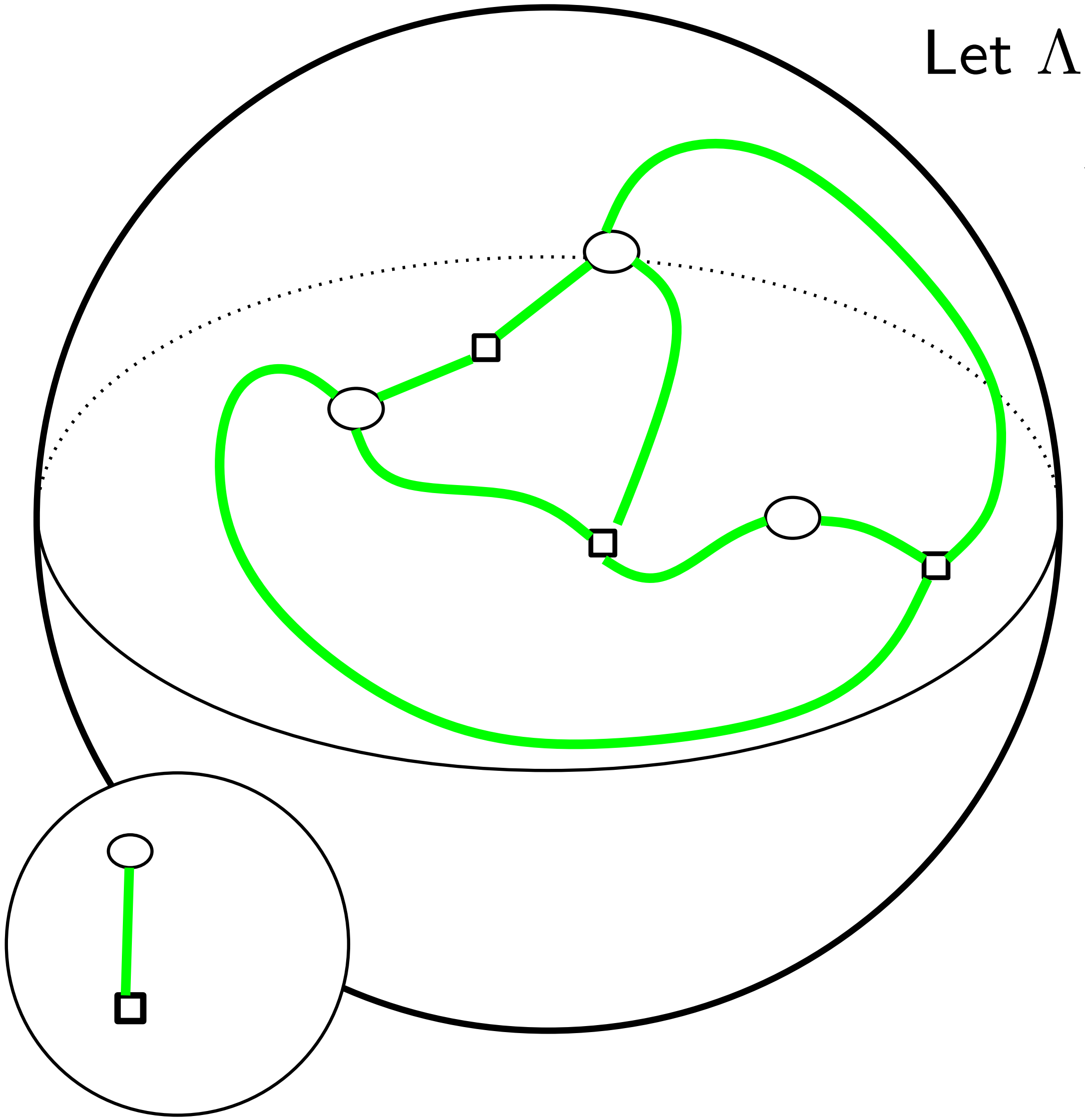

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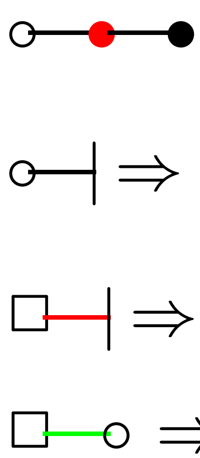

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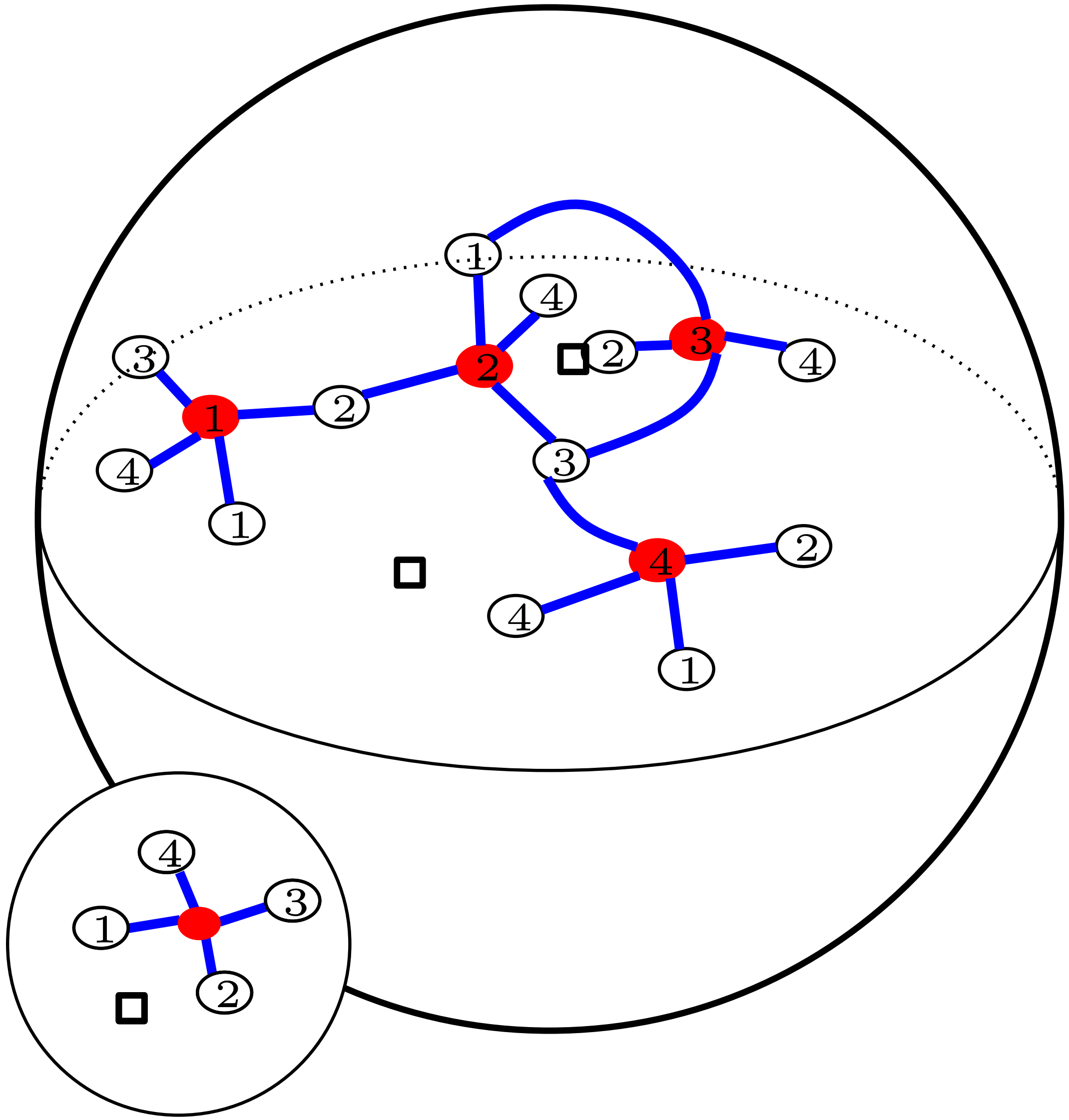
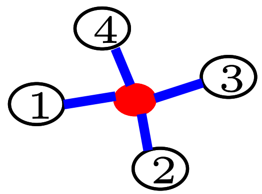
Keep the same covering, give different representations (data structures)

$\Rightarrow$ all these are bijections

Variants of star-constellations





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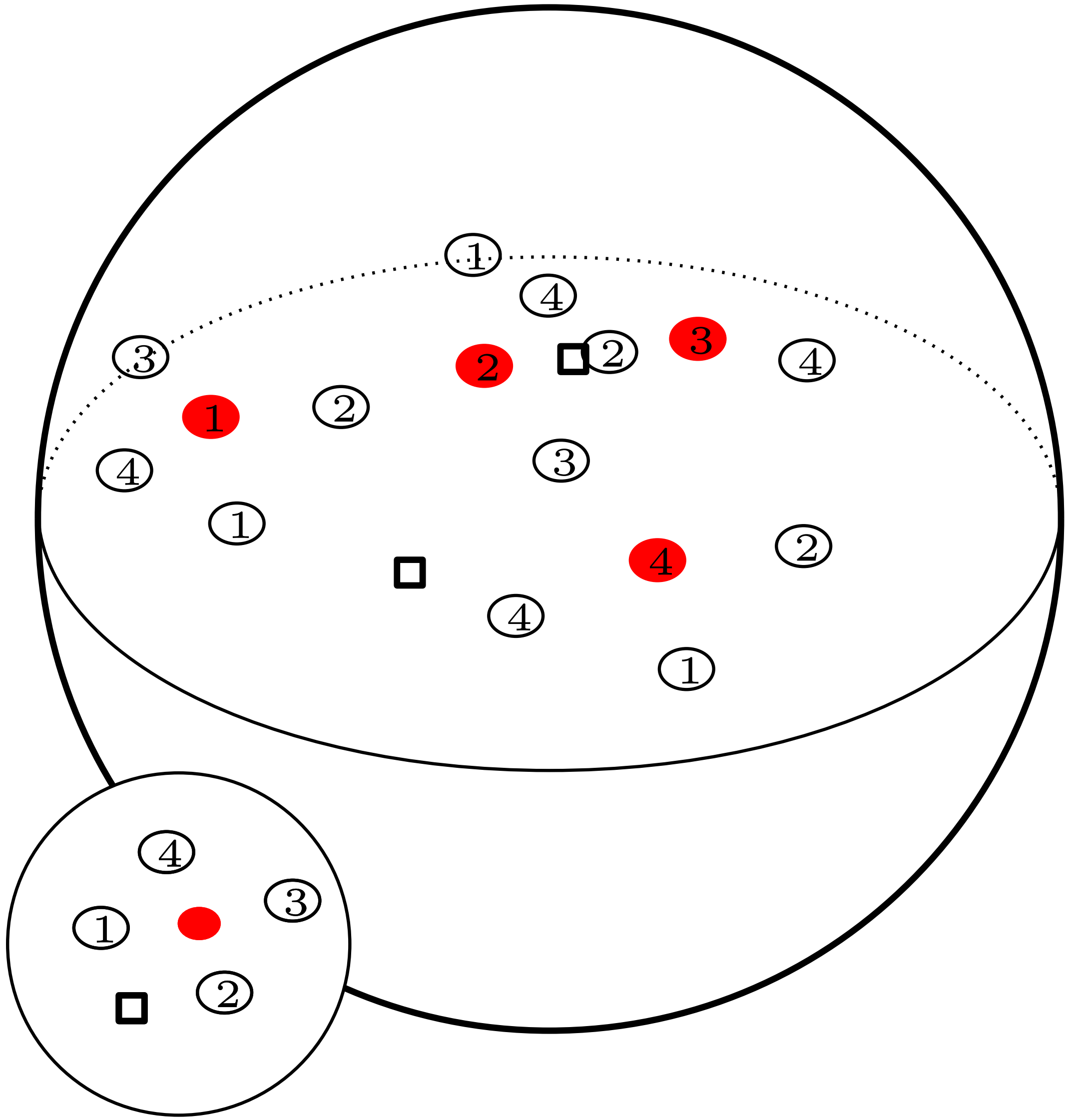

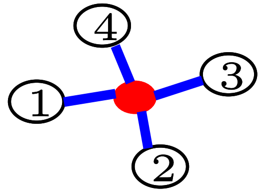

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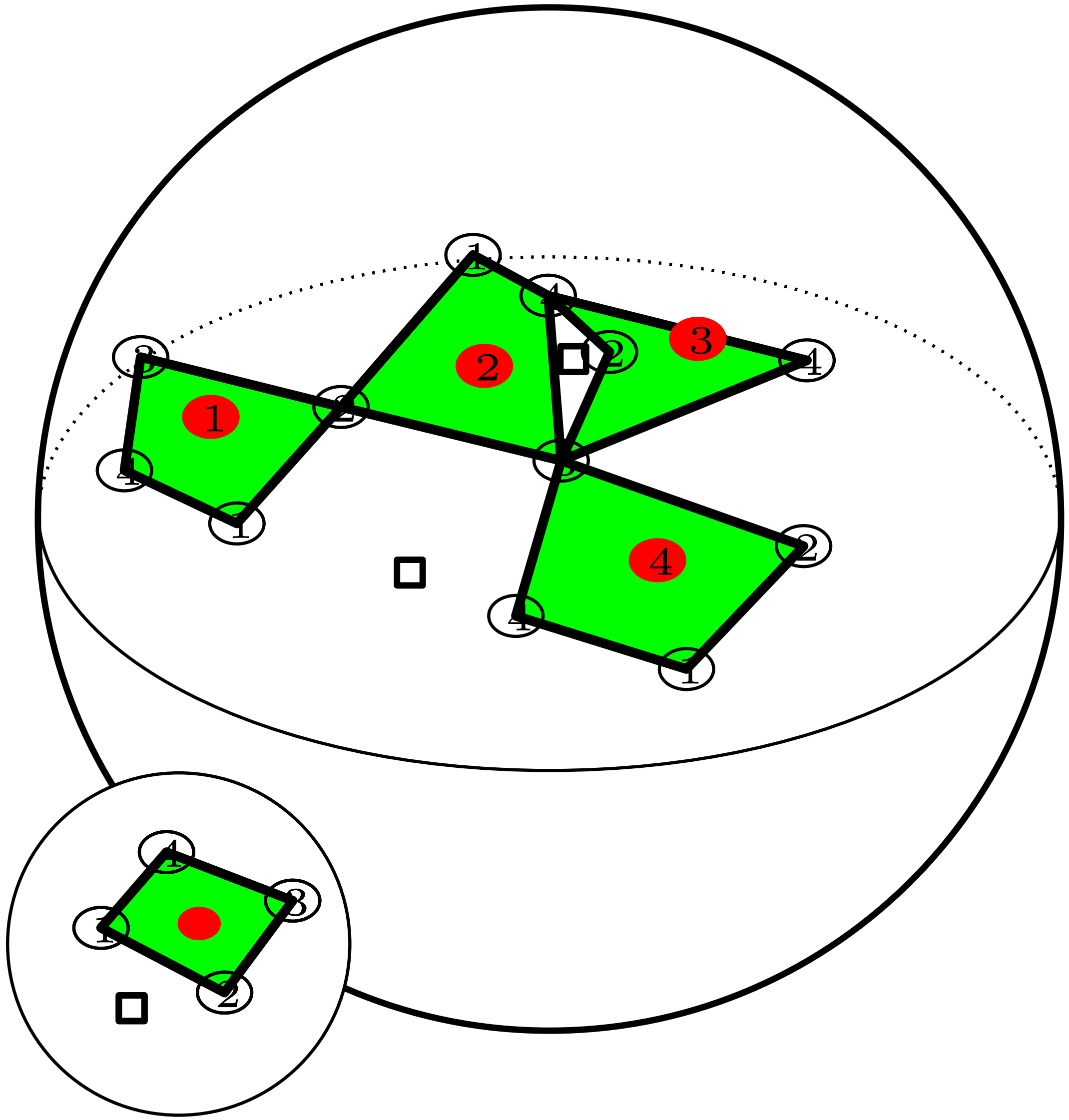

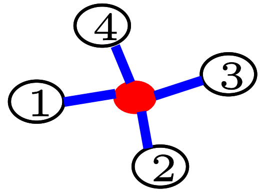

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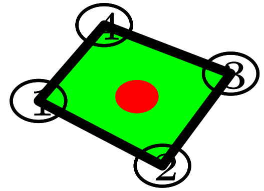

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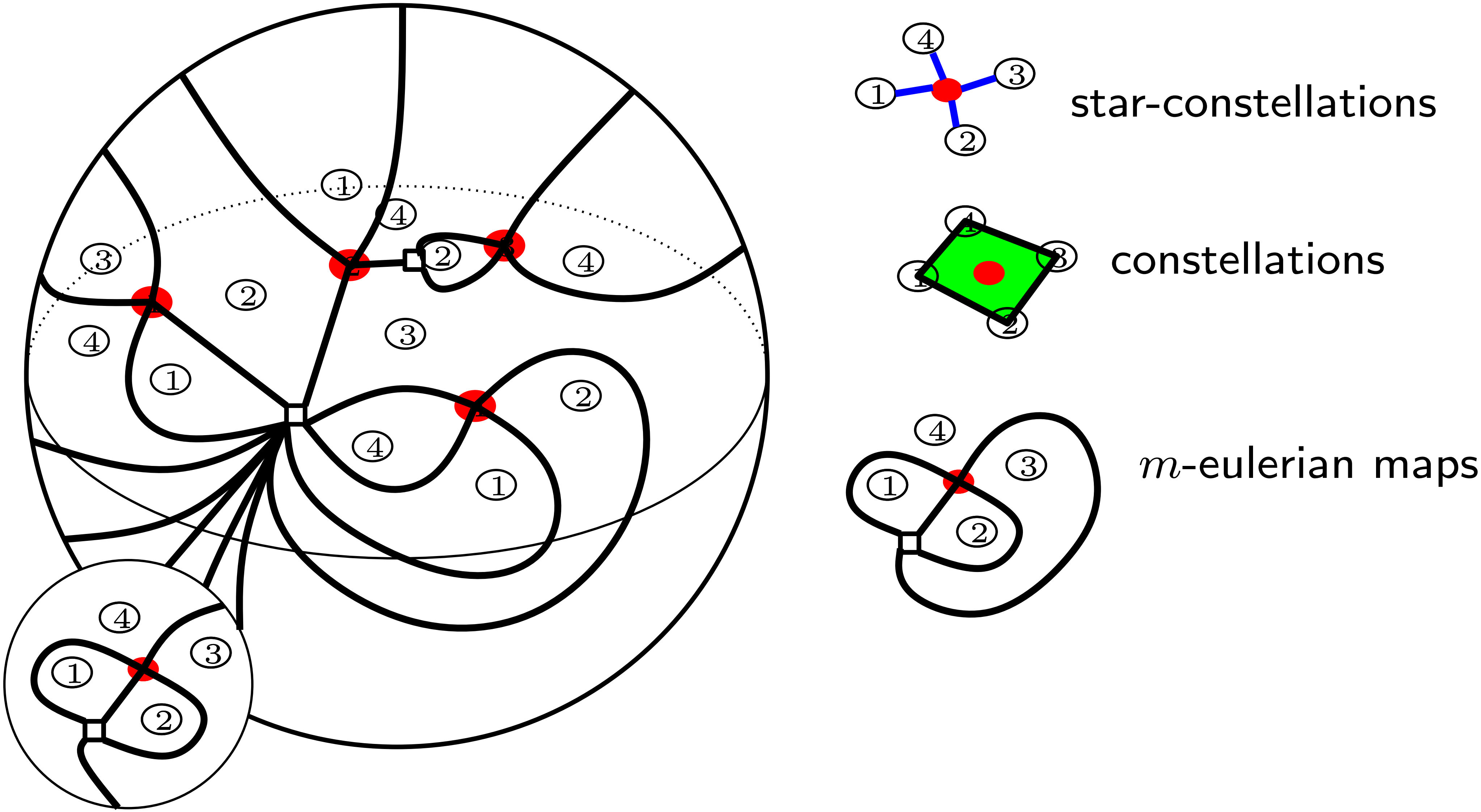
Variants of star-constellations



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Today

Factorizations, maps and ramified coverings

- Permutations, factorizations and increasing maps
- Hurwitz original motivation, ramified coverings
- Ramified coverings provide bijections "for free"

Later...

Orientations and decompositions of maps into trees

Applications to Hurwitz numbers

A formula for general factorizations [BMS00]

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition of n, and $\ell = \sum_i \ell_i$.
The number of m-uple of permutations $(\sigma_1, \ldots, \sigma_m)$ such that

- (factorization) $\sigma_1 \cdots \sigma_m = \sigma$ with cycle type λ
- (transitivity) $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$
- (minimality) the total rank of factors is $\sum_i r(\sigma_i) = n + \ell - 2$

is

$$m \frac{((m-1)n-1)!}{(mn-(n+\ell-2))!} \cdot n! \cdot \prod_i \frac{1}{\ell_i!} \binom{mi-1}{i}^{\ell_i}$$

Proofs:

(Bousquet-Mélou–Schaeffer 2000) (Goulden–?? 2009)

(bijection + inclusion/exclusion)(gfs and differential eqns)

$\lambda = n$, factorizations of n-cycles: $\frac{1}{(mn+1)} \binom{mn+1}{n} \cdot (n-1)!$

$\lambda = 1^n$, identity factorizations: $\frac{m}{(m-2)n+2} \frac{(m-1)^{n-1}}{(m-2)n+1} \binom{(m-1)n}{n} \cdot (n-1)!$

Our aim in the rest of the lectures

Prove the following two results using two bijective methods:

Factorization in transpositions:
$\lambda = 1^n$, factorizations of the identity: $n^{n-3} \cdot (2n-2)!$

Need to count fully increasing quadrangulations

Factorizations in arbitrary factors:
$\lambda = 1^n$, factorizations of the identity: $m \frac{(mn-n-1)!}{(mn-2n+2)!} (m-1)^n$

Need to count $(m+1)$-constellations.

The two methods extend to general λ.

The second method extends to non minimal factorizations (higher genus)

Second lecture

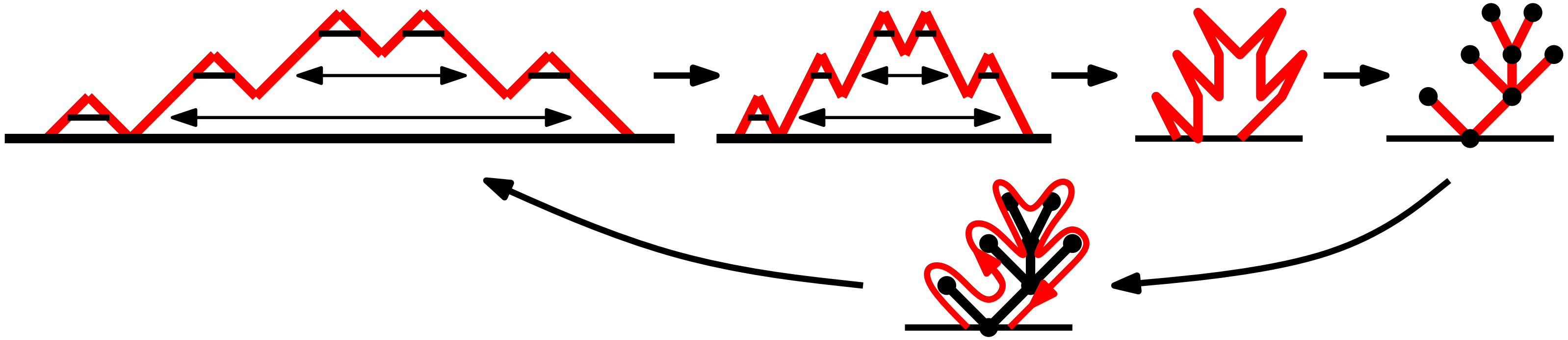
Orientation and the decomposition of maps into trees

- A quick reminder about trees
- General idea: decompose a map into two trees
- 2 strategies explain (almost) all known bijections
    - minimal orientations and direct opening
    - left accessible orientations and the master bijection

A quick reminder about trees

Dyck paths and plane trees

Dyck path of length $2n$ = contour of a plane tree with n edges



The Dyck code of a tree is obtained during the walk around it upon:

– writing u the first time a vertex is visited (up steps)

– writing d the last time a vertex is visited (down steps)

Encodings of trees by words

We shall need two other classical codes:

Encodings of trees by words

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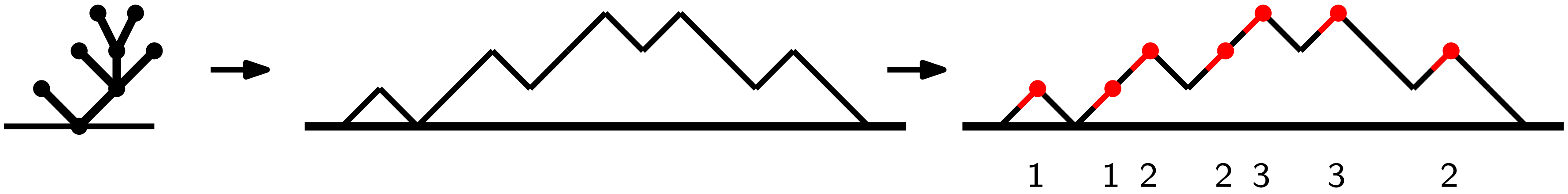

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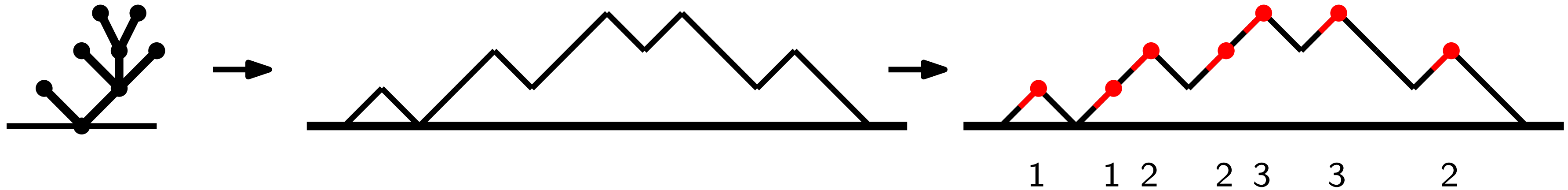
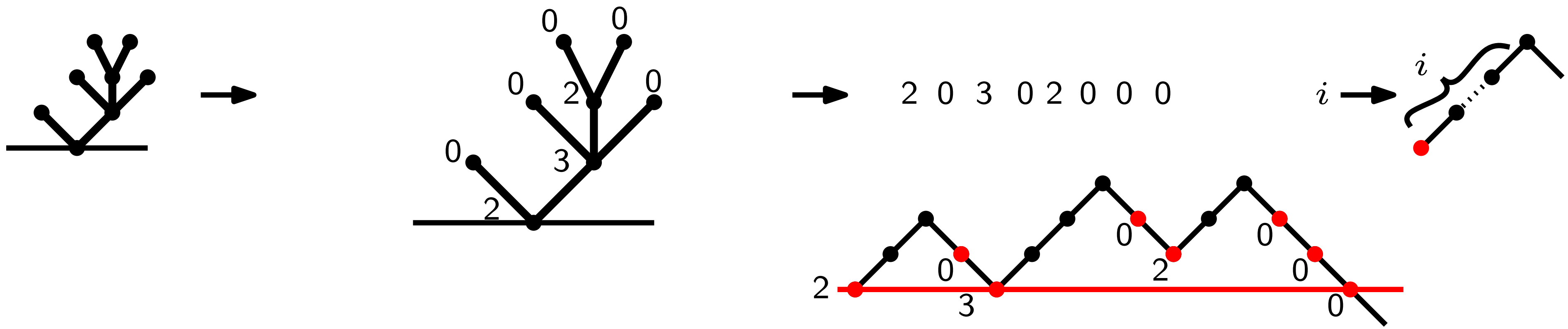


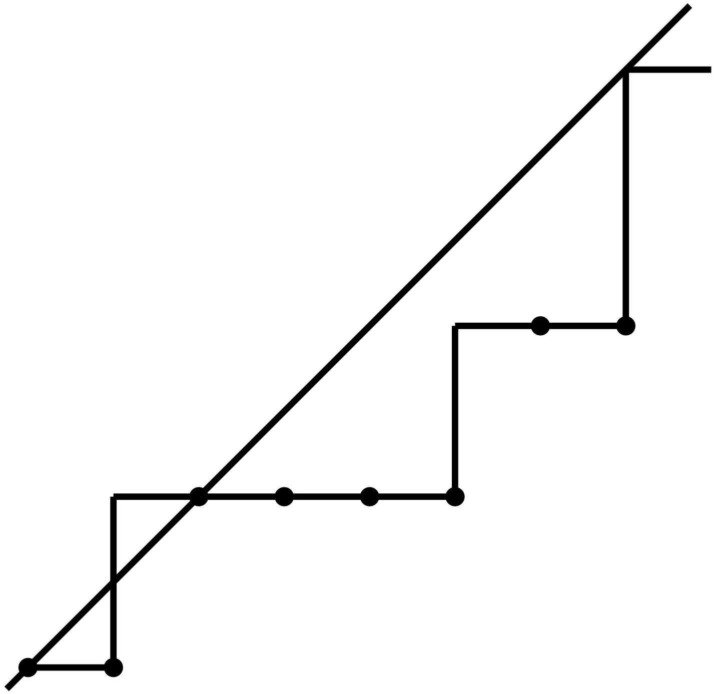
– degree code: write the degree of each vertex during its first visit



Cycle lemma and parking functions

after 45^o rotation: let $P(2n)$ denote paths from $(0,0)$ to $(n,n+1)$ ending by an horizontal step

$$|P(2n)| = \binom{2n}{n}$$



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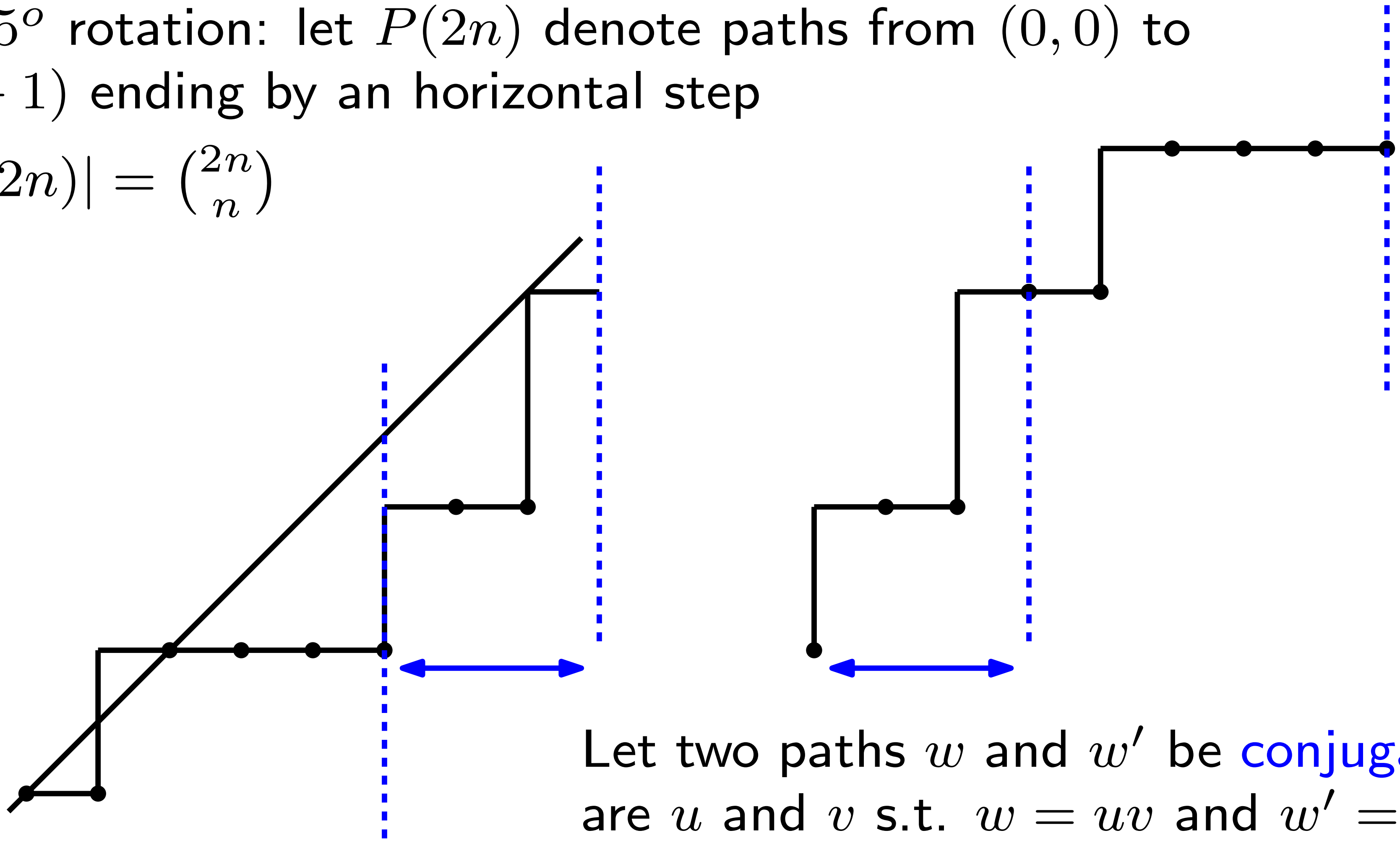

Let two paths w and w' be conjugate if there are u and v s.t. $w = uv$ and $w' = vu$.

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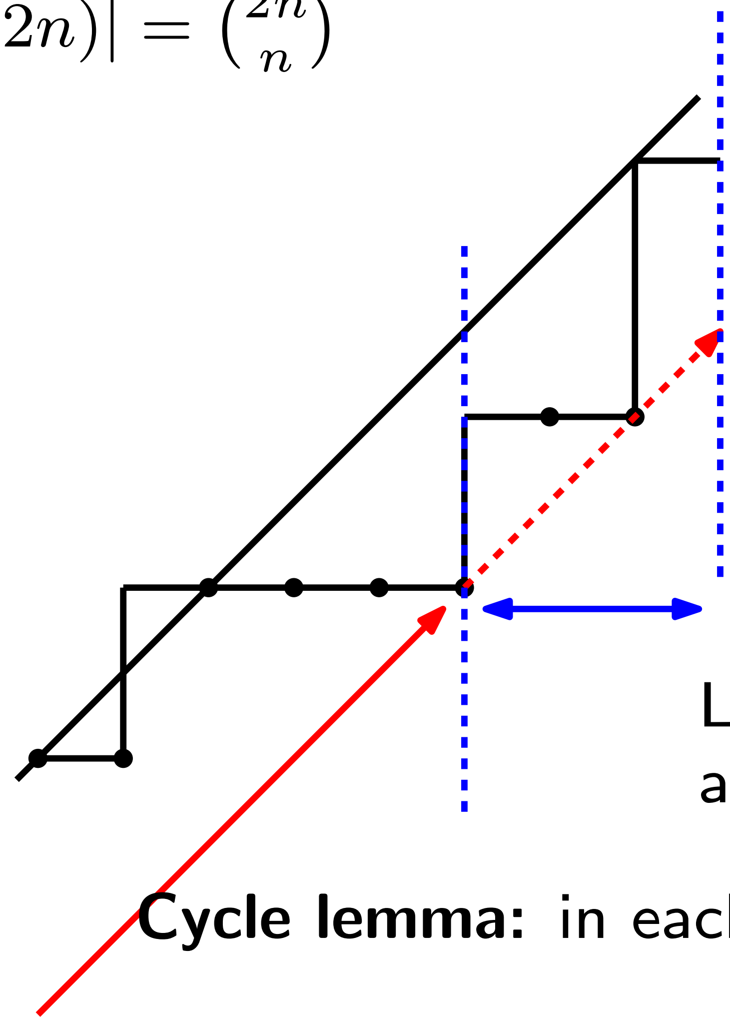

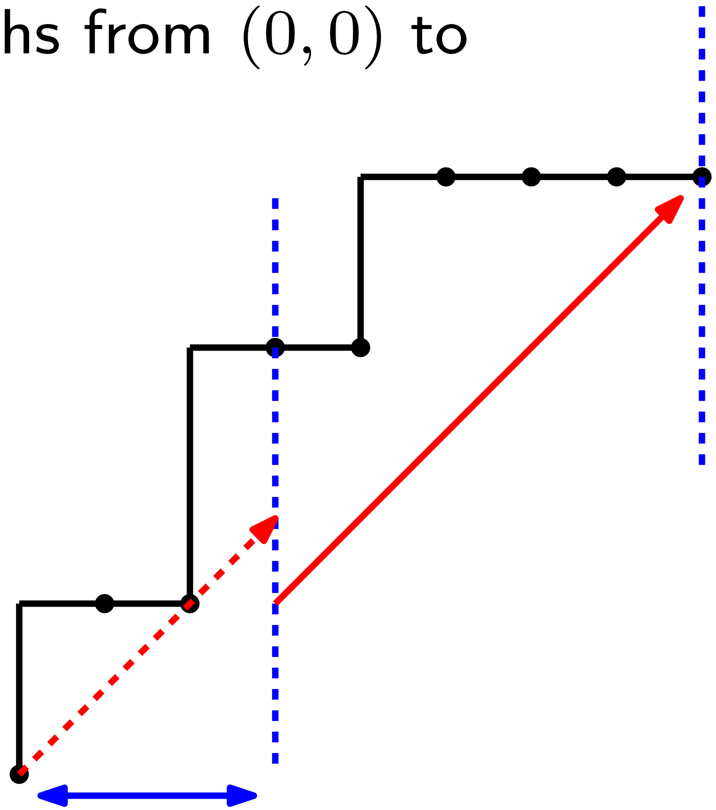

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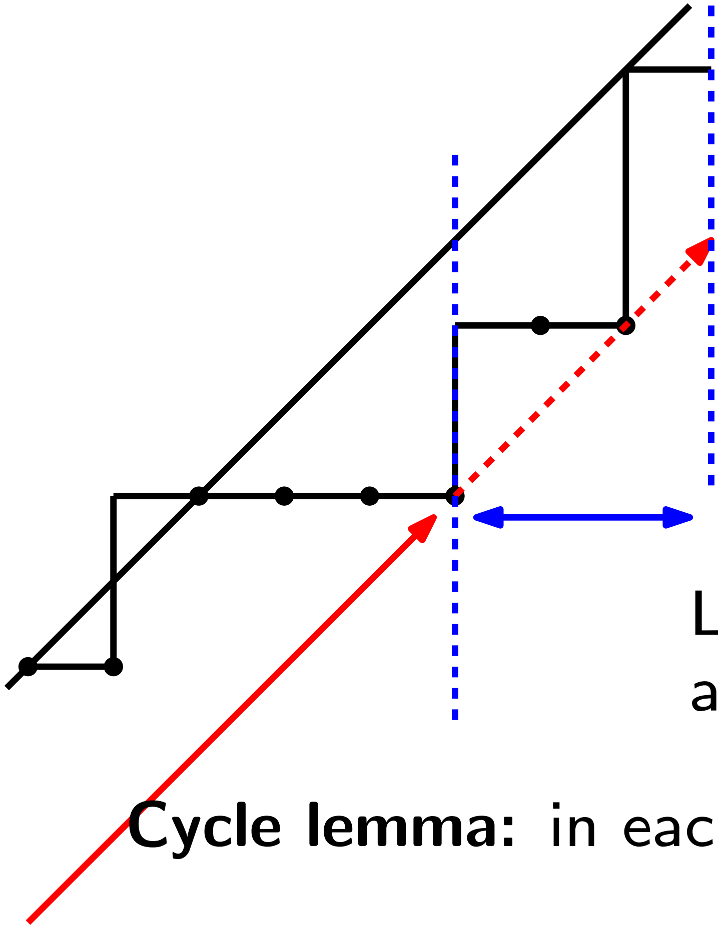

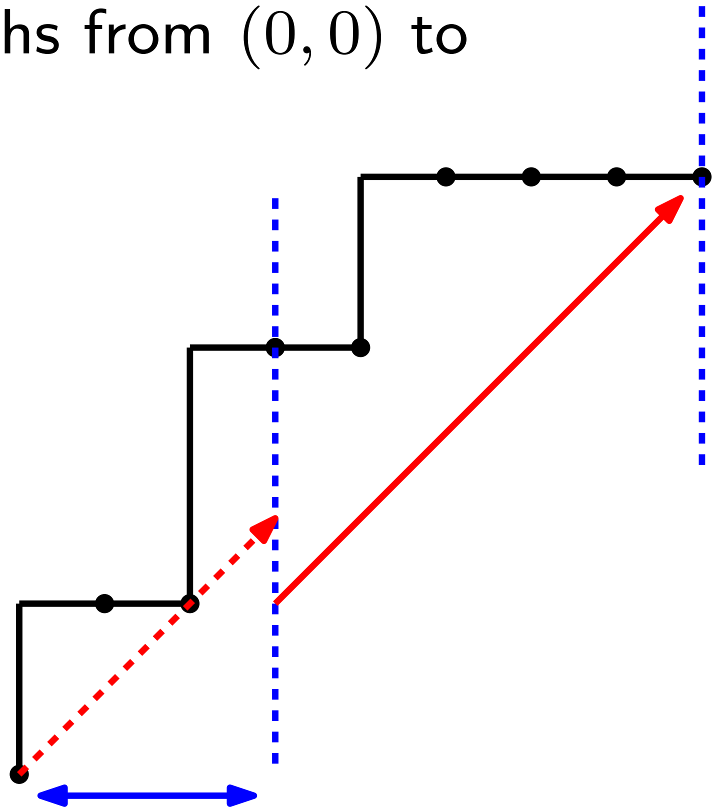

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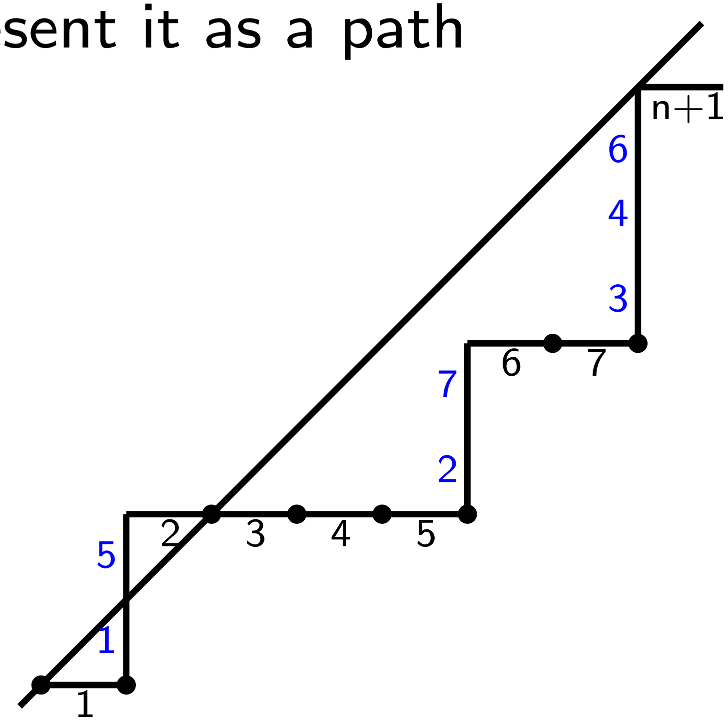
$n+1$ paths in $P(2n)$ yield 1 path in $D(2n)$: $\quad (n+1)|D(2n)| = |P(2n)|$.

Cycle lemma and parking functions

take a fonction f of $[n] \longrightarrow [n+1]$

1	2	3	4	5	6	7
2	6	8	8	2	8	6

represent it as a path



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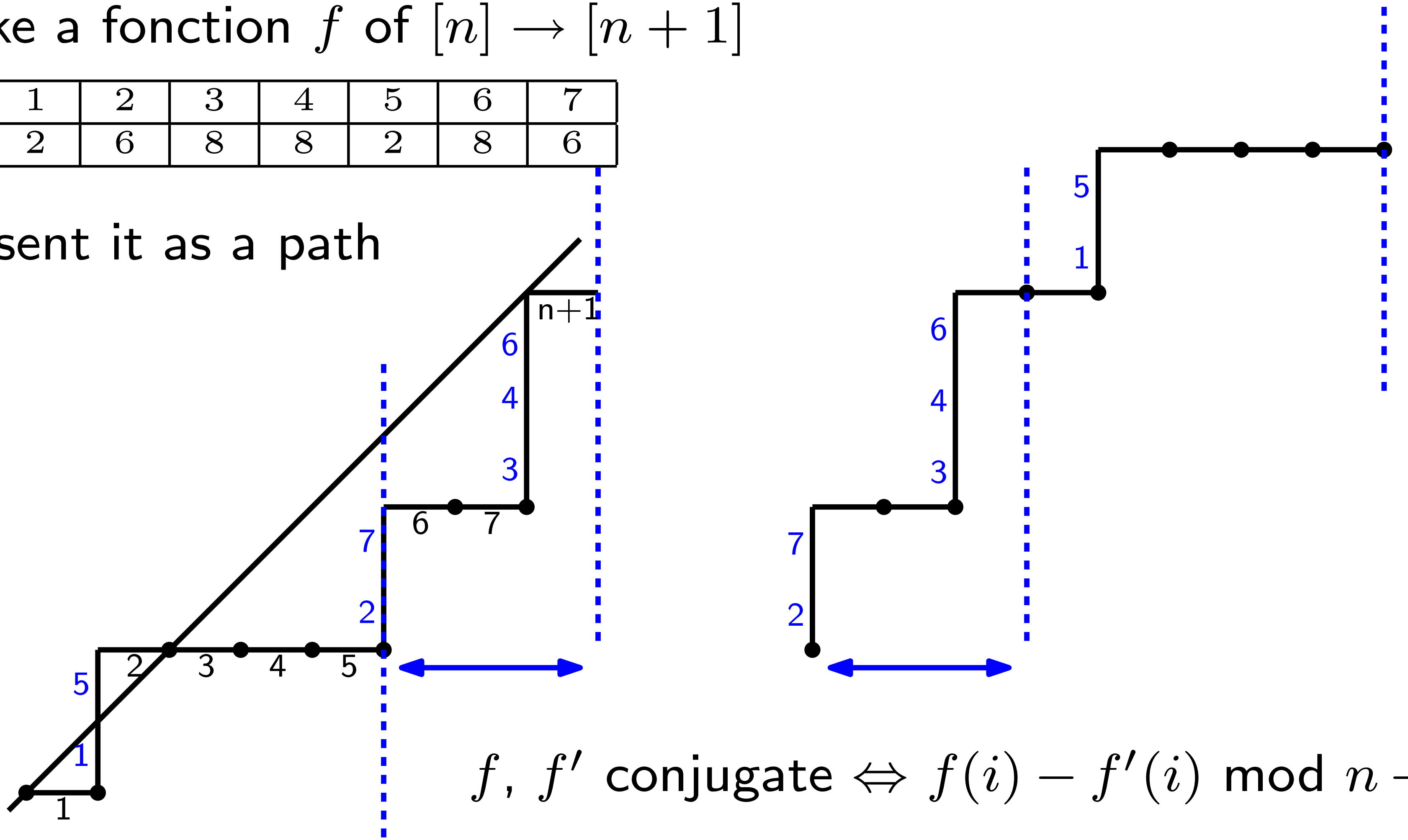

f, f' conjugate $\Leftrightarrow$ $f(i) - f'(i)$ mod $n+1 =$ cte

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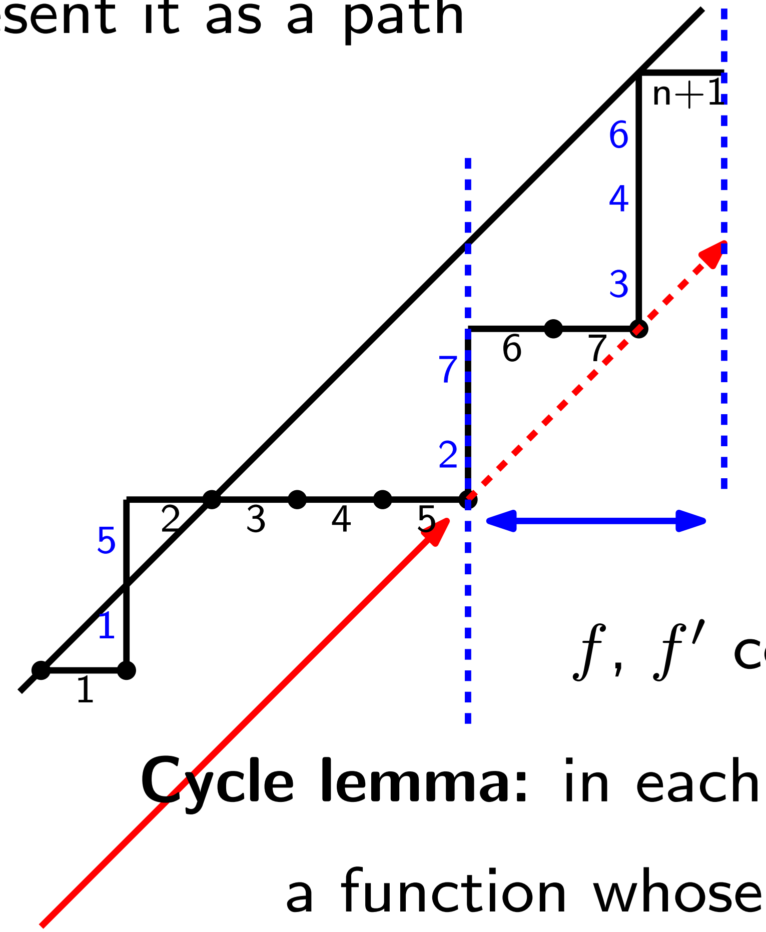

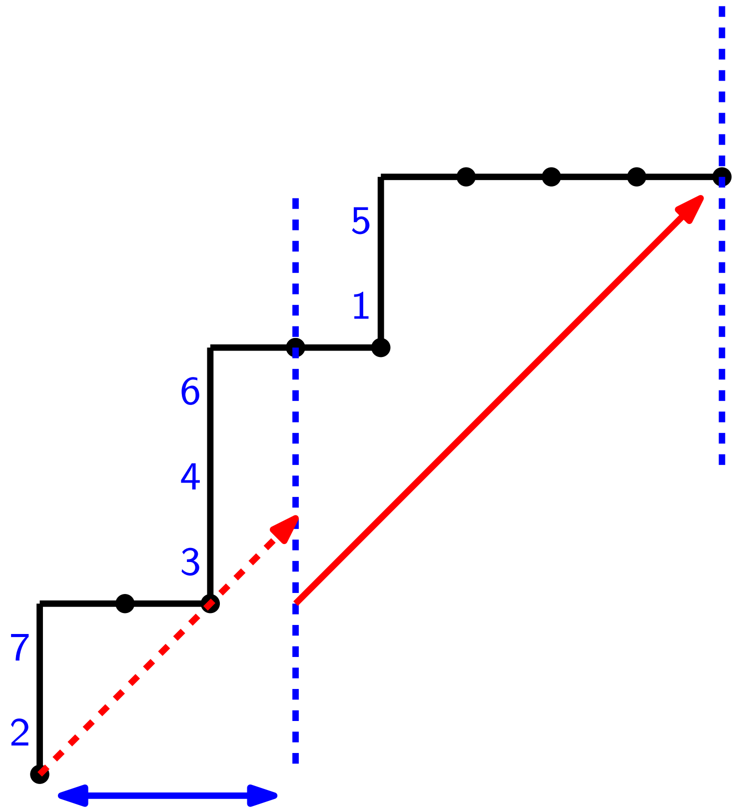

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Cycle lemma: in each conjugacy class exactly one path is positive

a function whose path is positive is a Parking function

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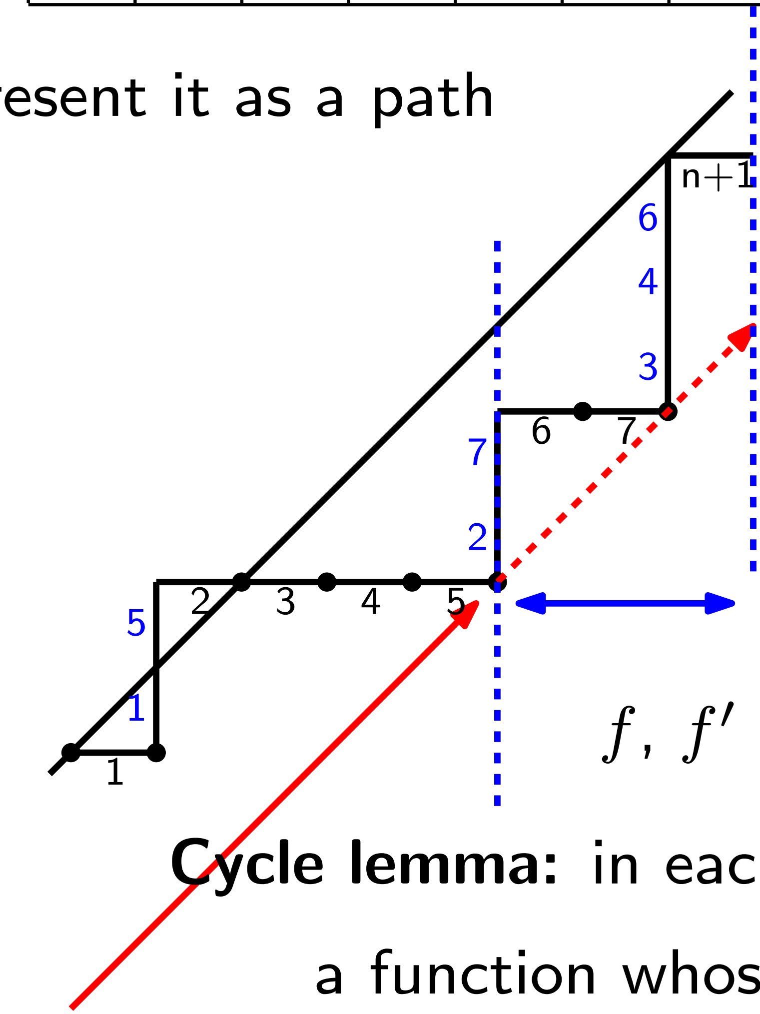

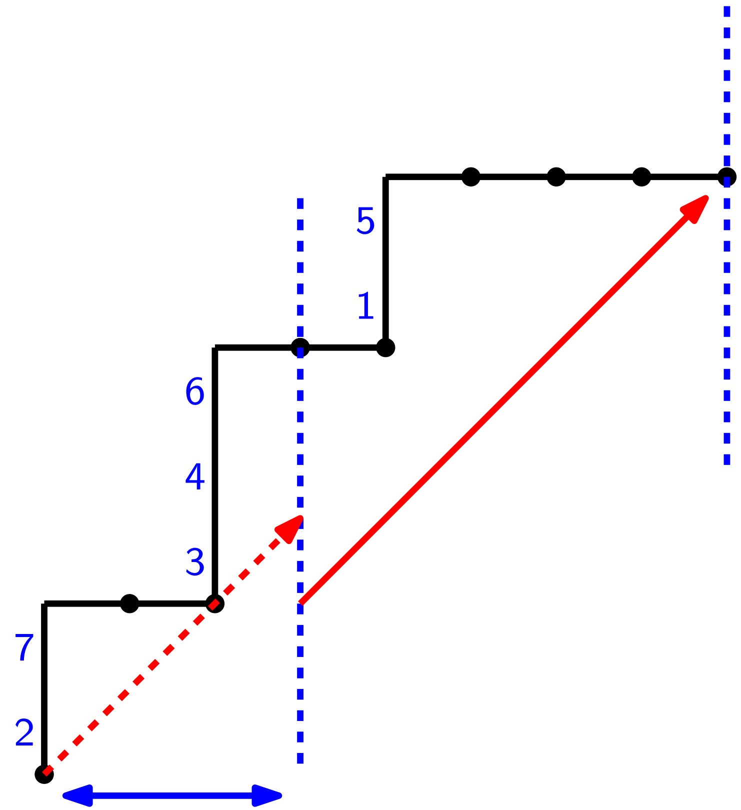

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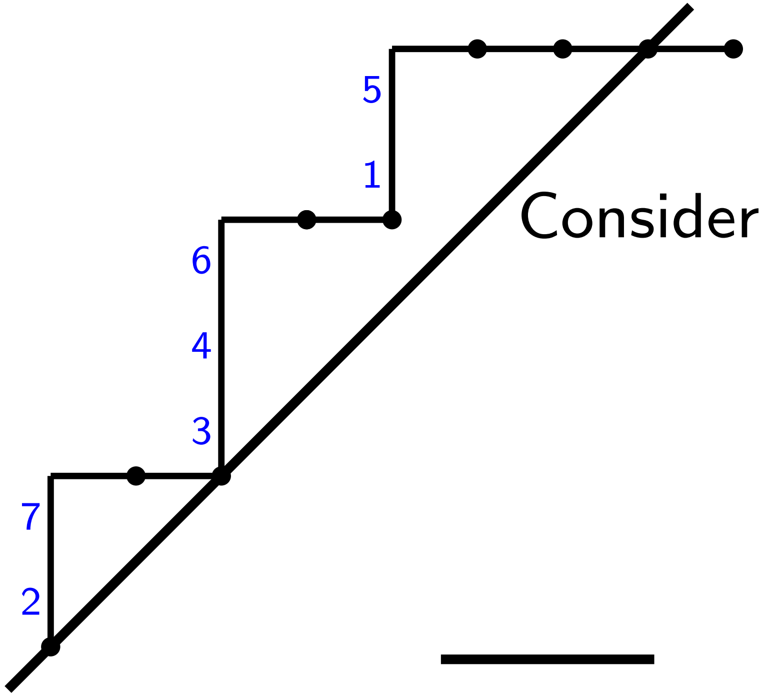
the number of Parking functions $[n] \to [n+1]$ is $\frac{1}{n+1}(n+1)^n = (n+1)^{n-1}$

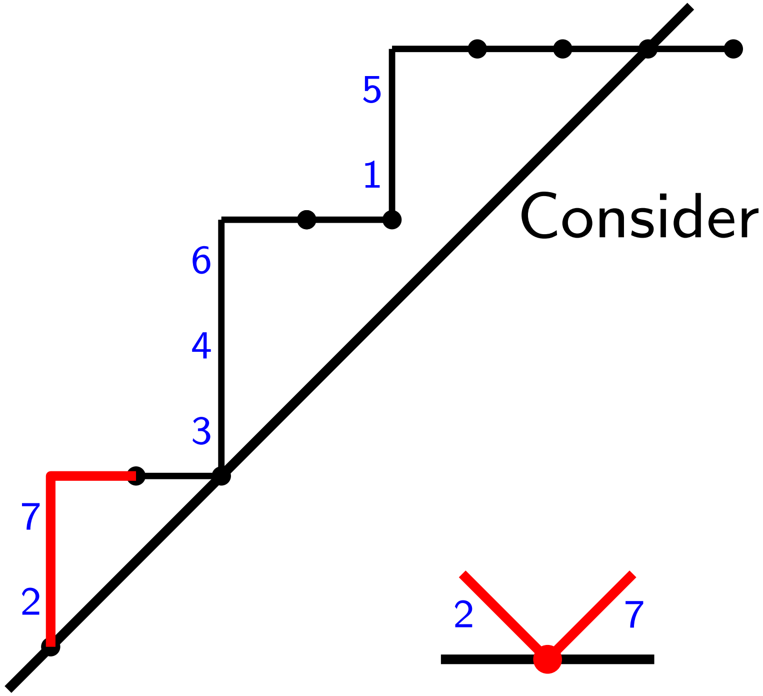
Parking function and codes of trees



Take a parking function

Consider the path as the degree code of a tree

Parking function and codes of trees



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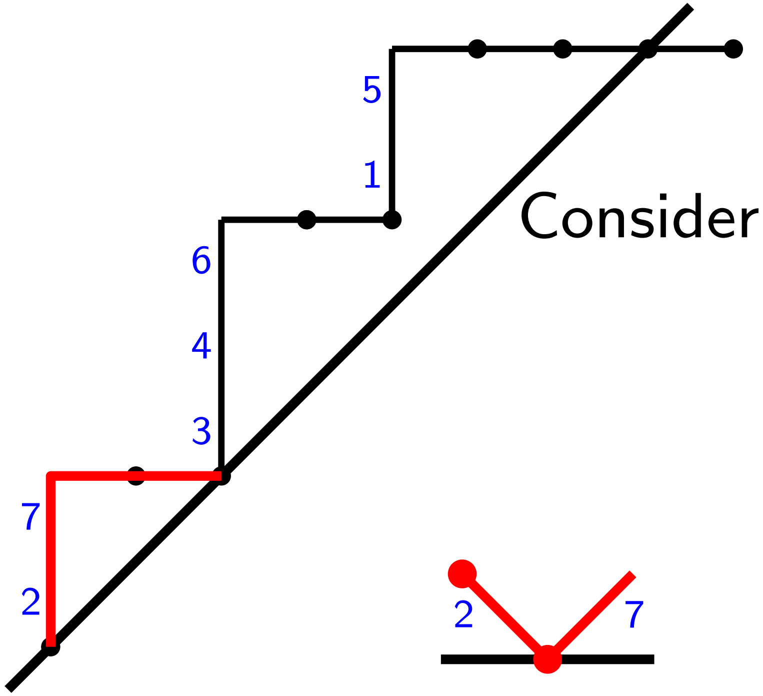

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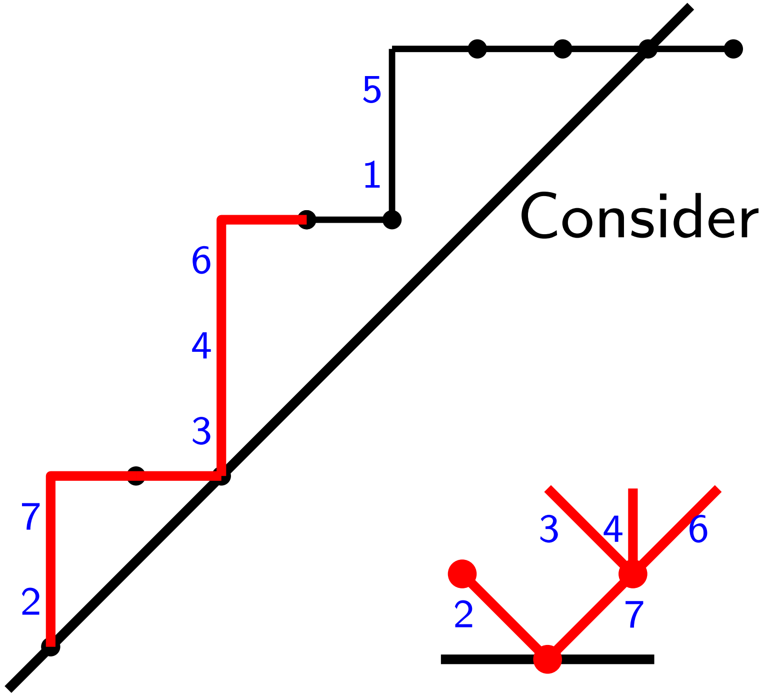

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Consider the path as the degree code of a tree

Parking function and codes of trees

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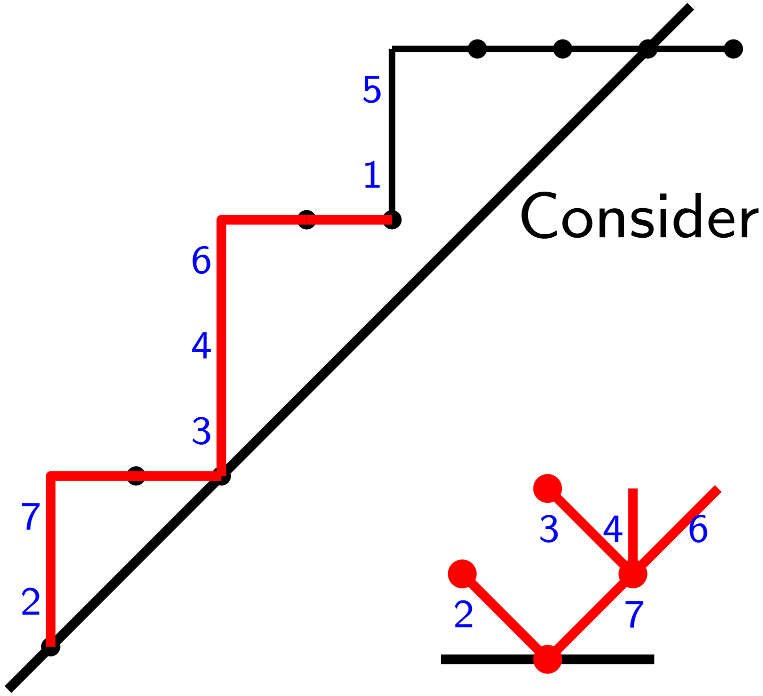

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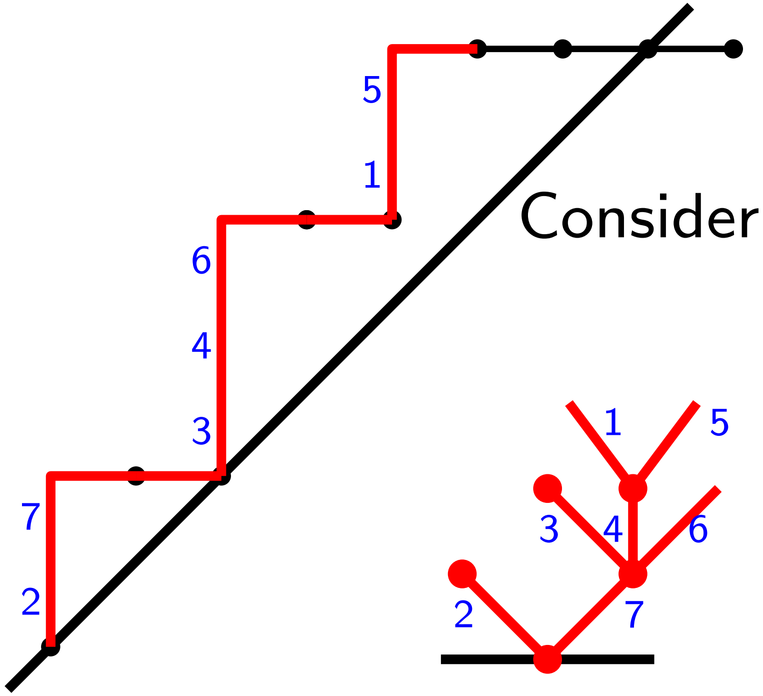

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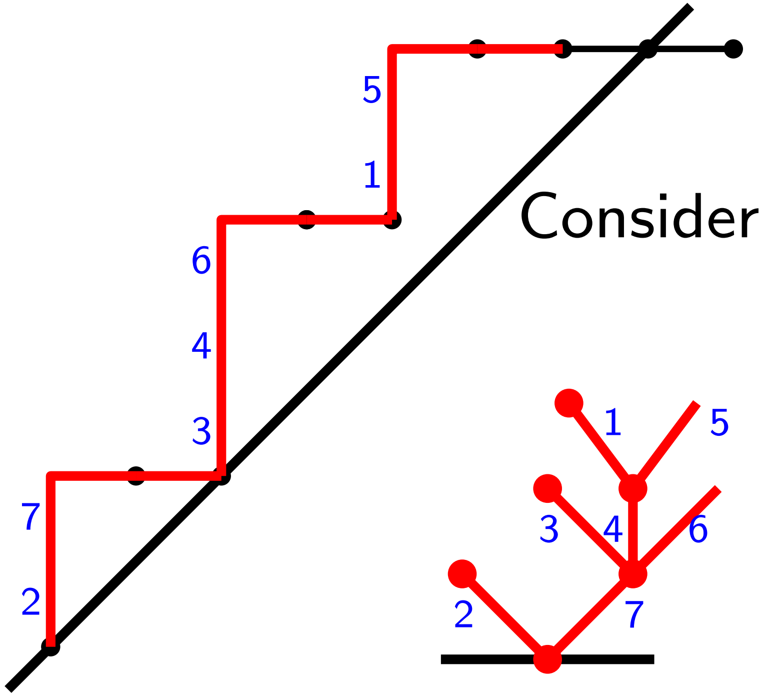

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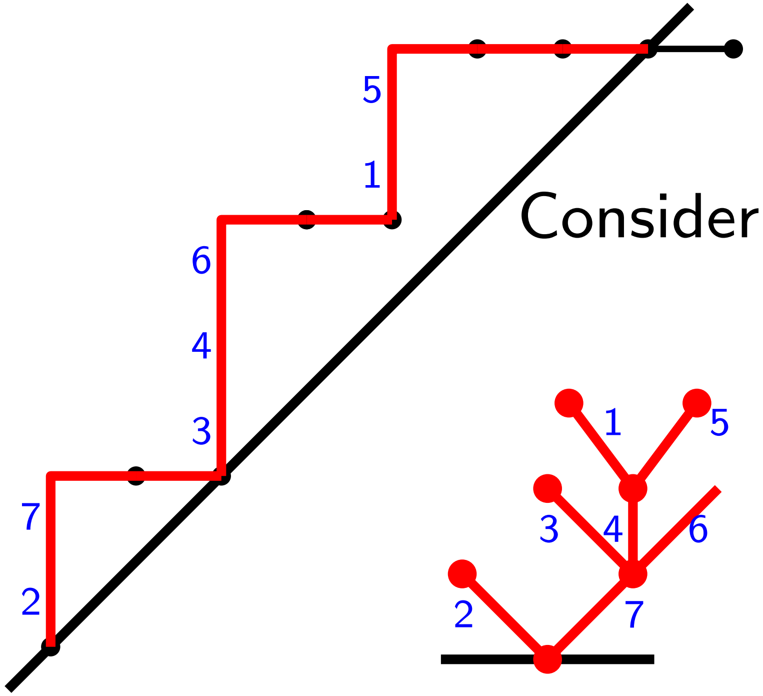

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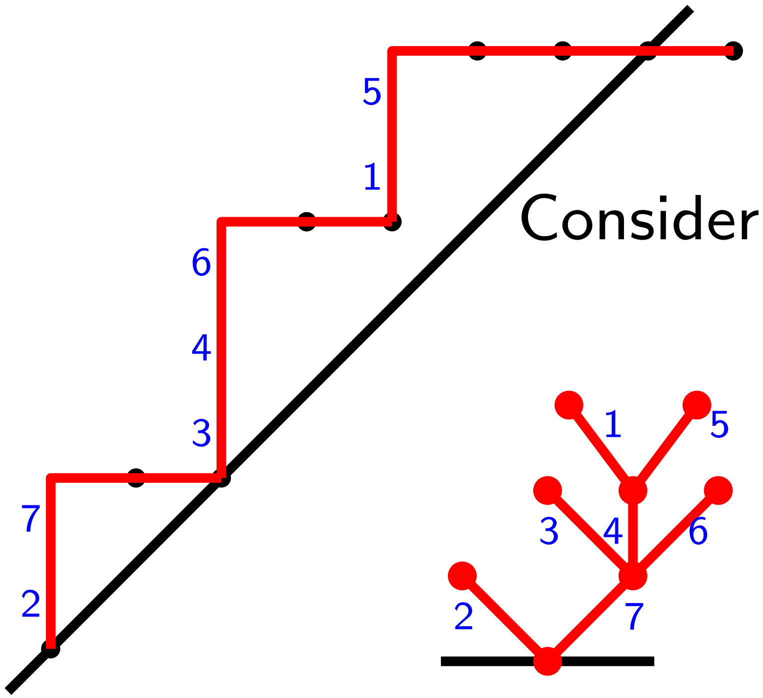

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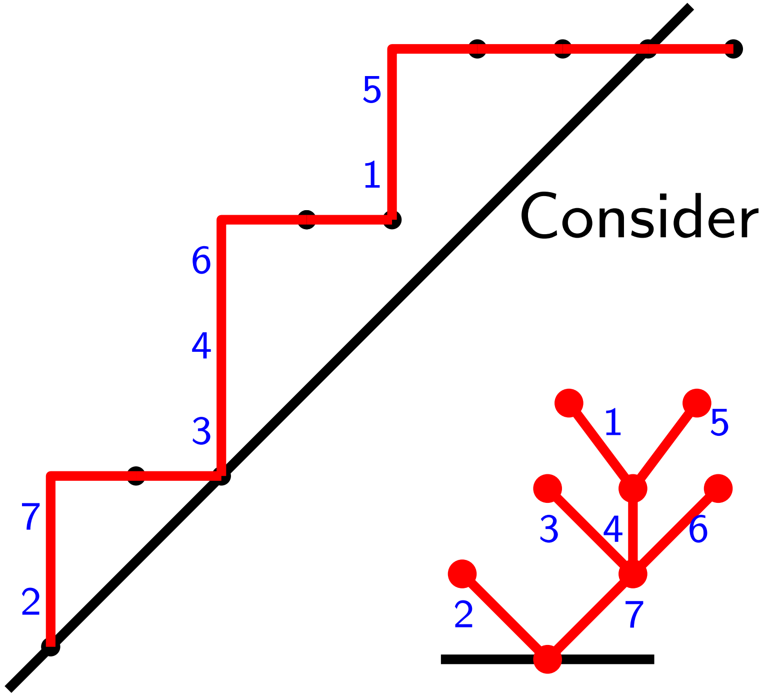

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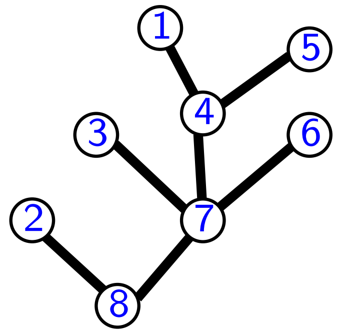
Consider the path as the degree code of a tree

Put label on vertices
(root gets $n+1$)



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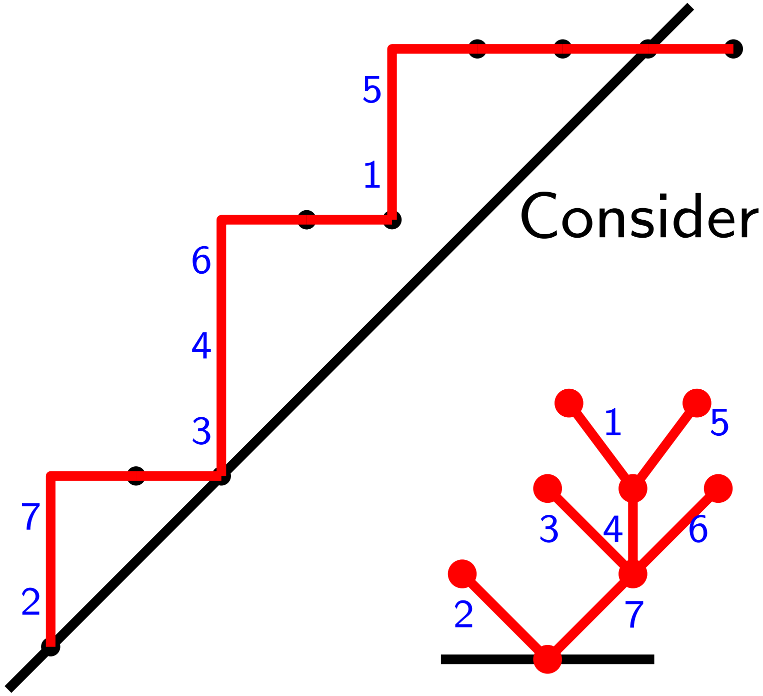

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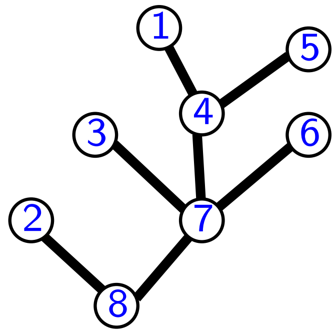
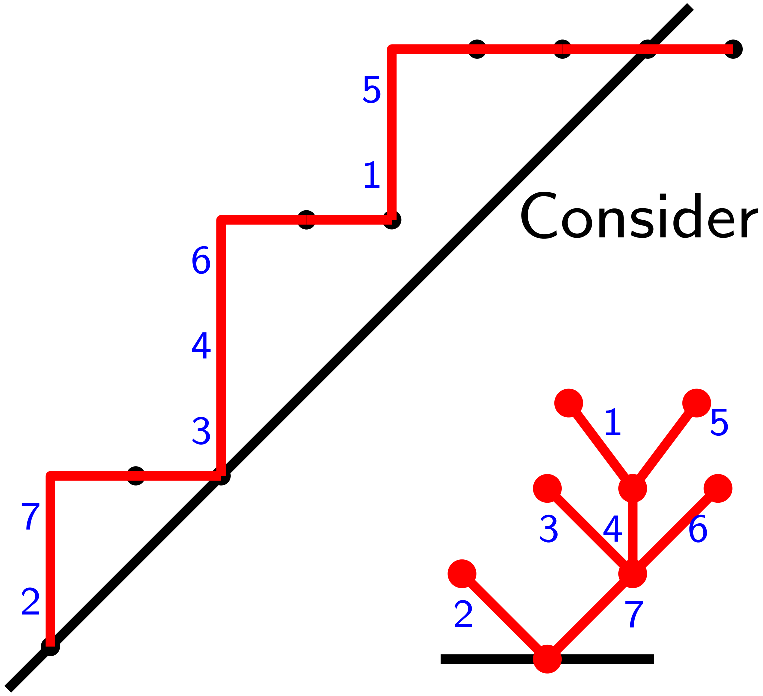


$(n+1)^{n-1}$ Cayley trees with $n+1$ labeled vertices

Parking function and codes of trees



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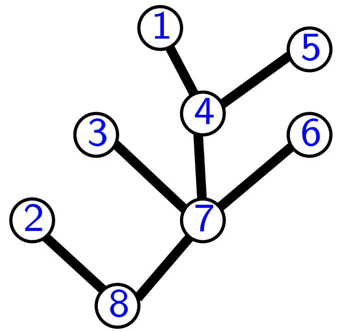

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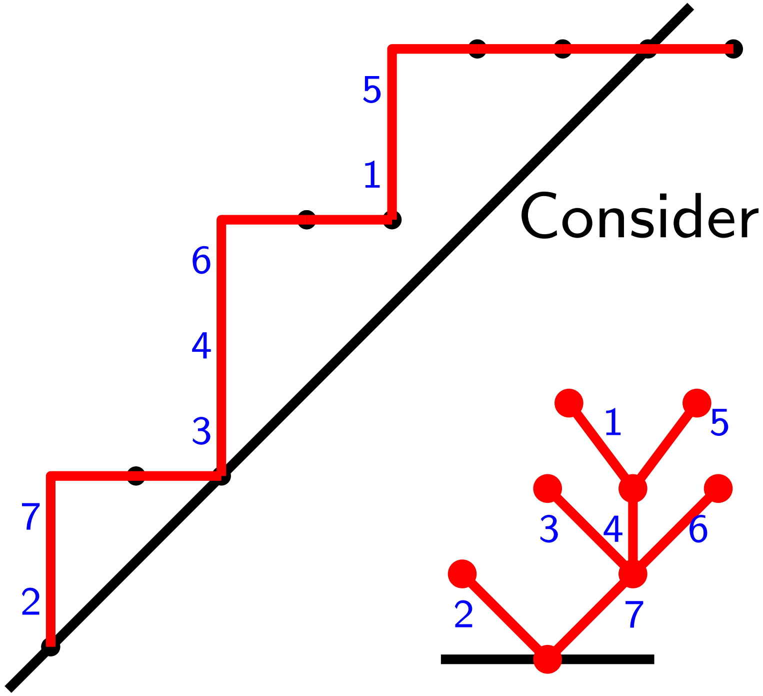

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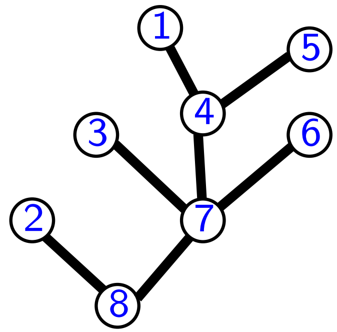

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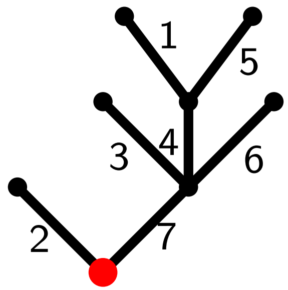


$(n+1)^{n-1}$ rooted Cayley trees with n indexed edges

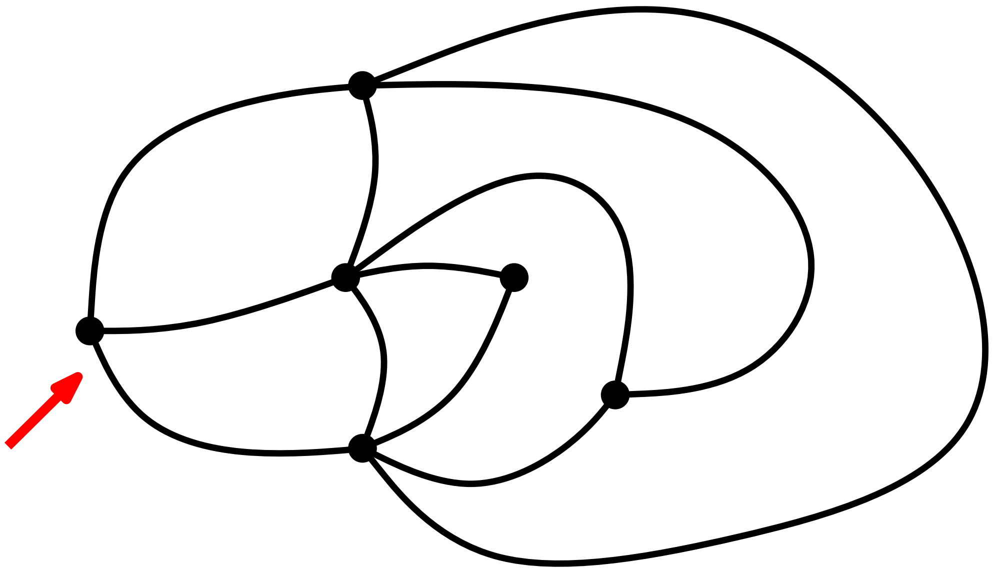
From maps to trees (I): tree-rooted maps

first strategy: Mullin primal dual decomposition

Planar maps, spanning trees and duality

Recall a planar map is a proper embedding of a connected graph on the sphere (considered up to homeomorphisms).

From now on, map means rooted planar map.



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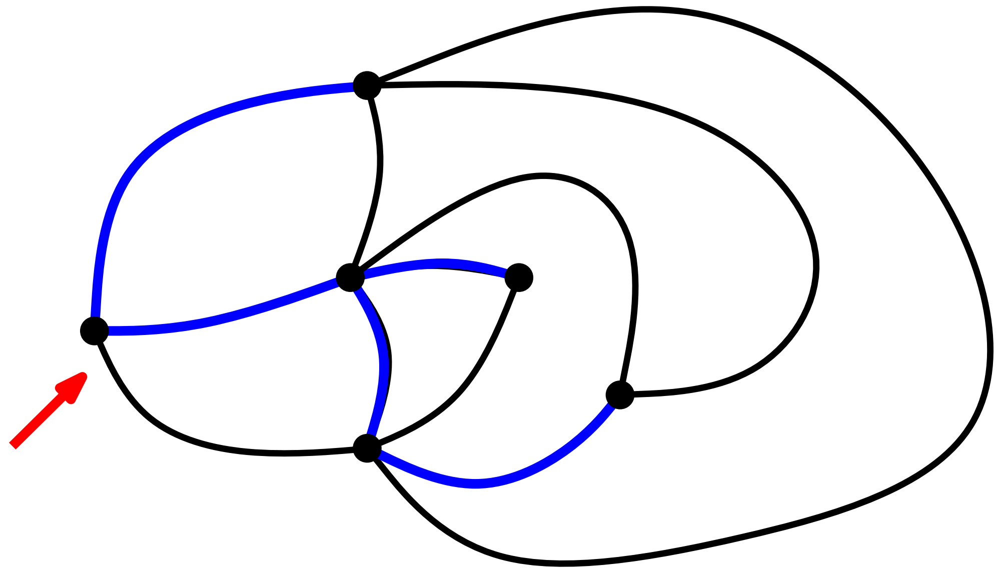

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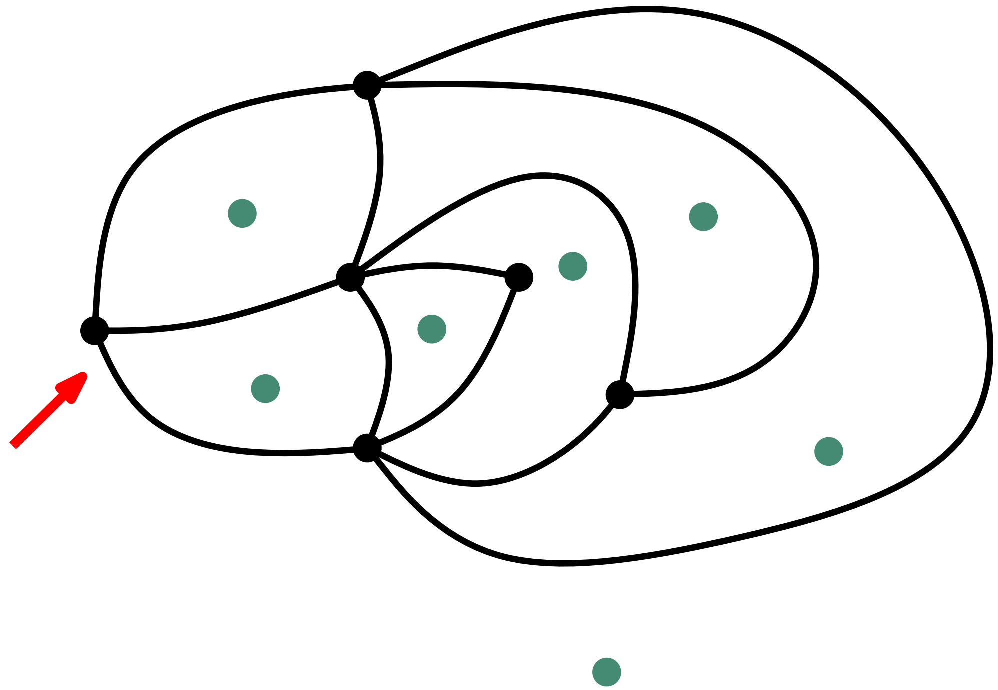

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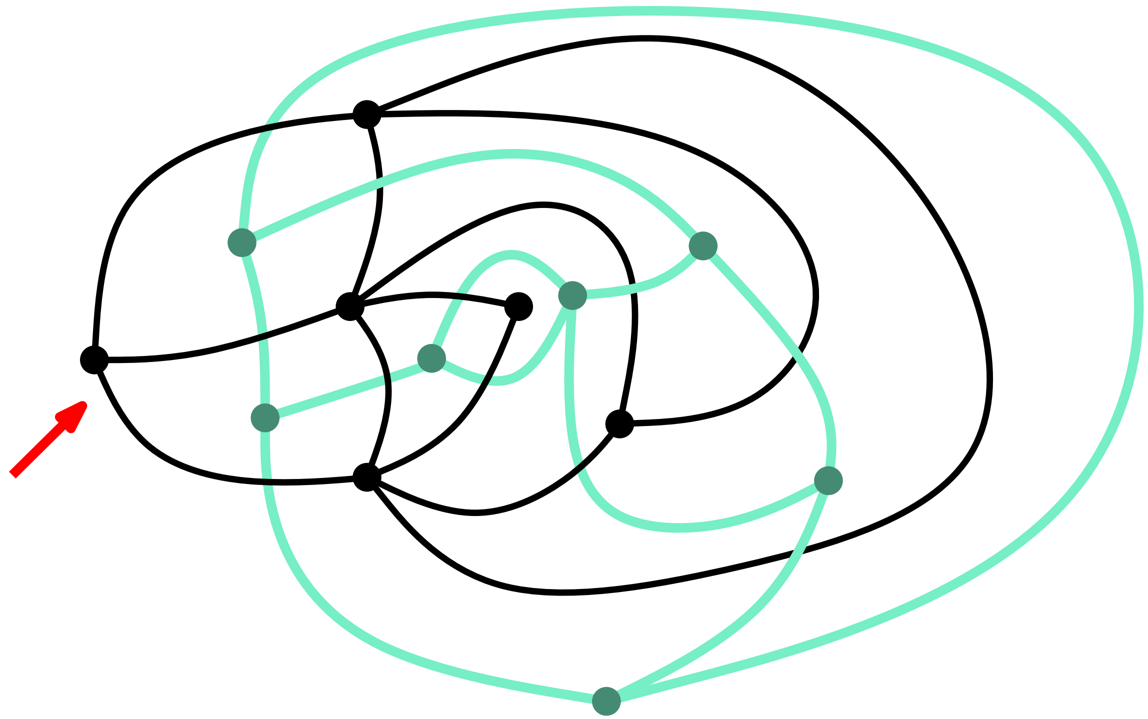

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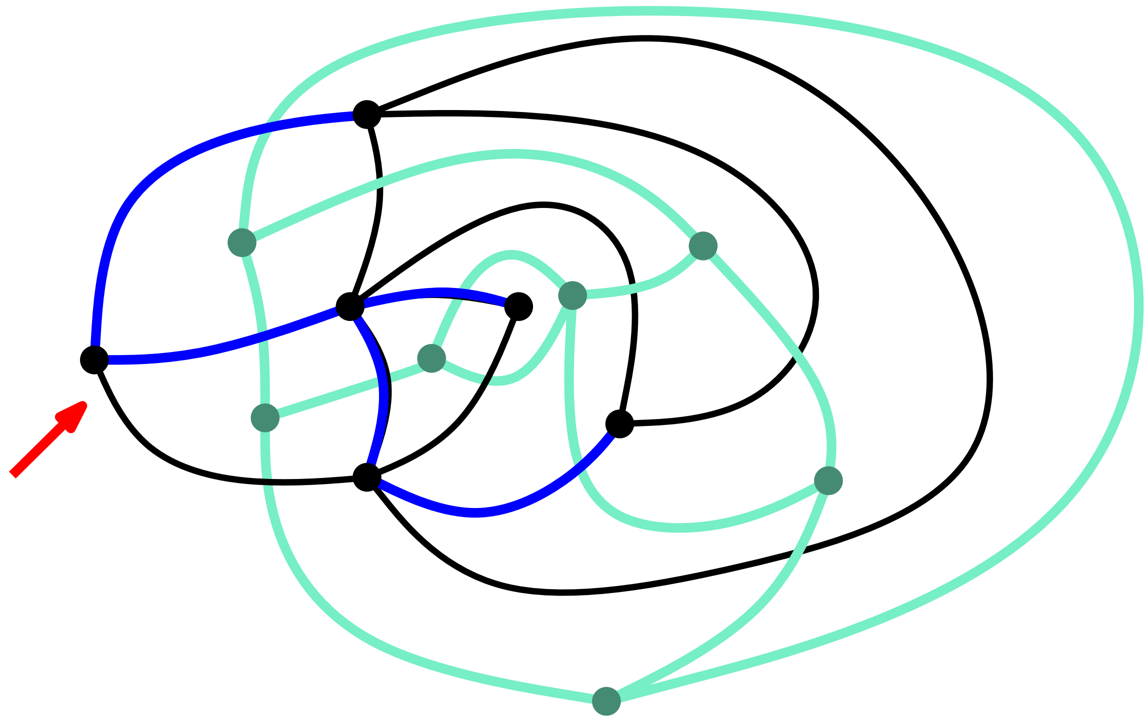
The dual map of a tree-rooted map is a tree-rooted map: it is naturally endowed with a dual spanning tree.



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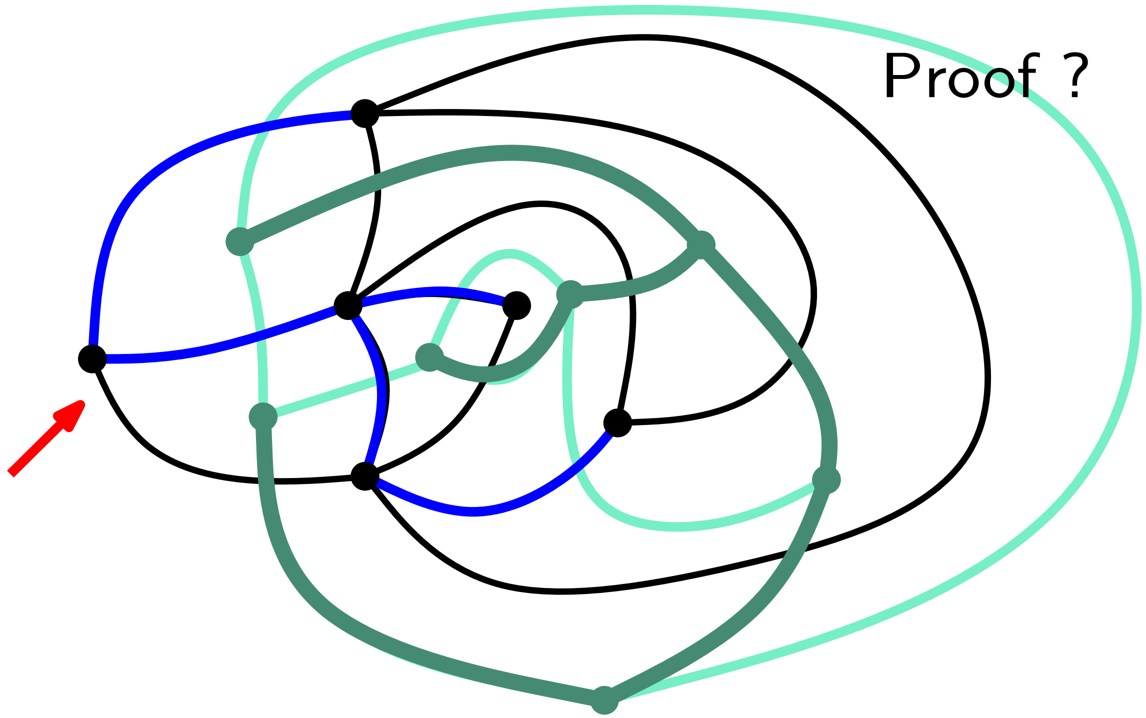

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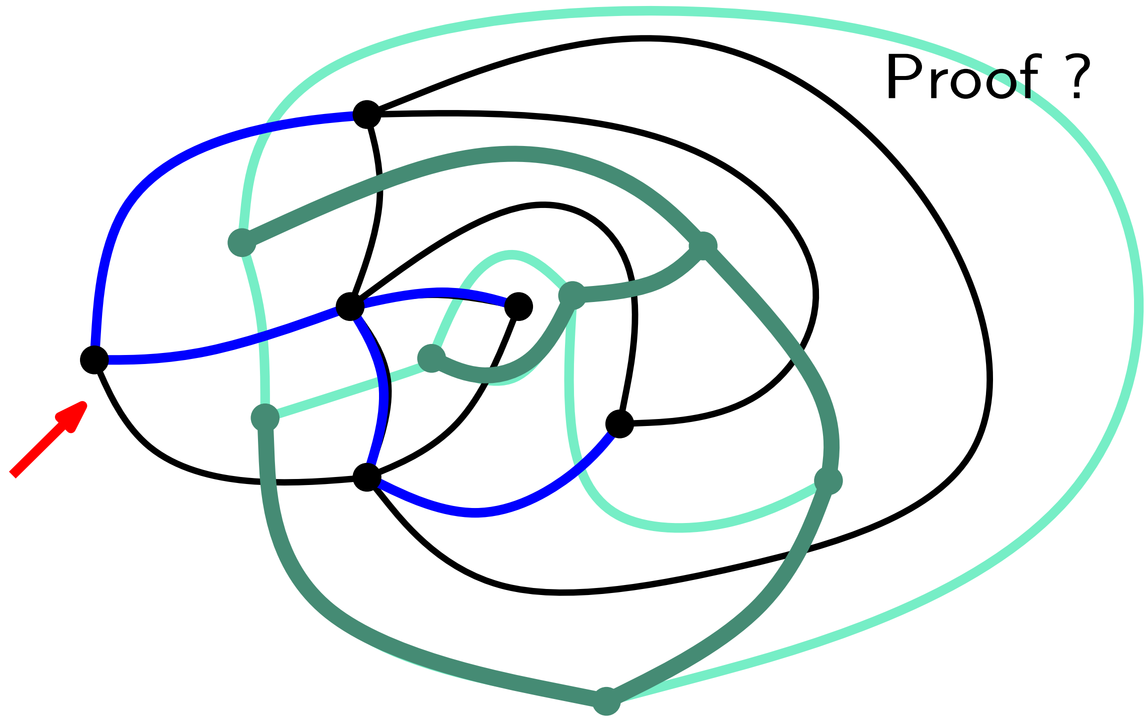
Euler's relation:

$$(\#\text{vertices-1})+(\#\text{faces-1}) = \#\text{edges}$$



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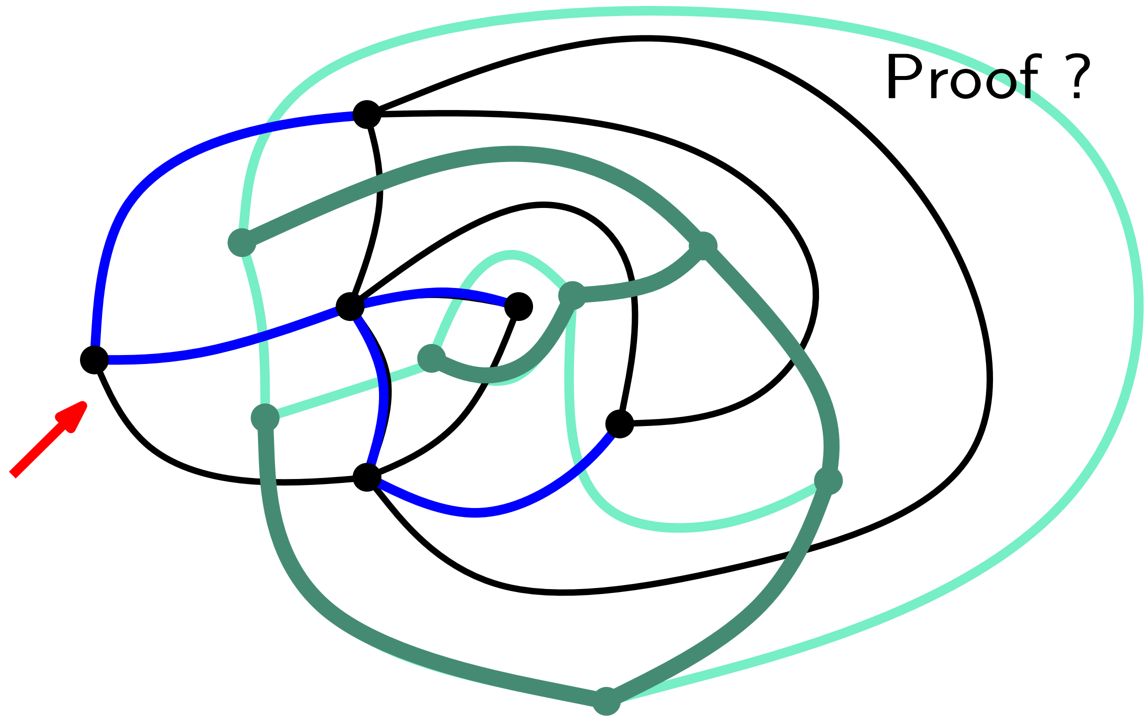
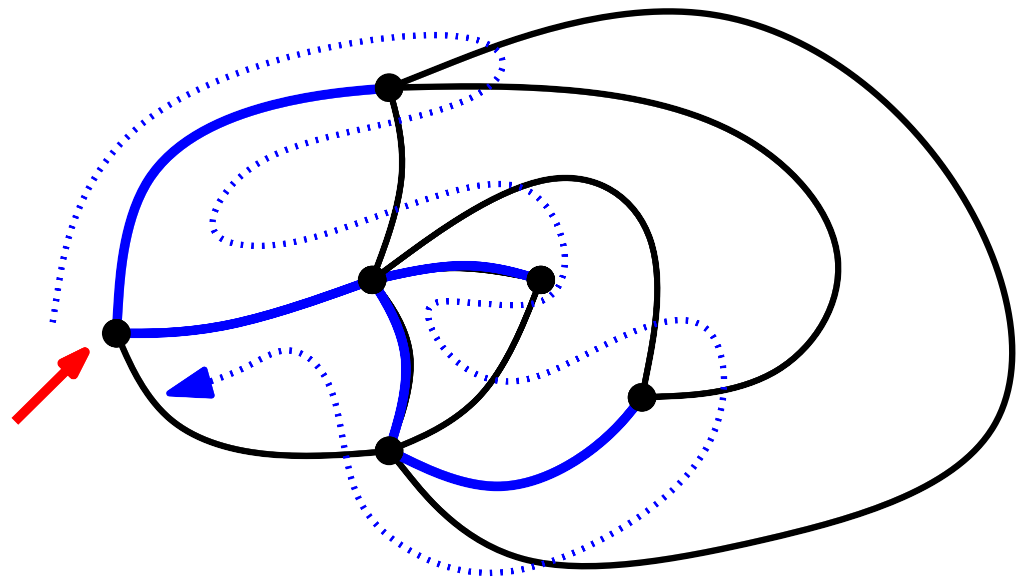
Proof?



Encoding tree-rooted maps with pairs of trees

Starting at a root corner
turn around the tree



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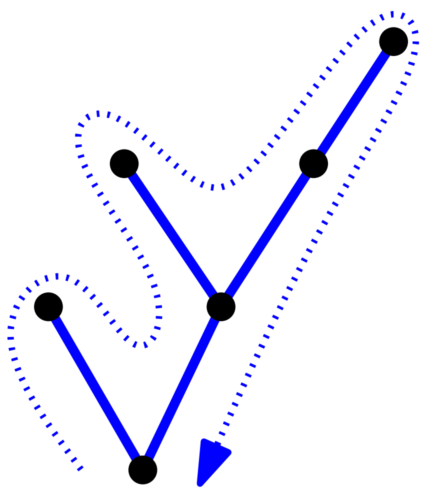

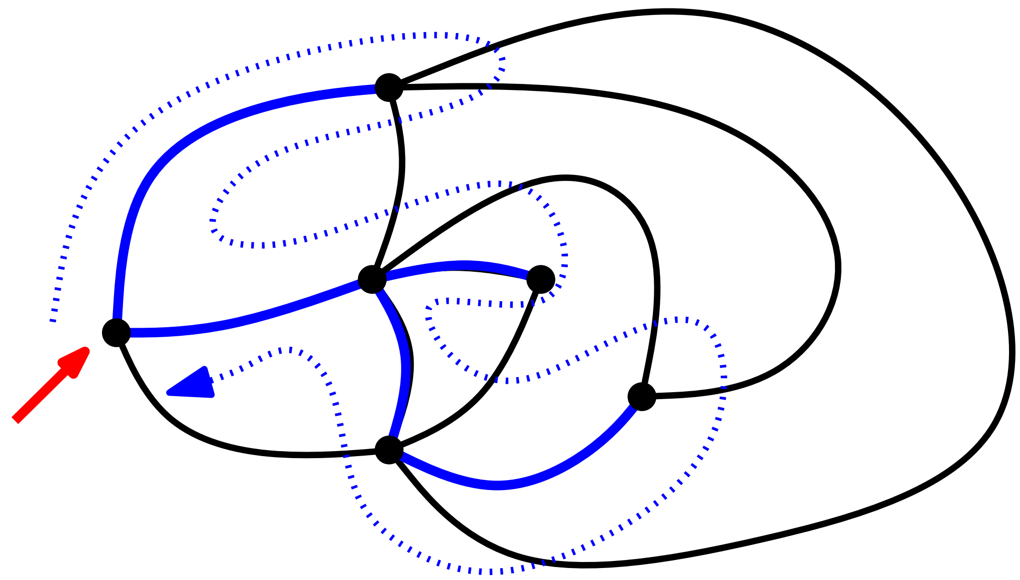

Encoding tree-rooted maps with pairs of trees

Starting at a root corner
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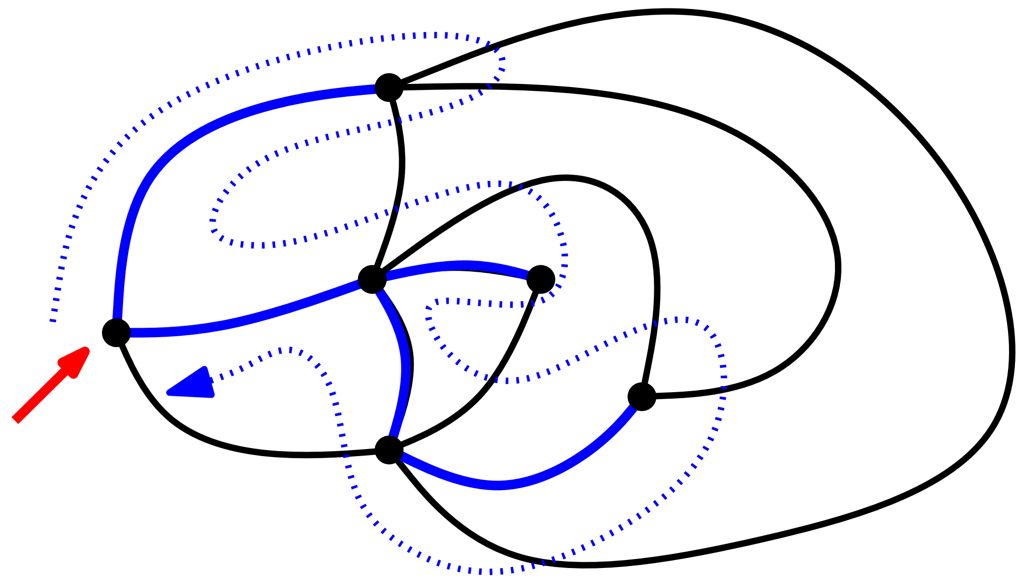
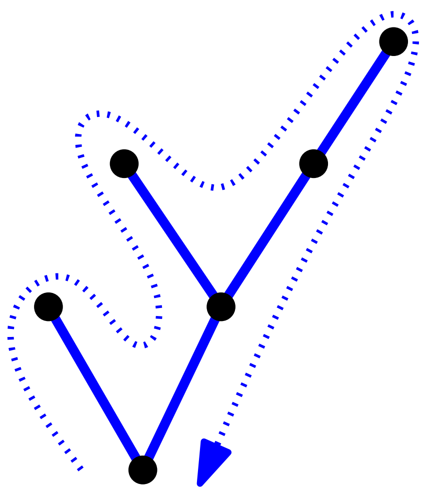
non visited edges = balanced
parenthesis word





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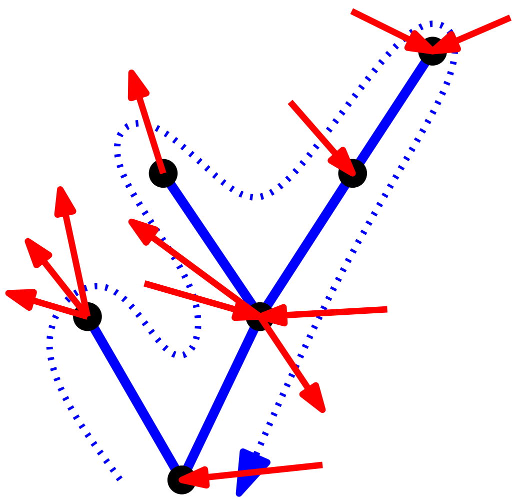

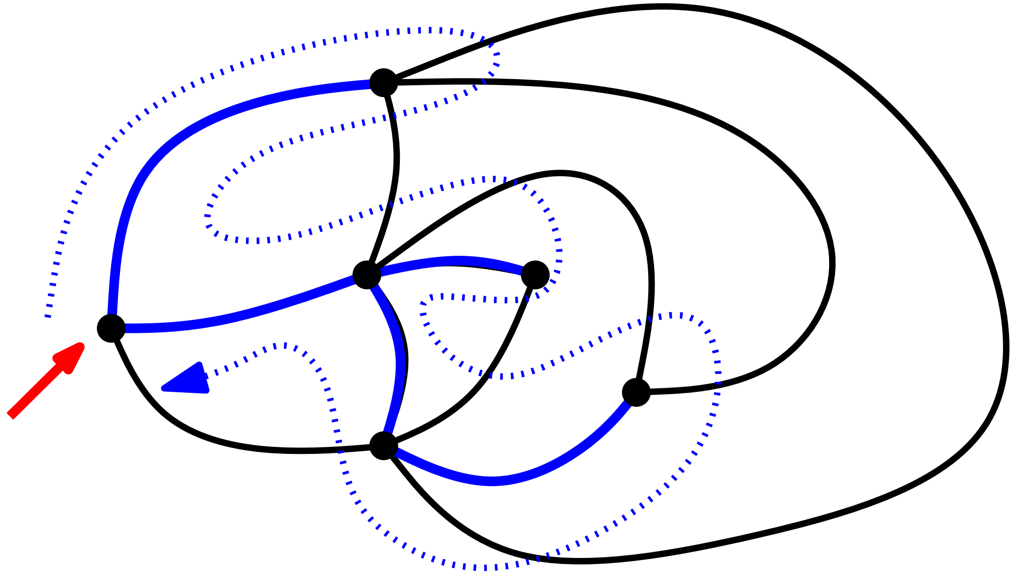

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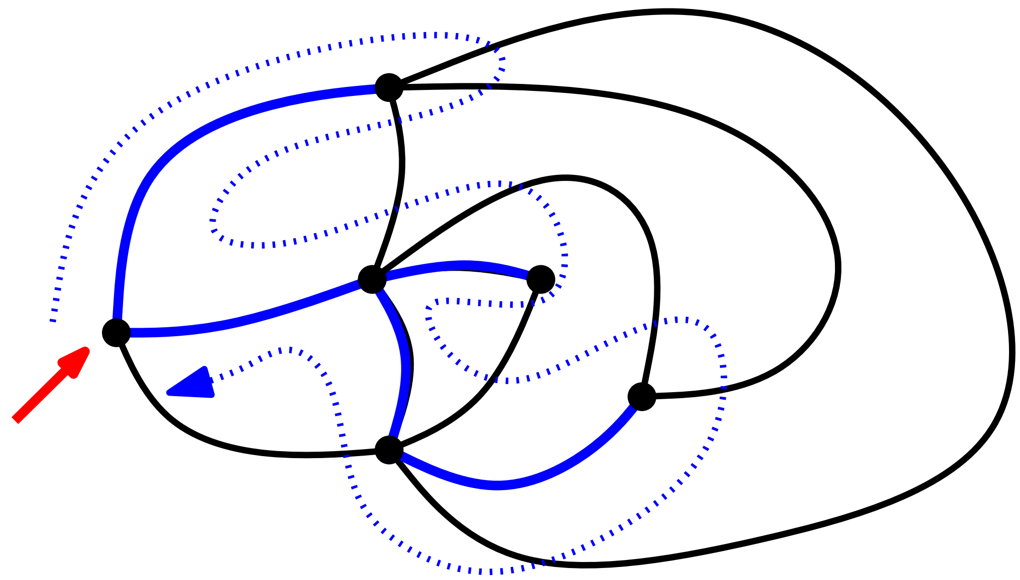

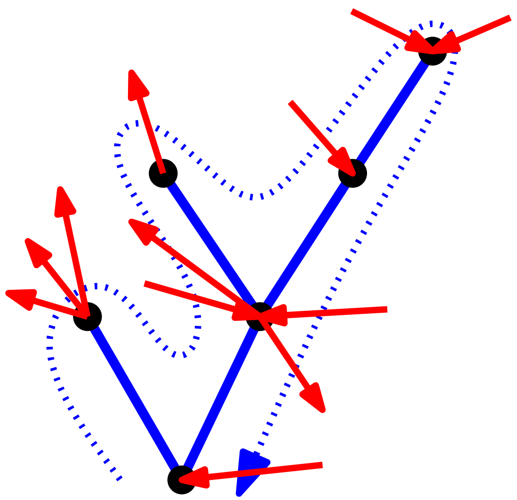

Code of the tree-rooted map = tree decorated by a balanced
parenthesis word

Encoding tree-rooted maps with pairs of trees

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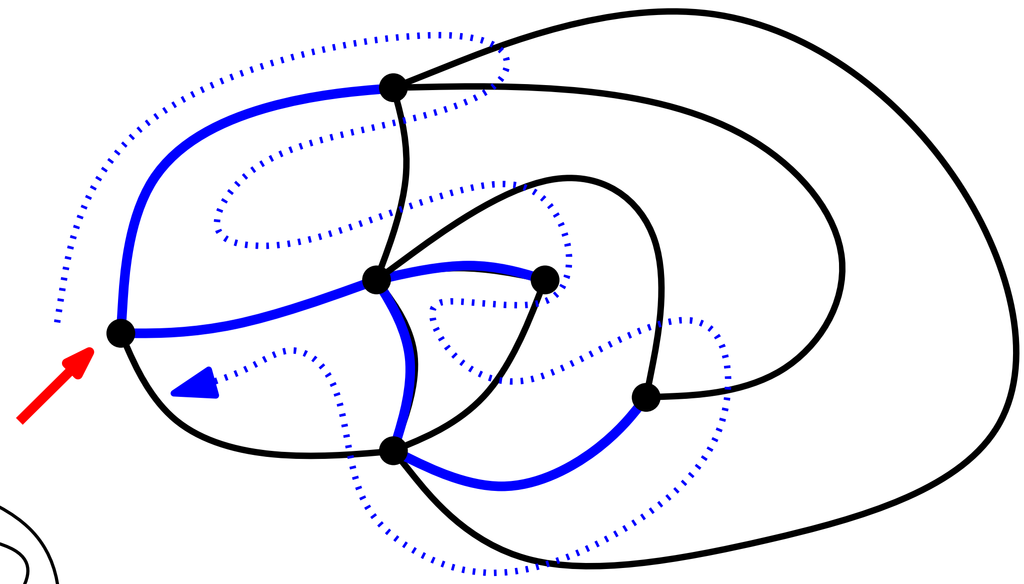

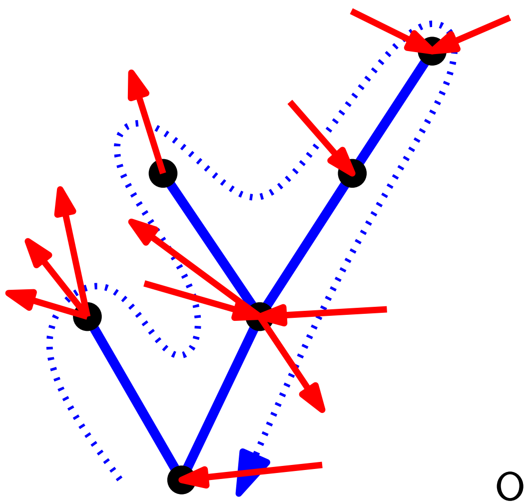

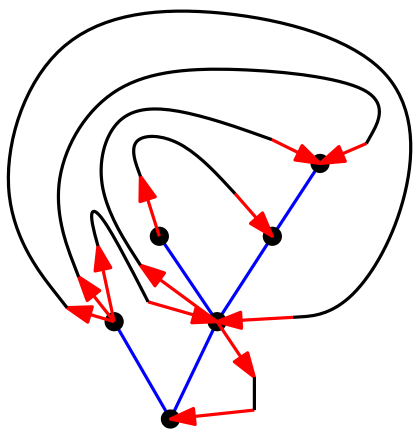

Observe that closure edges turn
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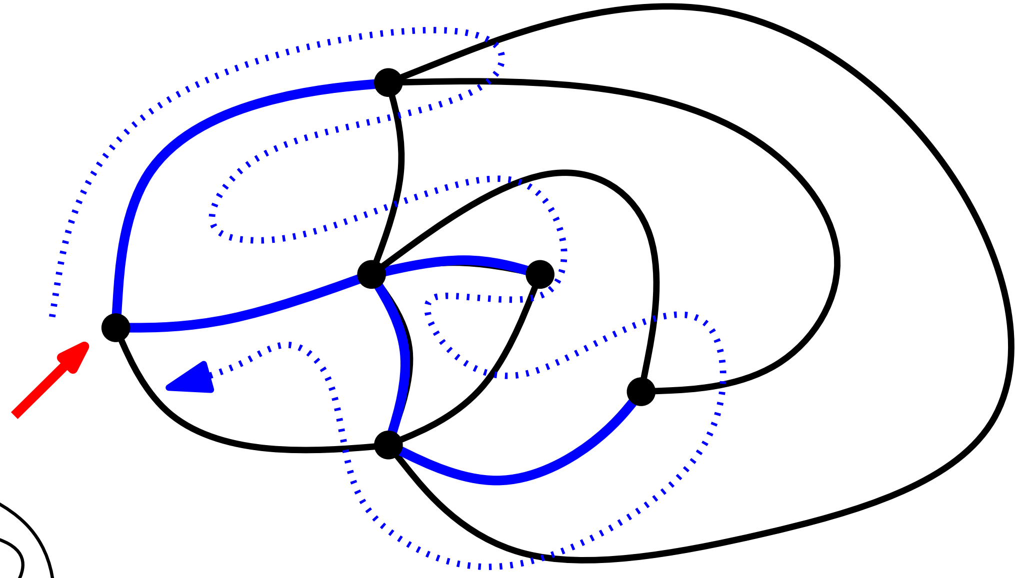

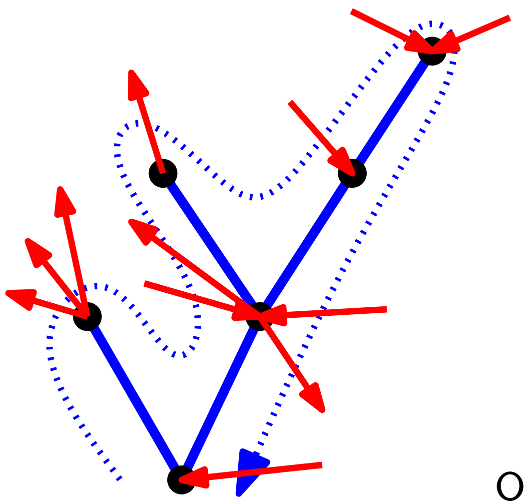

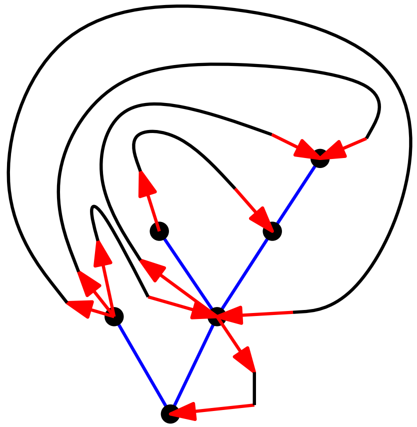

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Using the code of the tree by
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$uuuududuuudududdddudd$

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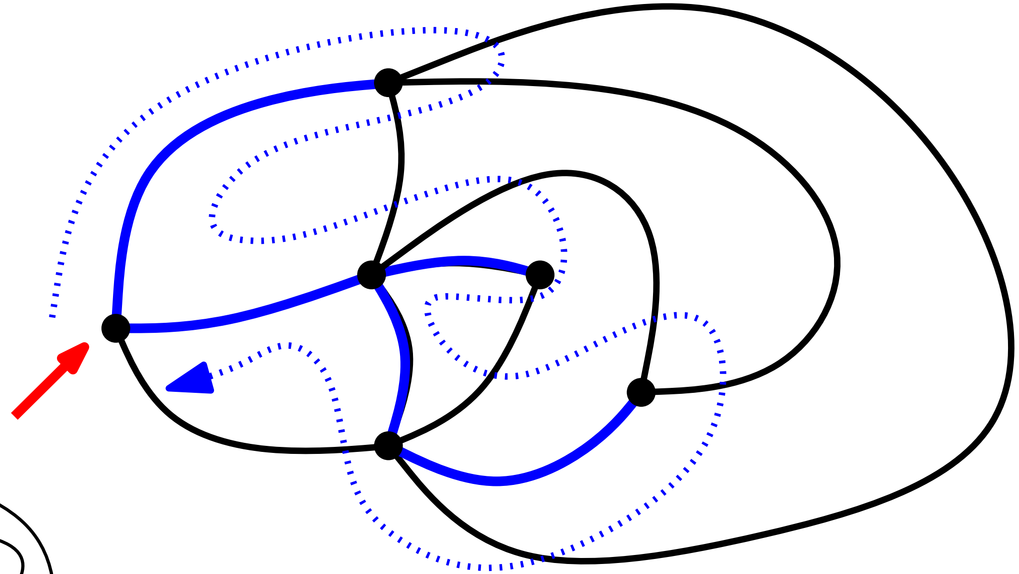

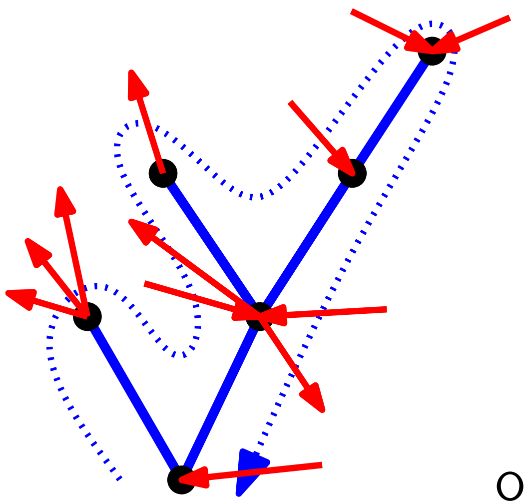

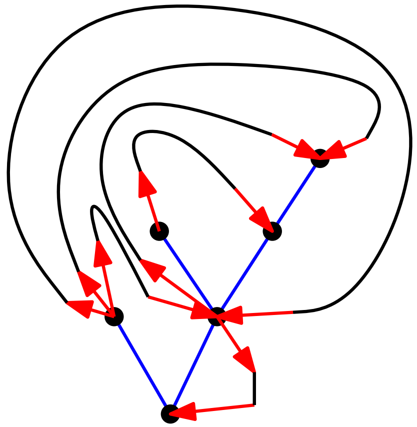

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Code of the tree-rooted map = tree decorated by a balanced
parenthesis word = shuffle of two Dyck words

Encoding tree-rooted maps with pairs of trees

Starting at a root corner
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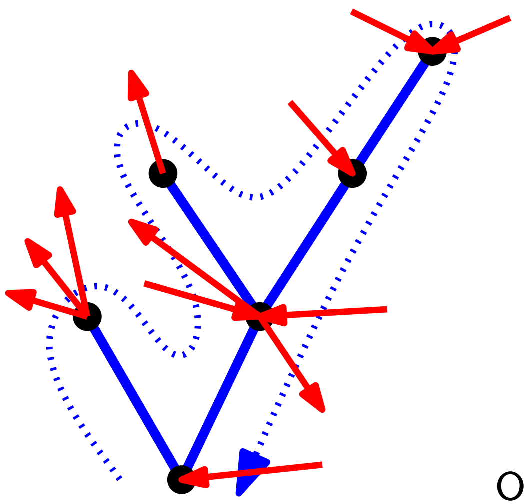
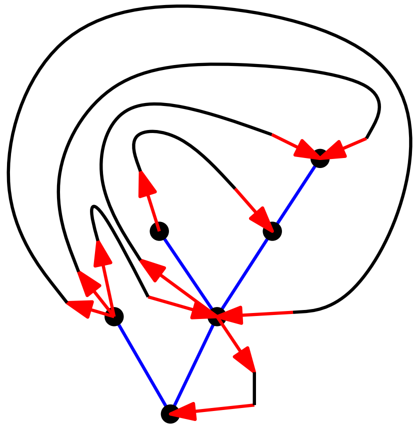
non visited edges $=$ balanced
parenthesis word





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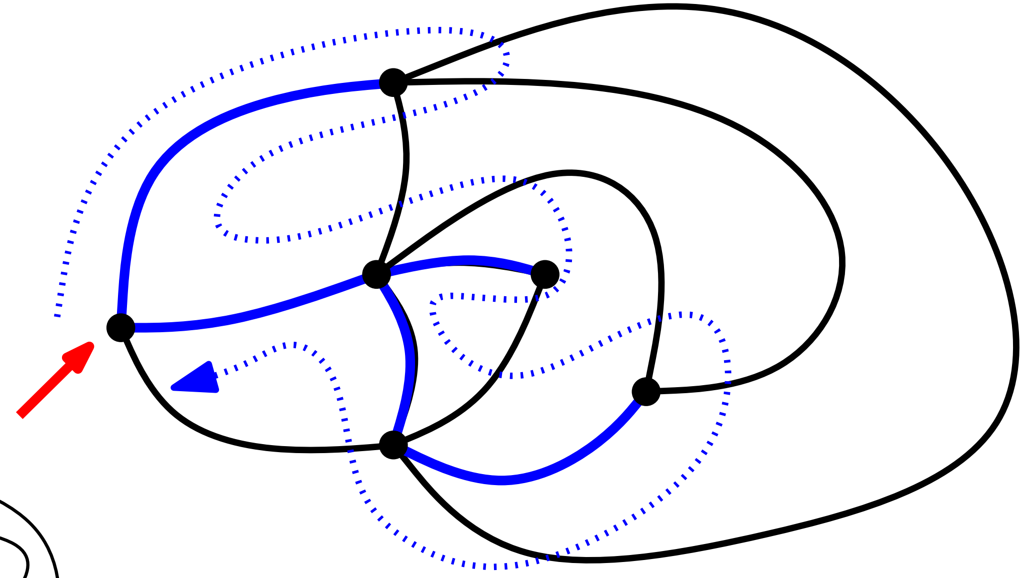

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The number of tree rooted planar maps with n edges is
$\sum_{i=0}^{n} \binom{2n}{i} C_i C_{n-i}$ where C_n denotes Catalan numbers.

From maps to trees (I): tree-rooted maps

first strategy: Mullin primal dual decomposition

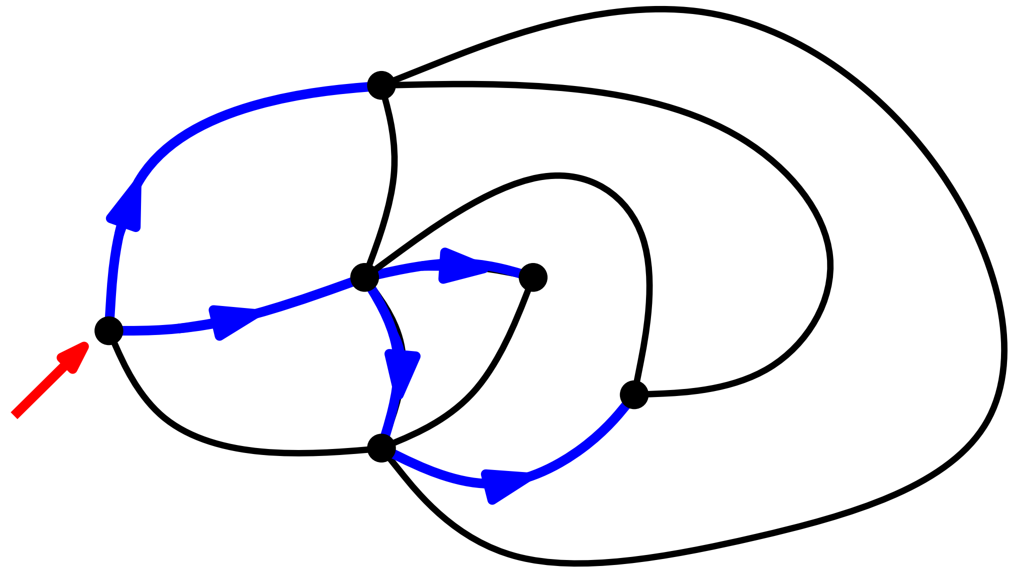
intermede: minimal orientations

second strategy: unfolding

} Bernardi's master bi-theorem

From tree-rooted maps to minimal accessible maps

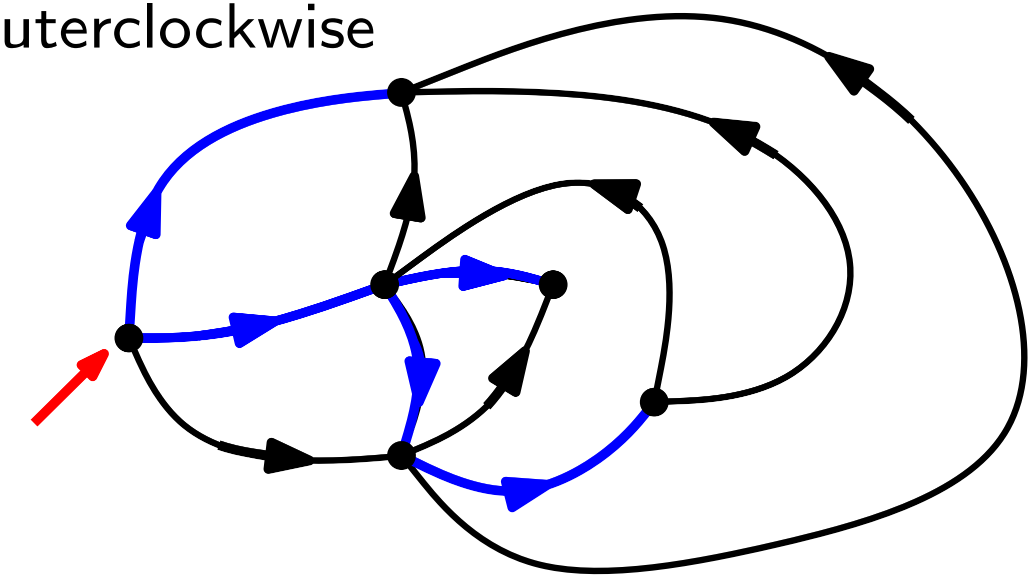
Orient the tree edges away from the root



From tree-rooted maps to minimal accessible maps

Orient the tree edges away from the root

Orient the other edges couterclockwise around the tree



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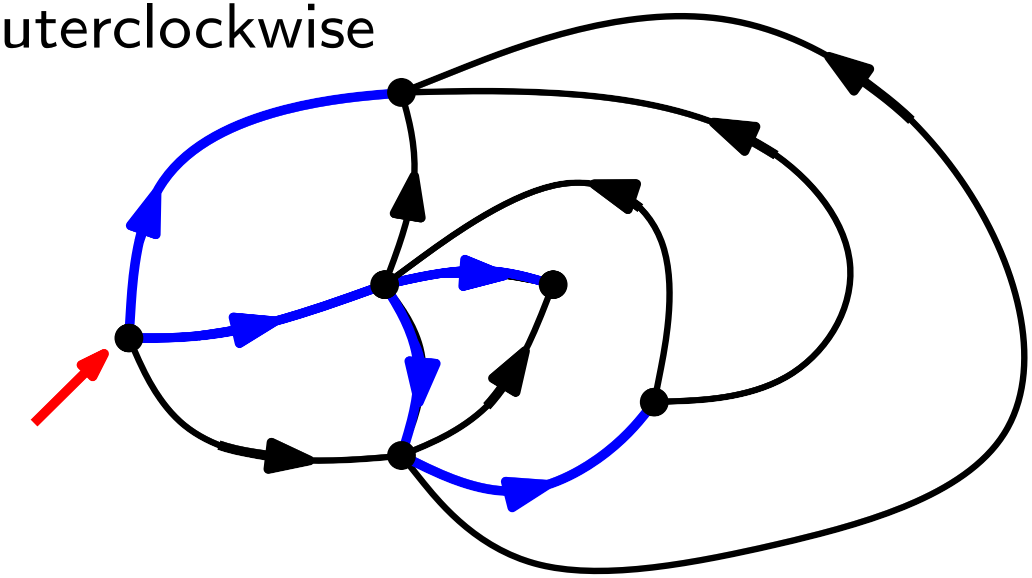

From tree-rooted maps to minimal accessible maps

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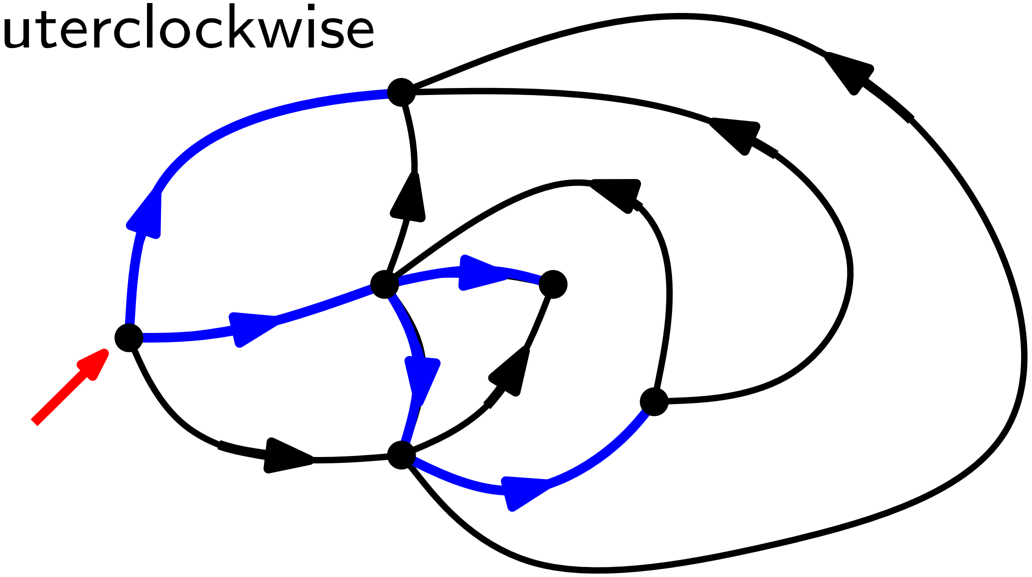
Orient the other edges couterclockwise around the tree

The resulting orientation has no clockwise circuit.



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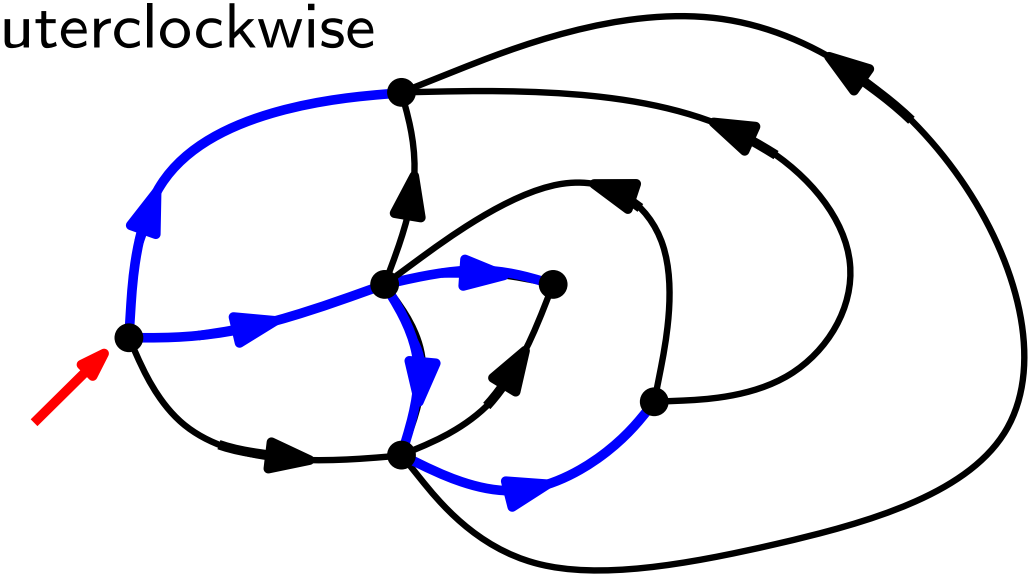

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It is called a minimal orientation (for the order induced by circuit reversal).

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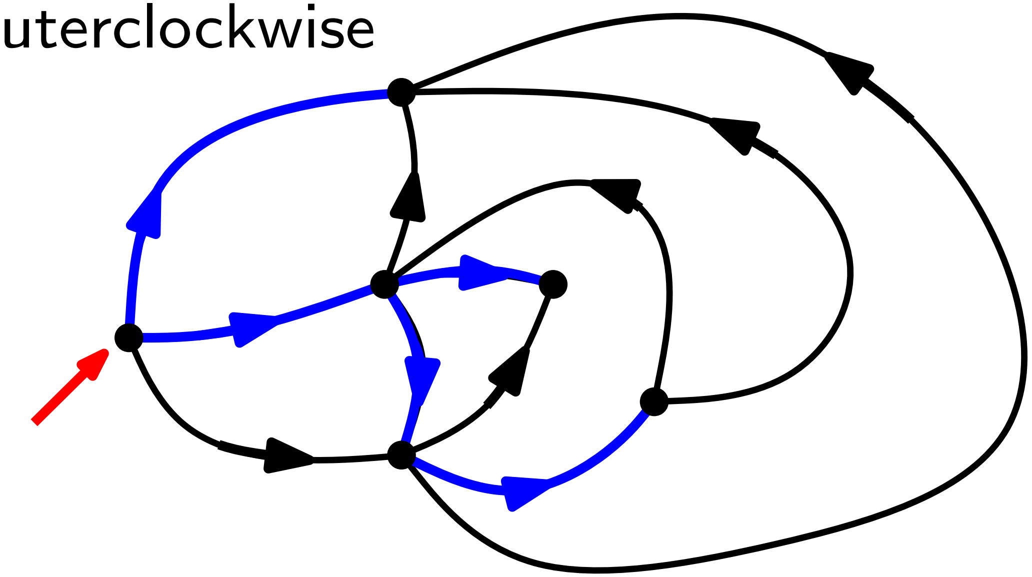

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A oriented map is accessible if every vertex can be reach by an oriented path from the root.

From tree-rooted maps to minimal accessible maps

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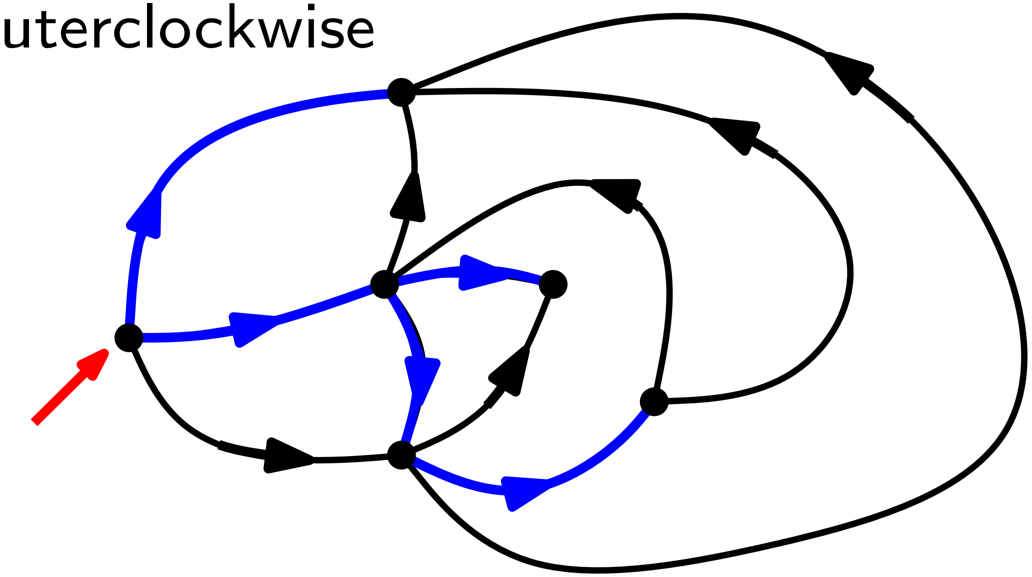

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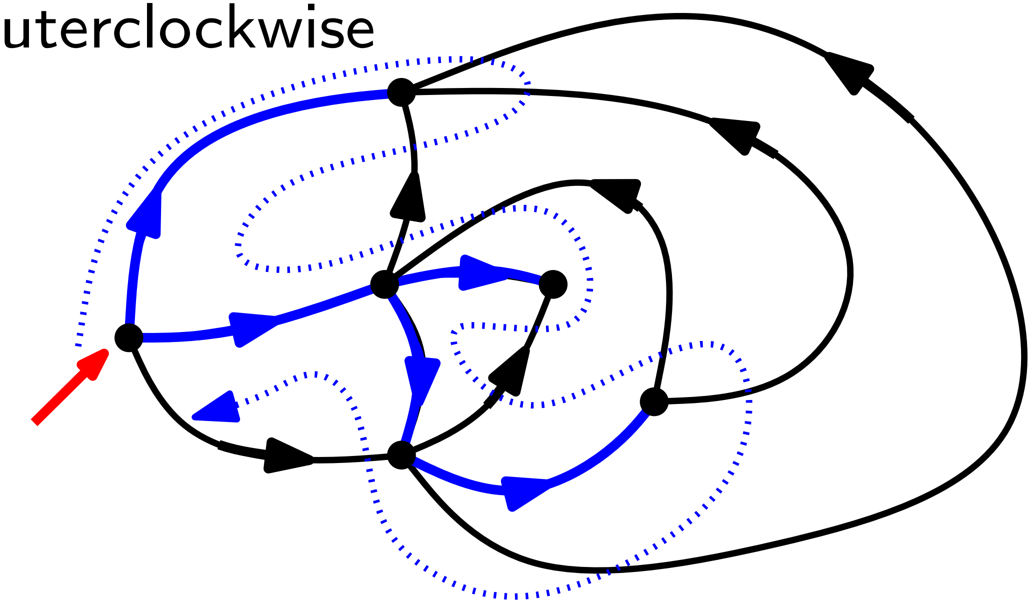
Theorem (Bernardi) This is a bijection between tree-rooted maps with n edges and minimum accessible maps with n edges

From tree-rooted maps to minimal accessible maps

Orient the tree edges away from the root

Orient the other edges couterclockwise around the tree



The resulting orientation has no clockwise circuit.

It is called a minimal orientation (for the order induced by circuit reversal).

A oriented map is accessible if every vertex can be reach by an oriented path from the root.

Theorem (Bernardi) This is a bijection between tree-rooted maps with n edges and minimum accessible maps with n edges

The tree is recovered by reconstructing its contour (or equivalently by leftmost depth first search).

From maps to trees (I): tree-rooted maps

first strategy: Mullin primal dual decomposition

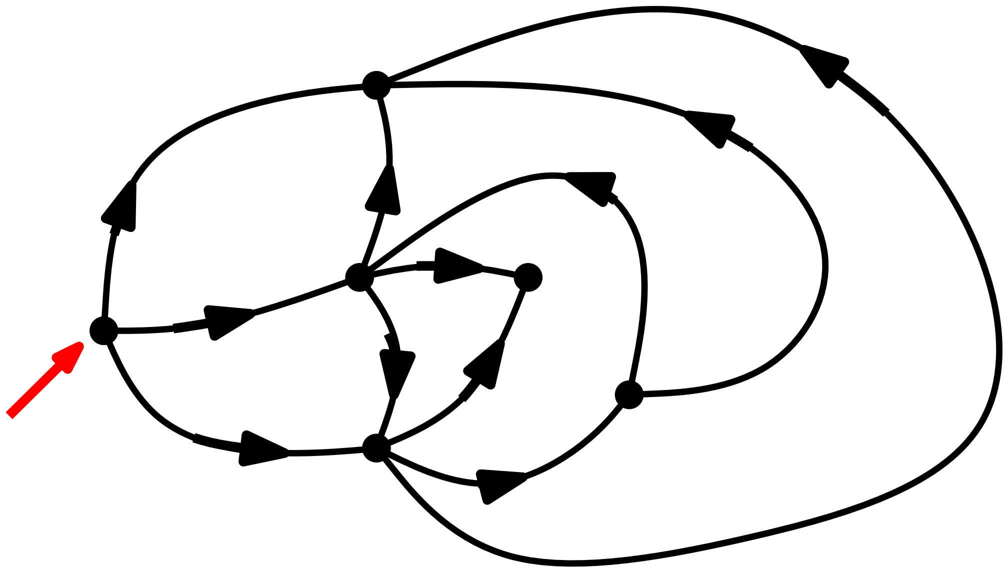
intermede: minimal orientations

second strategy: unfolding

Bernardi's master bi-theorem

Bernardi's decomposition of minimal accessible maps

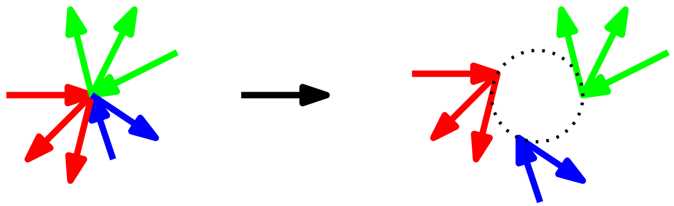
Consider a minimal accessible map

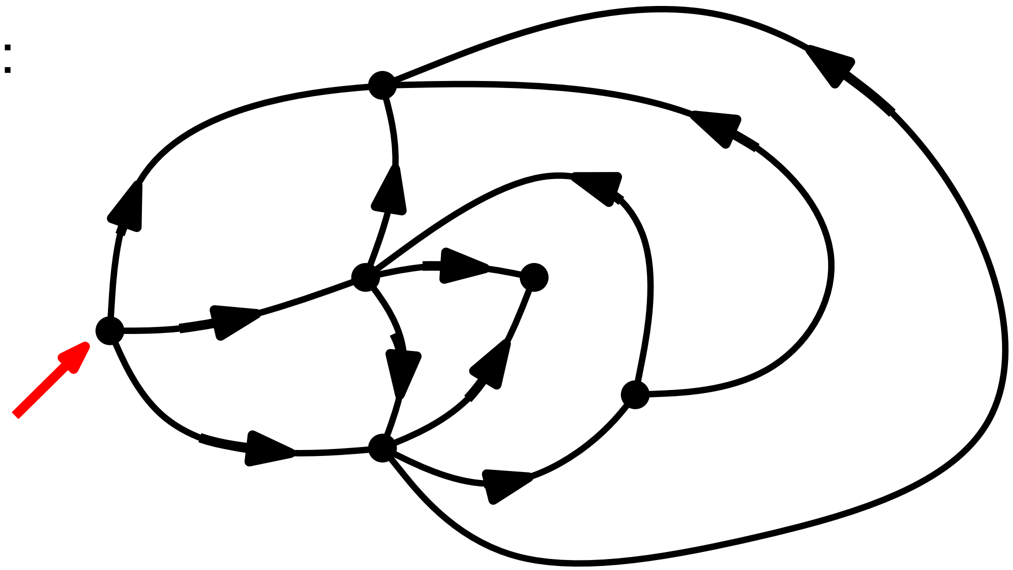


Bernardi's decomposition of minimal accessible maps

Consider a minimal accessible map

Define its vertex unfolding:





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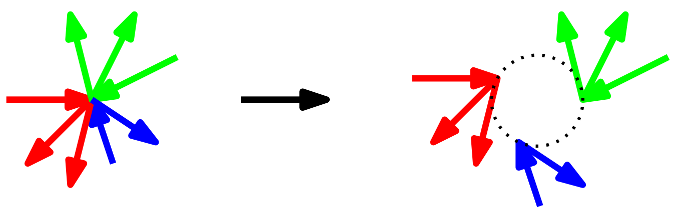
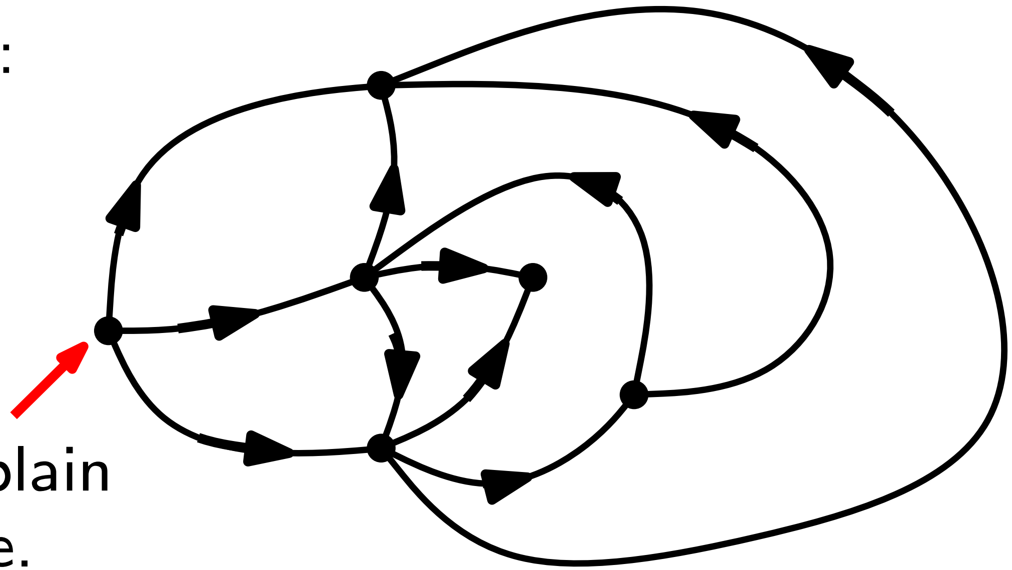
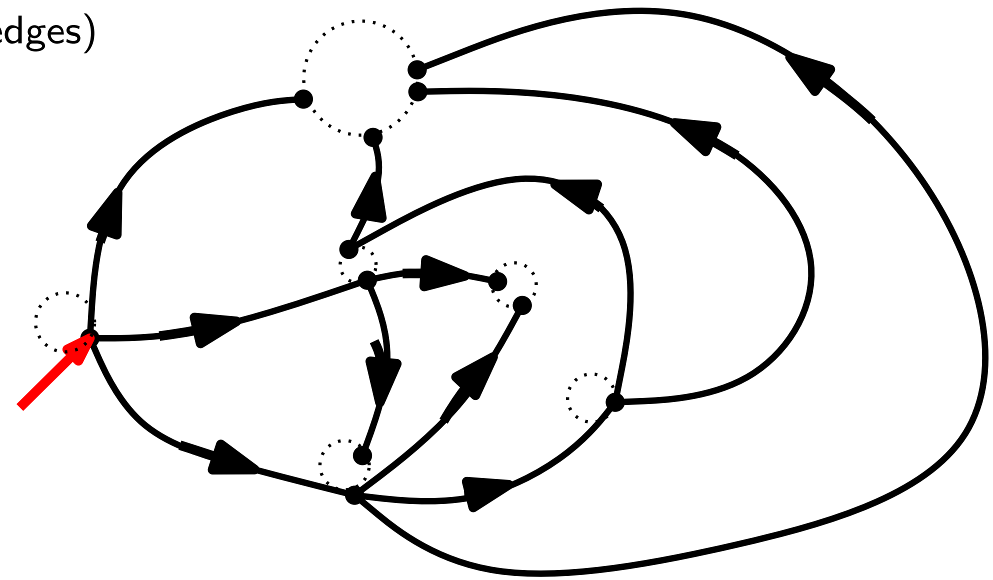




In the unfolded map, the plain edges form a spanning tree.

(clockwise cycles are ruled out by external edges)

(a counterclockwise cycle would be non accessible from the outside)



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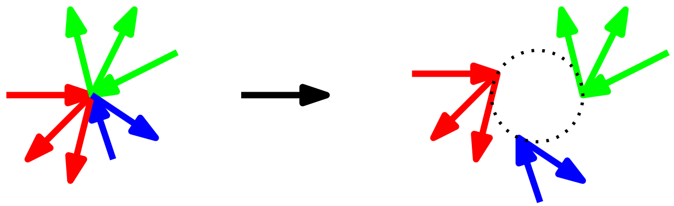

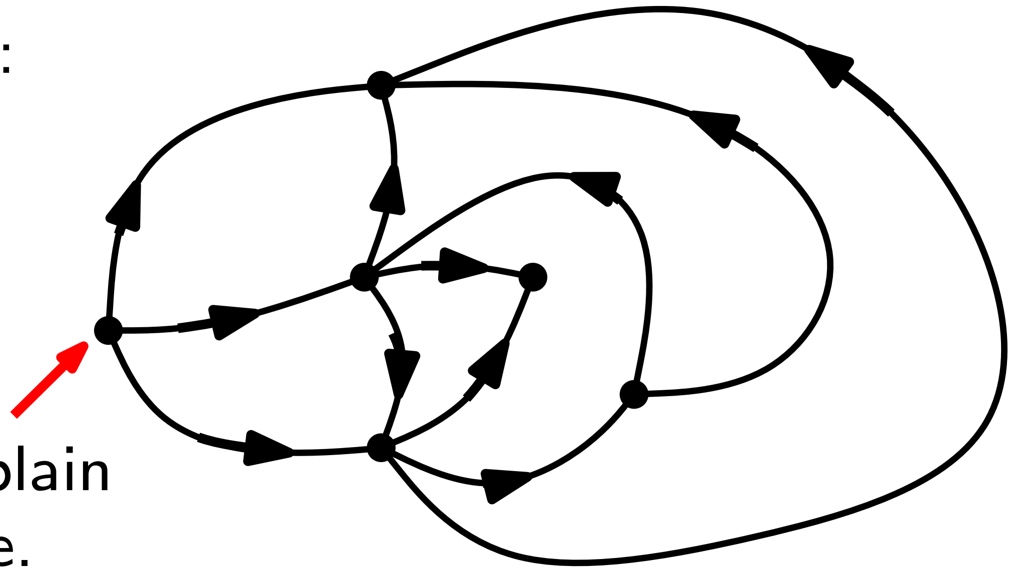

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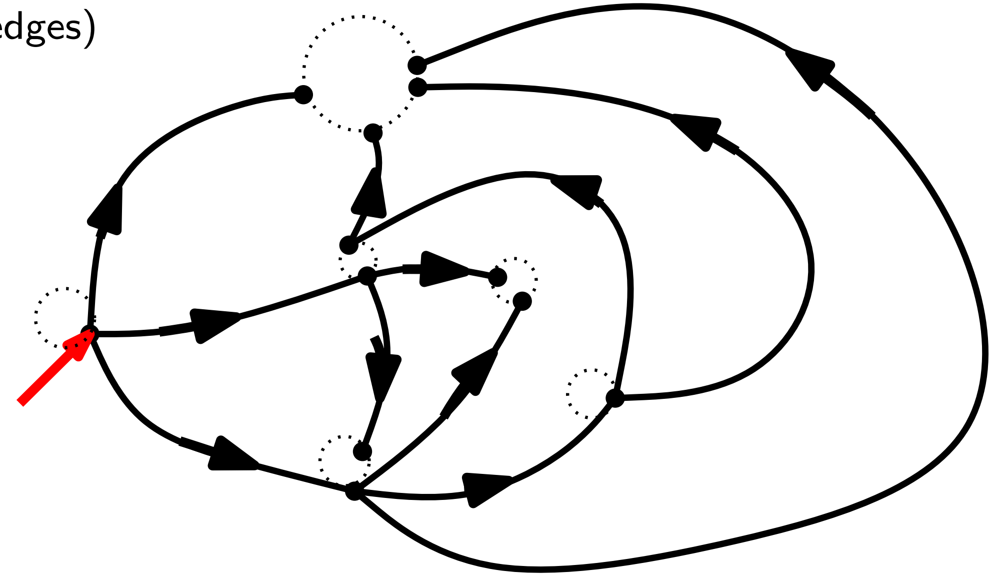

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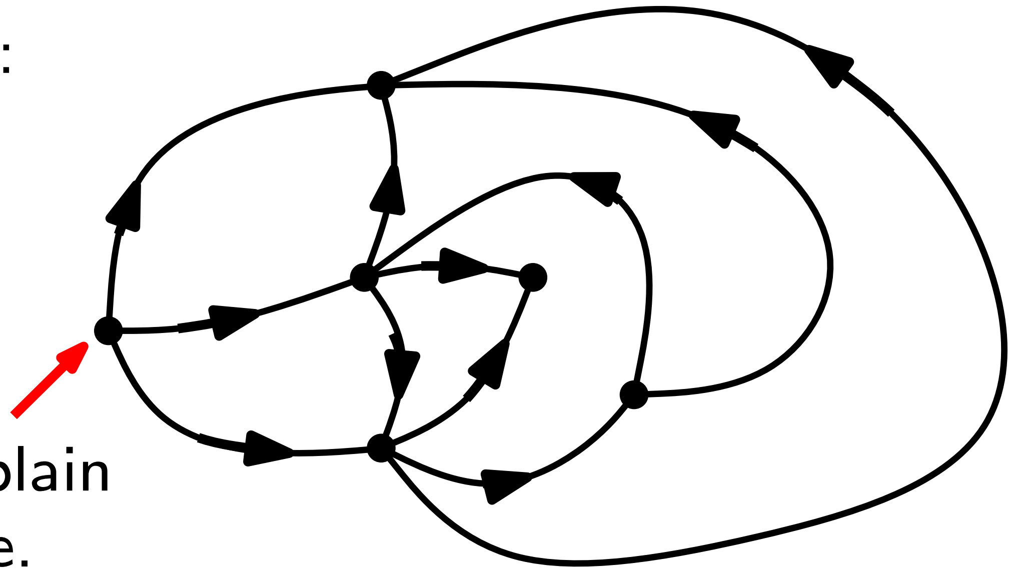

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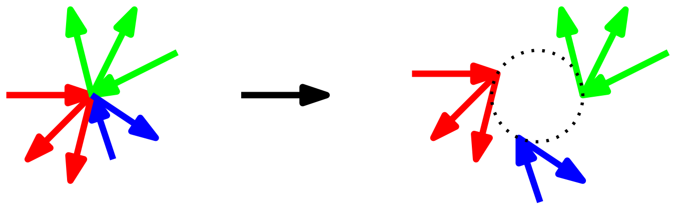

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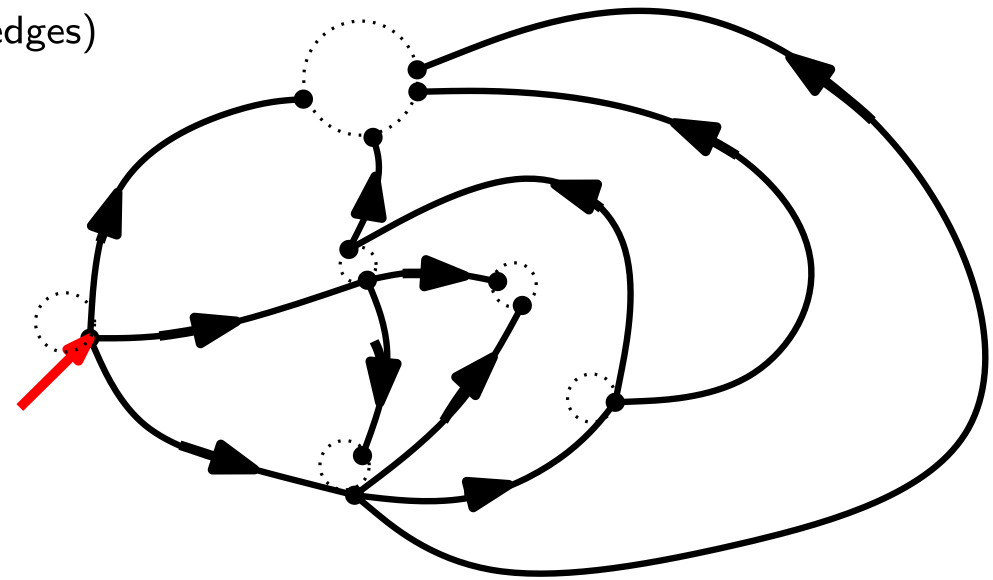

The unfolded map is tree-rooted

The dual tree is naturally bicolored

Bernardi's decomposition of minimal accessible maps

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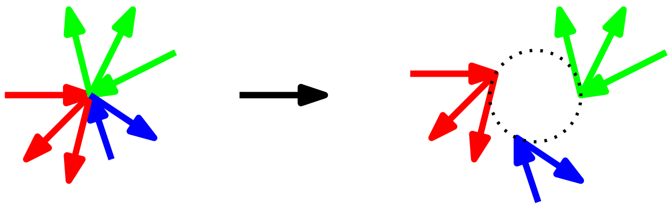

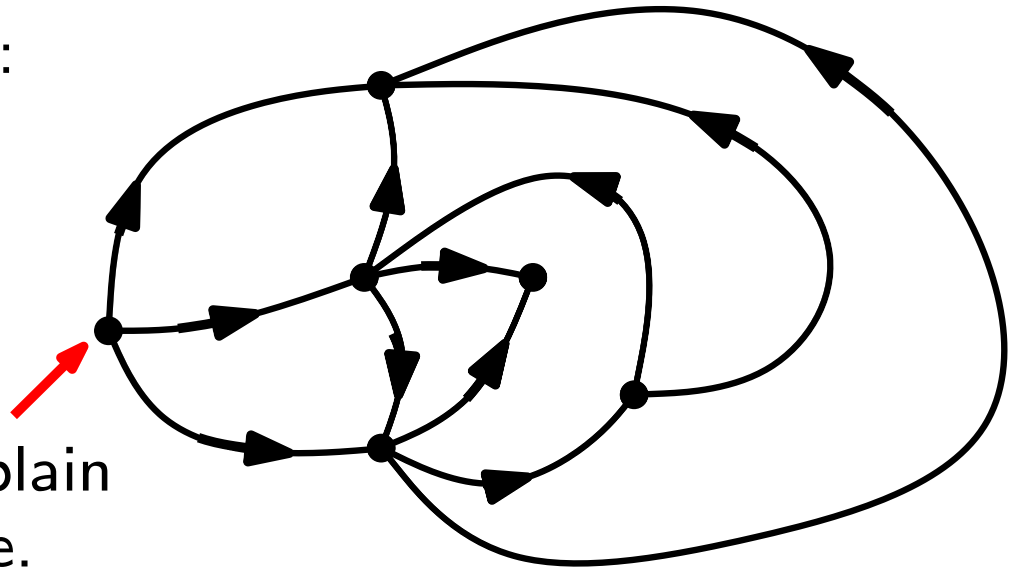

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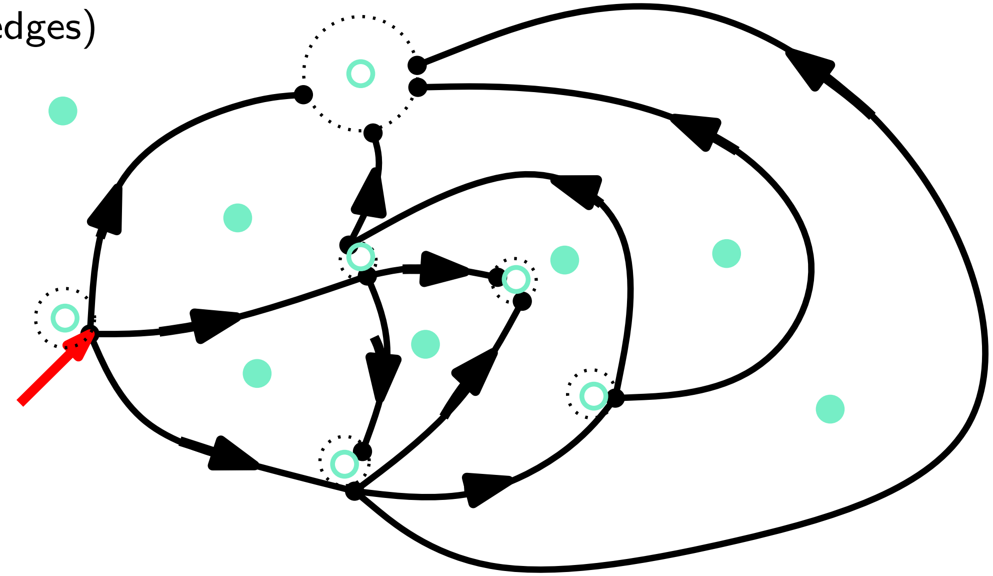

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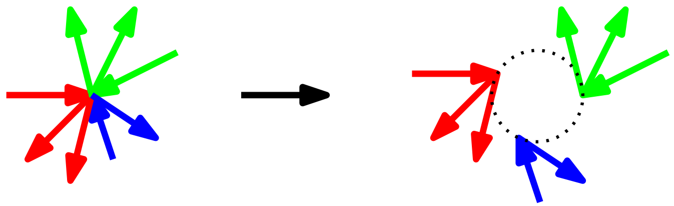

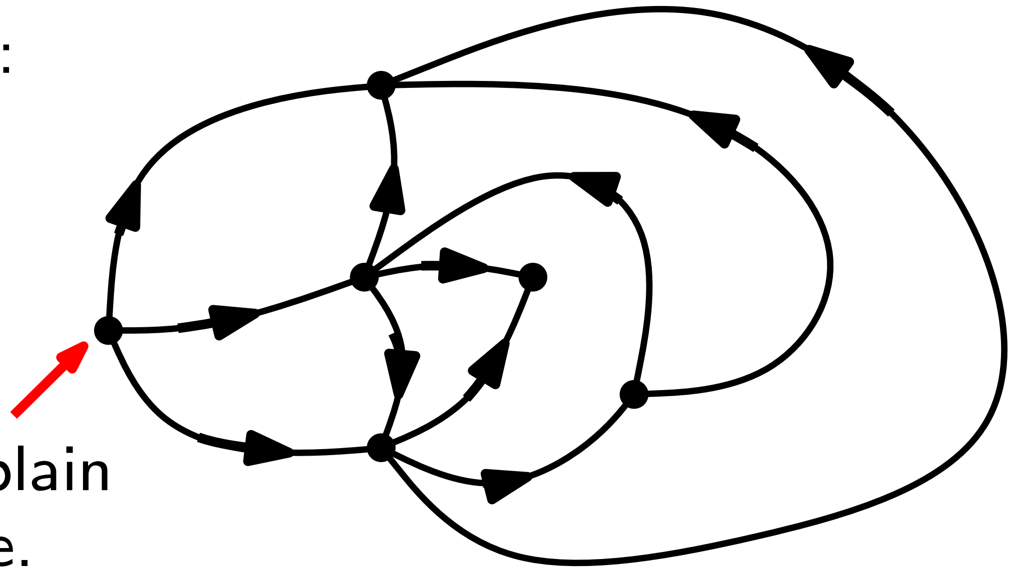

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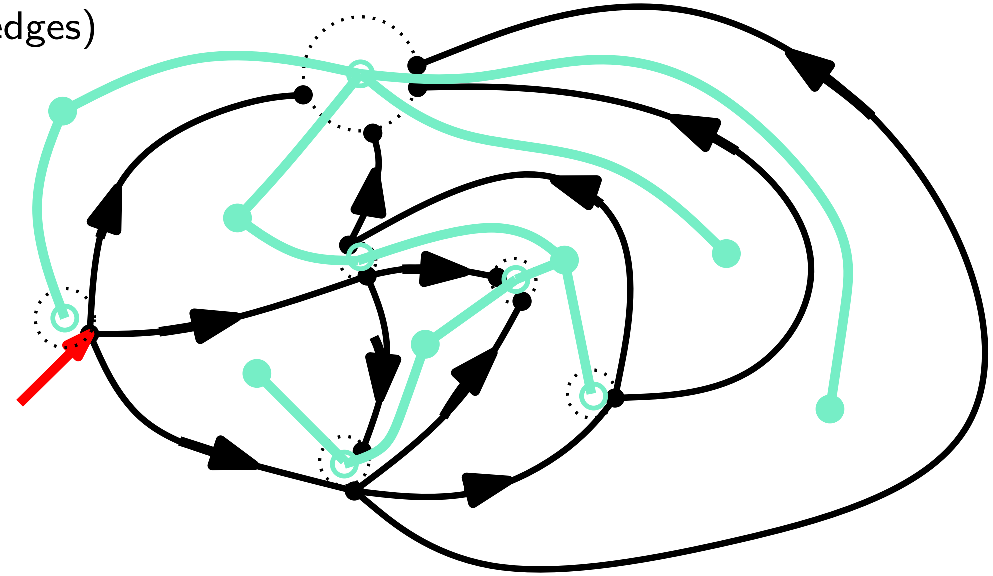

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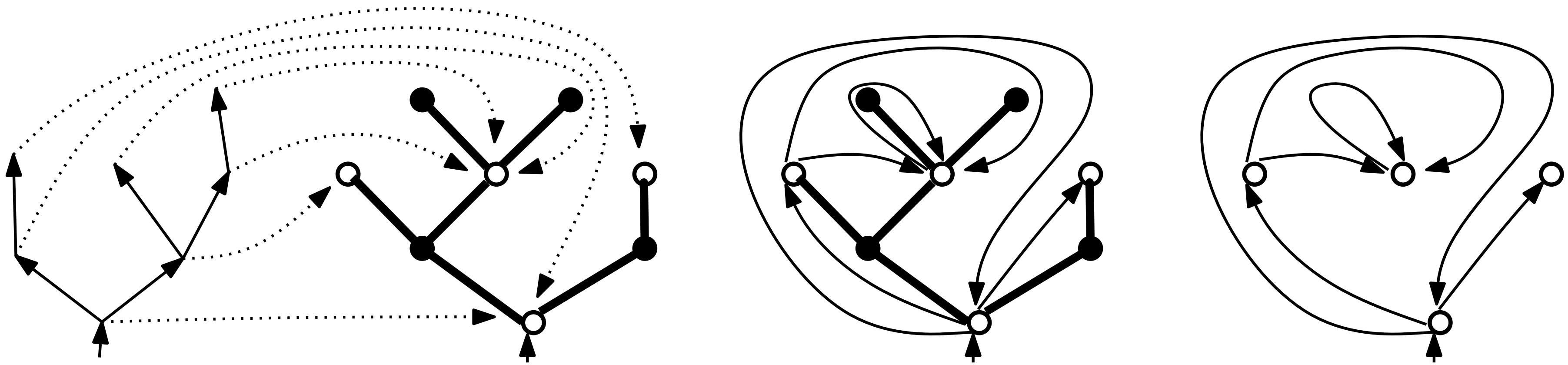
Bernardi's master bijection for tree-rooted maps

The primal and dual trees of the unfolded maps are glued canonically (no shuffling of the codes required)



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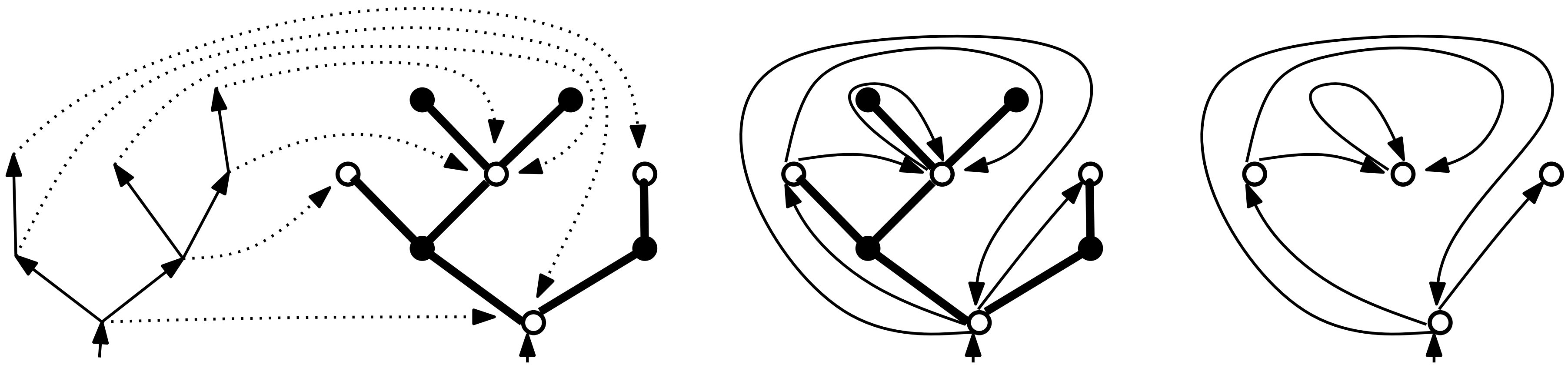
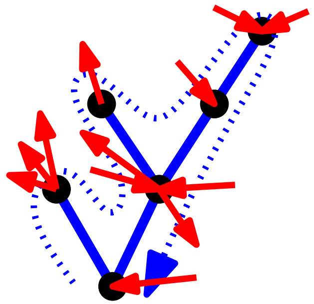


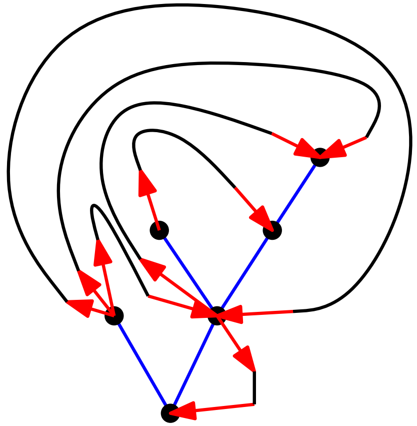
Conversely gluying an arbitrary tree with n edges with an arbitrary tree with $s + f = n + 2$ vertices yields a left-accessible map

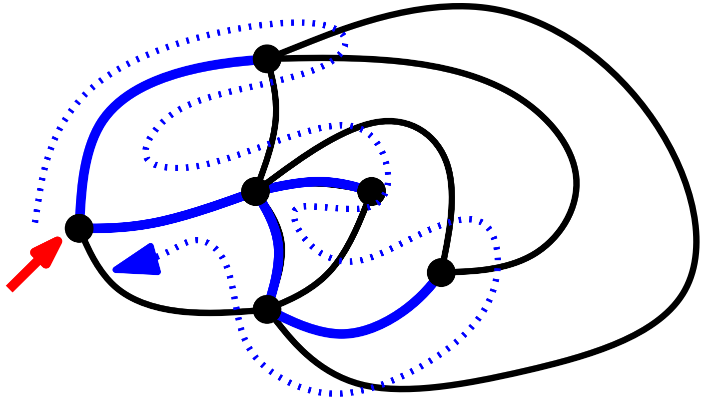
Theorem(Bernardi) This is a bijection between such pairs of trees and minimal accessible maps with n edges, (and tree-rooted maps via previous Theorem).

Corollary: $\sum_{i=0}^{n} \binom{2n}{i} C_i C_{n-i} = C_{n+1} C_n$

Summary: two strategies for tree-rooted maps







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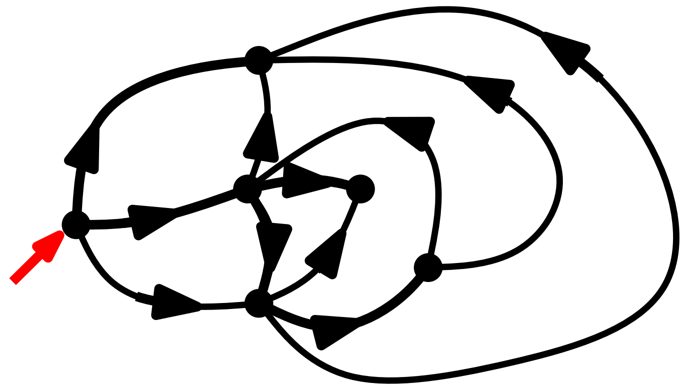
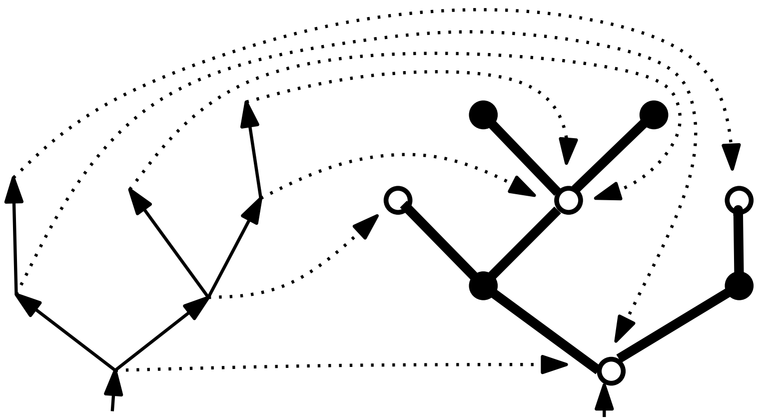
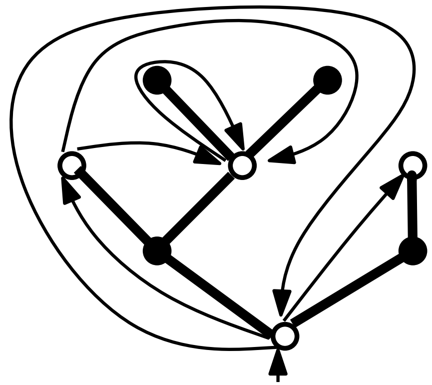
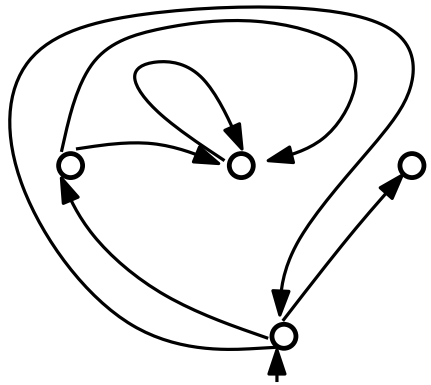








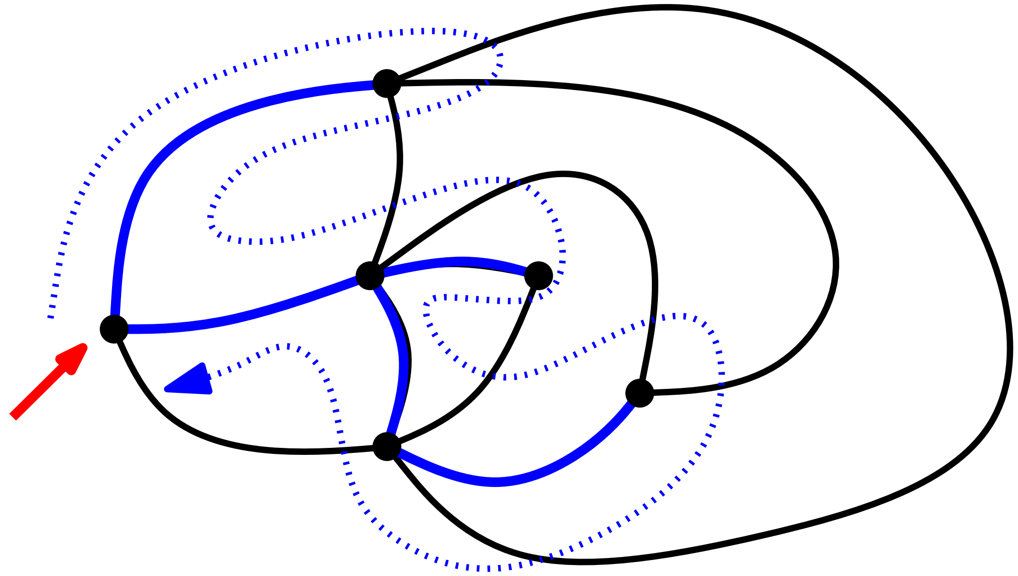
From maps to trees (II): eulerian maps

first strategy: blossoming trees

Encoding rooted maps with trees

Let us recycle the first idea used for tree-rooted maps

using a canonical spanning tree



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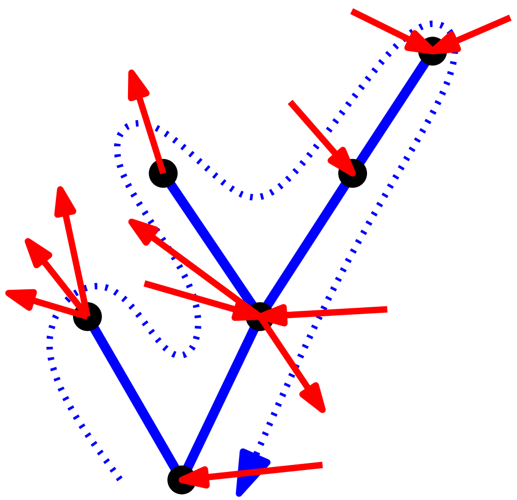

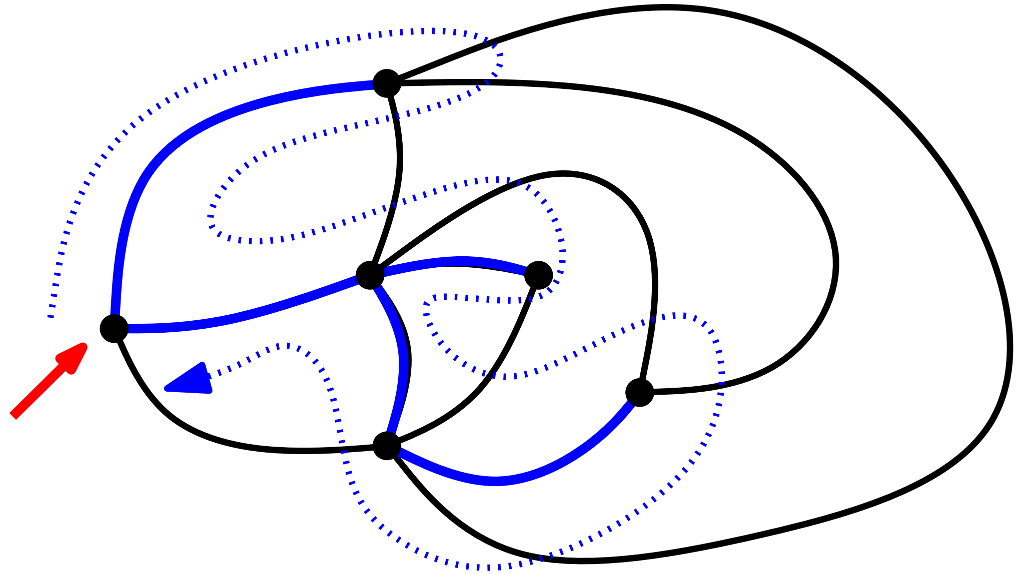

Then write the code of the primal tree on the chosen canonical tree

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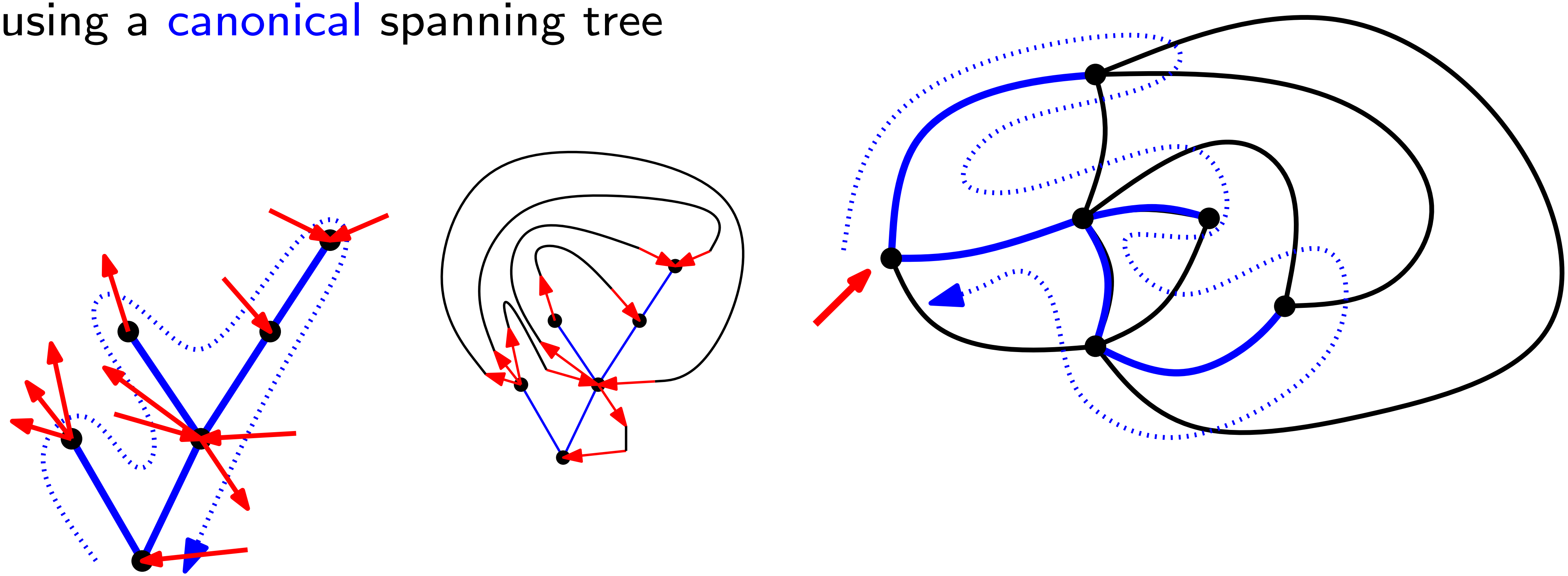

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The map is recovered from the code by *closure*.

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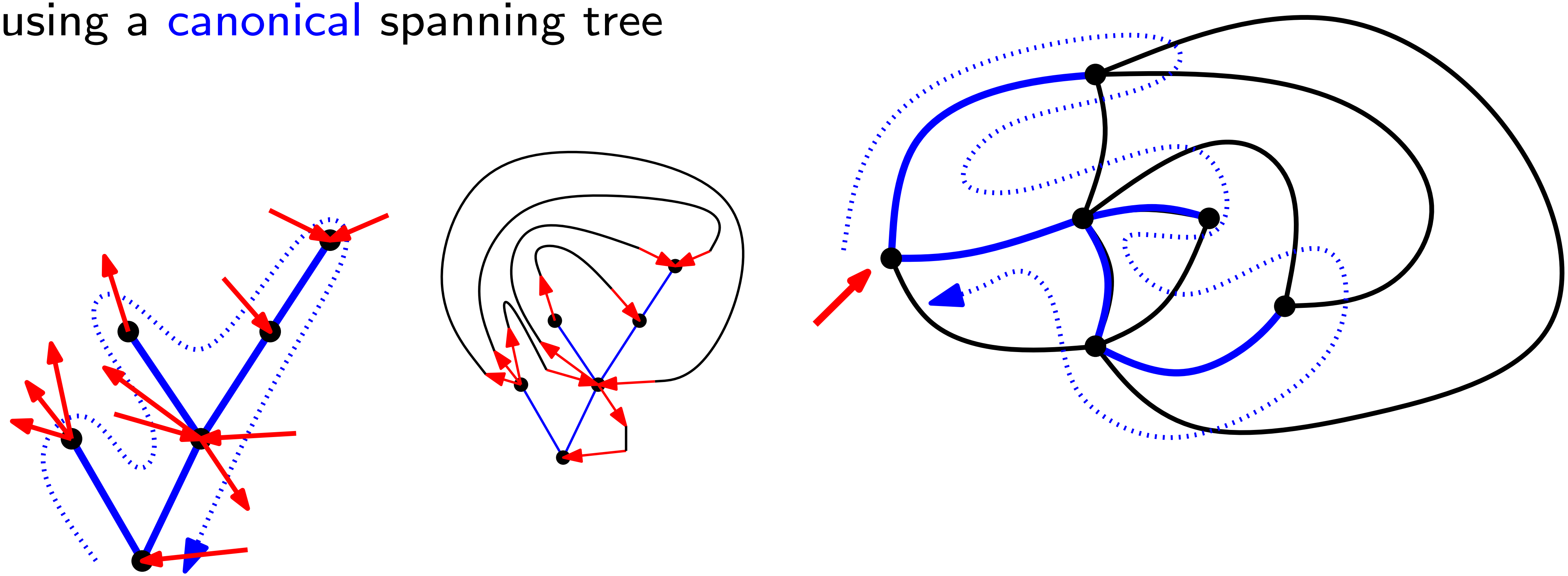

Then write the code of the primal tree on the chosen canonical tree

The map is recovered from the code by *closure*.

Our code of the map will be a canonical decorated tree

Question is "How do we choose the canonical spanning tree ?"

Minimal orientations and canonical spanning trees

Idea: Use Bernardi's first bijection the other way round:

Choose a minimal accessible orientation to get a spanning tree

Our pb becomes:

How to choose a canonical accessible minimal orientation?

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How to choose a canonical accessible minimal orientation?

A function $\alpha : V \rightarrow \mathbb{N}$ is feasible on a plane map M if there exists an orientation of M such that for each vertex v the outdegree of v is $f(v)$.

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Theorem (Felsner 2004). Let α be a feasible function on a plane map M. Then α has a unique α-orientation without clockwise cycles.

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Fact: For many subclasses $\mathcal{F}$ of planar maps, there exists an $\alpha_{\mathcal{F}}$ s.t.:

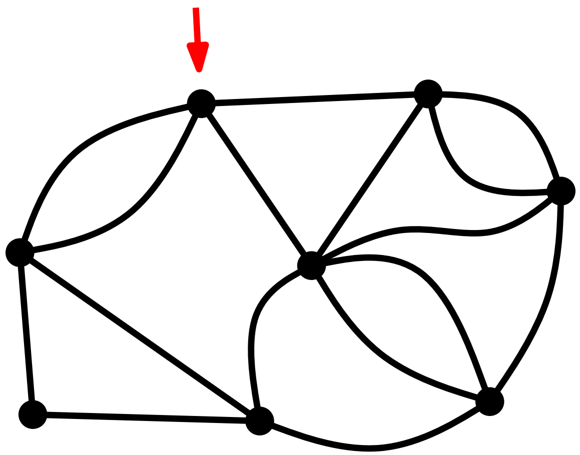
A planar map is in $\mathcal{F}$ if and only if it admits an $\alpha_{\mathcal{F}}$-orientation.

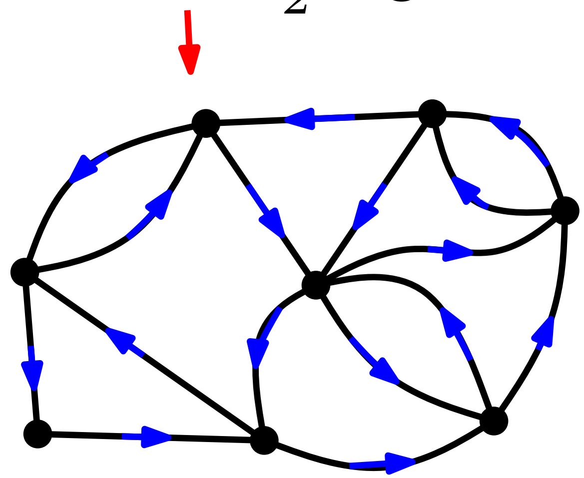
The example of eulerian maps

A map is eulerian if it admits a cycle that visits every edge exactly once.

Let $\frac{1}{2}$deg denote the $\frac{1}{2}$degree function.

Proposition. A map is eulerian if and only if its admits a $\frac{1}{2}$deg-orientation.

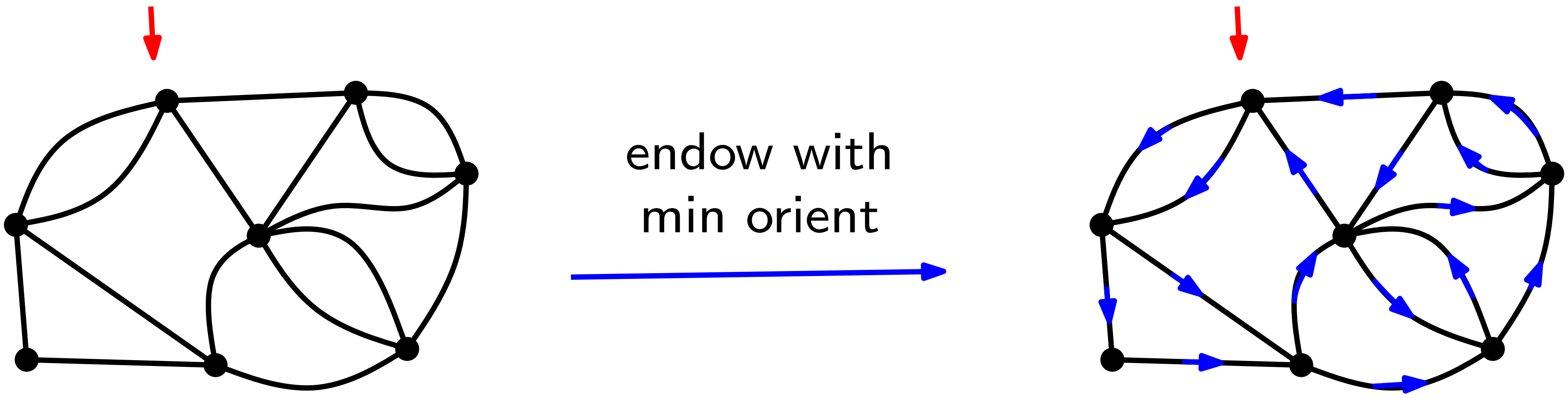




a map is 2-connected $\Leftrightarrow$ it admits a bipolar orientation
$\Leftrightarrow$ its quadrangulation admits
an orientation with $\alpha(v) = 2$

a map is a simple triangulation $\Leftrightarrow$ it admits an orientation with $\alpha(v) = 3$

The example of eulerian maps



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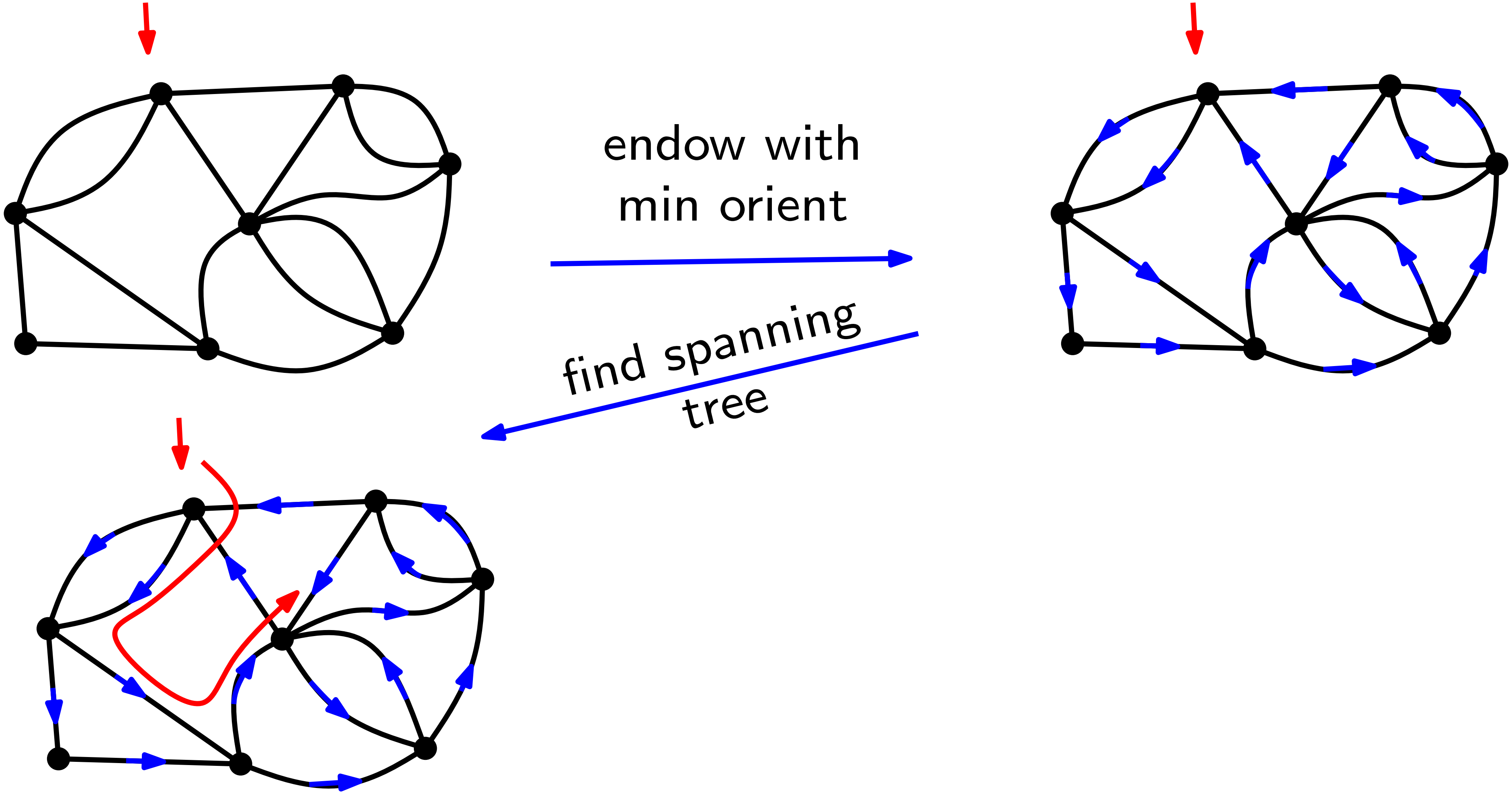

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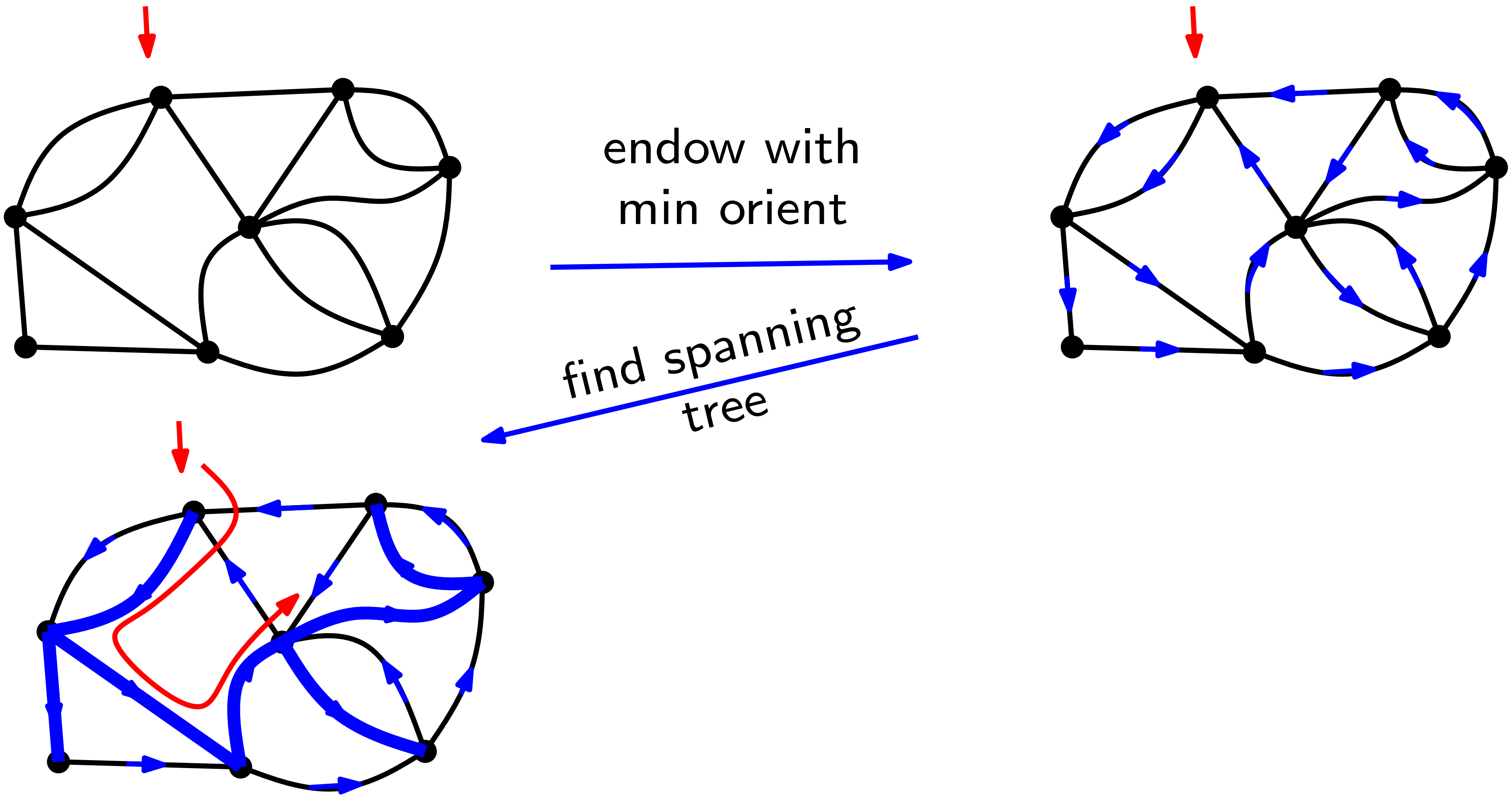

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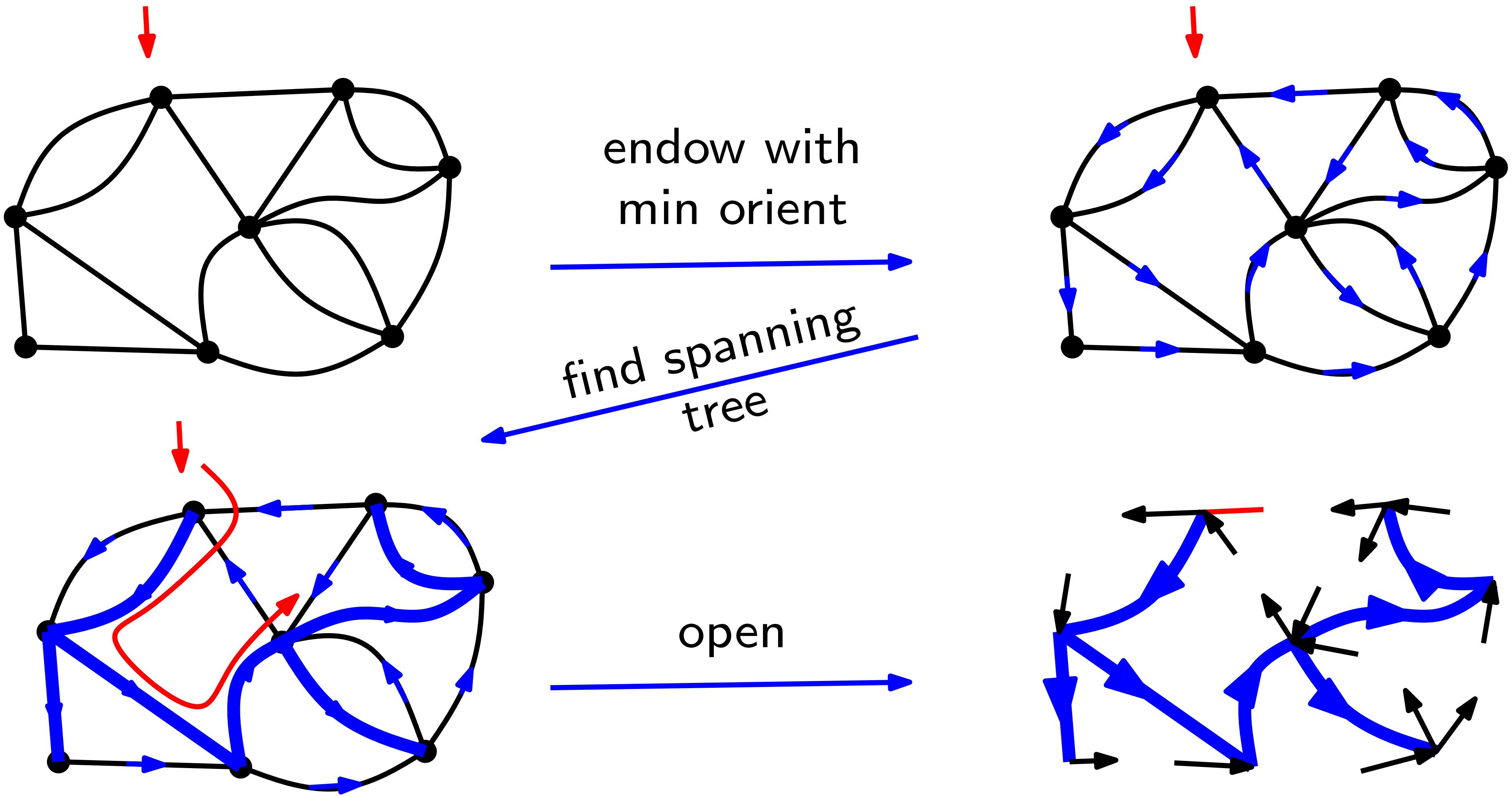

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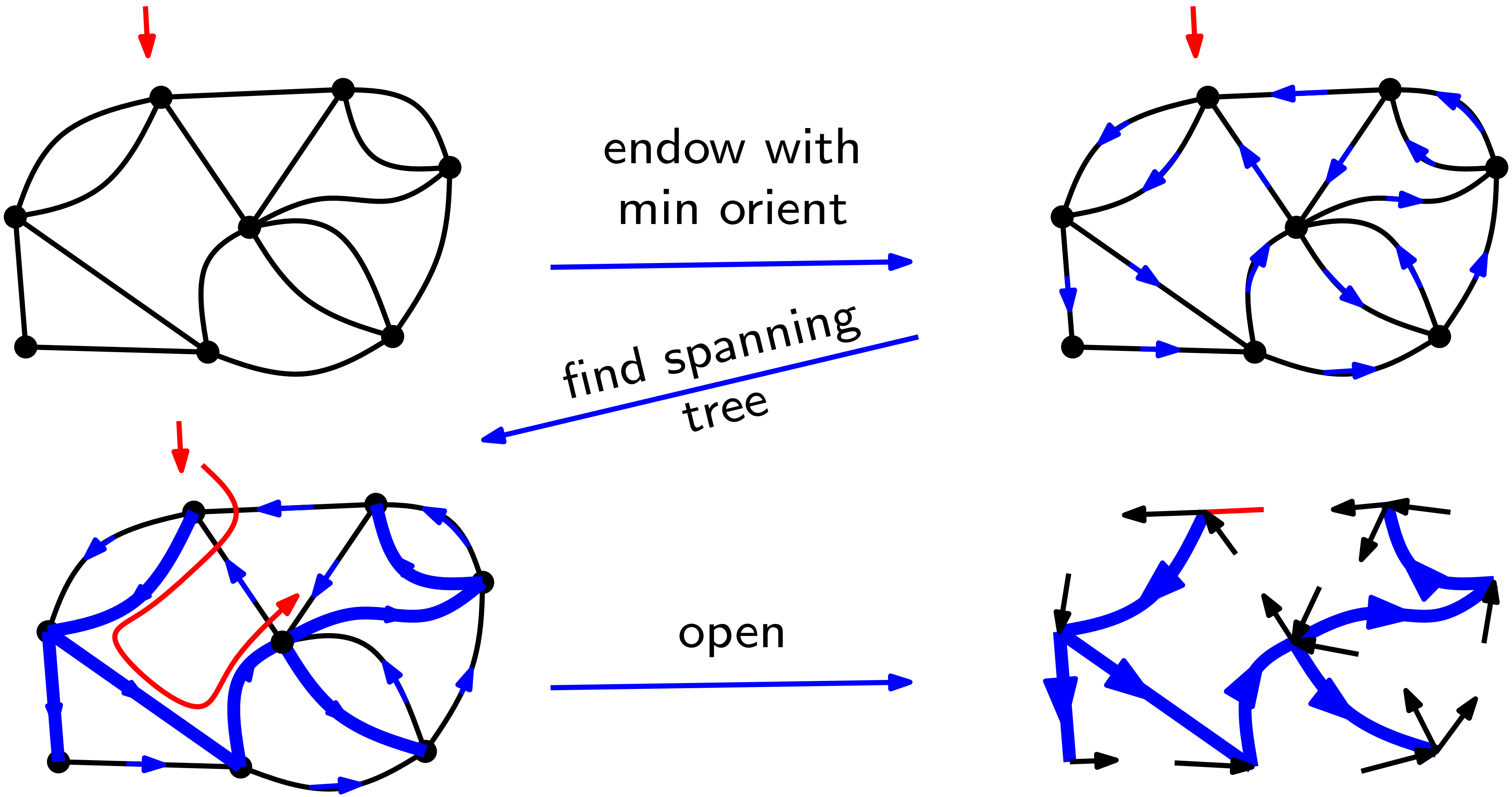

Corrolary. This is a bijection between eulerian map with d_i vertices of degree i and rooted* plane trees with d_i vertices of total degree $2i$ s.t.

- a vertex of total degree $2i$ has $i-1$ incoming half-edges

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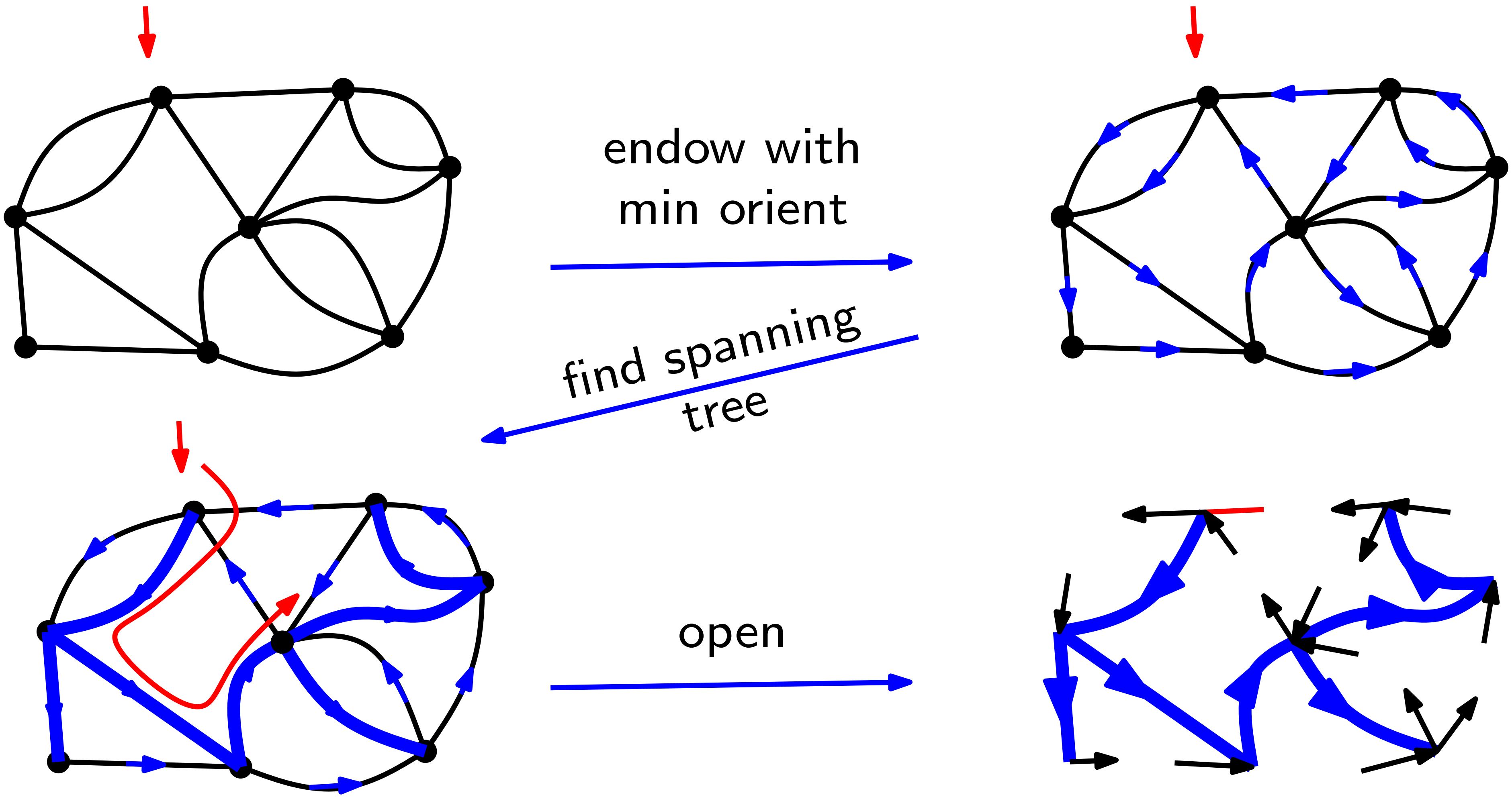

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- the tree is balanced (half-edges must form balanced parentheses)

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- the tree has 1 incoming half edge per vertex

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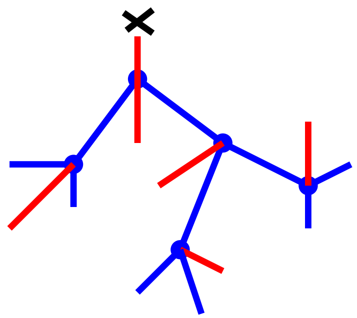

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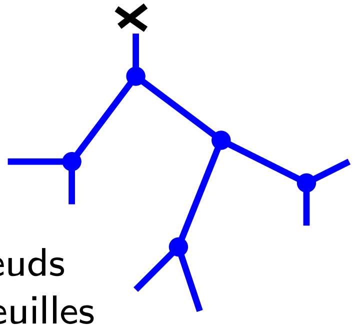
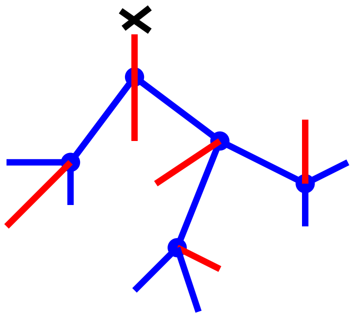


n nœuds
$n+2$ feuilles

$$\frac{1}{n+1}\binom{2n}{n}$$



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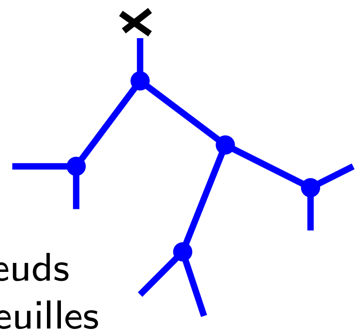

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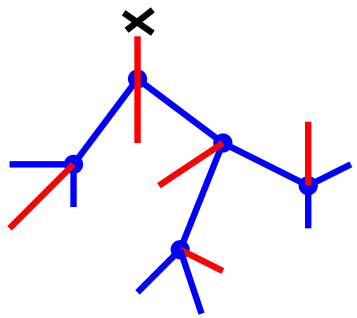

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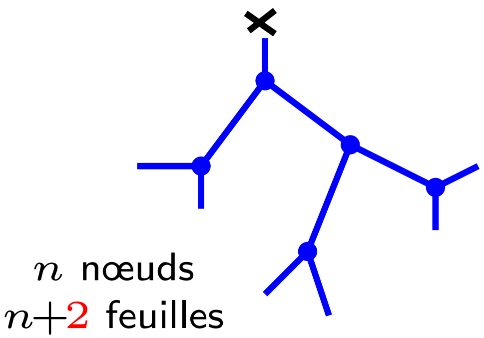

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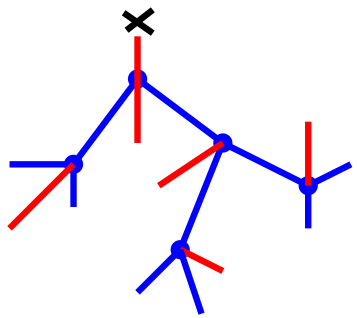

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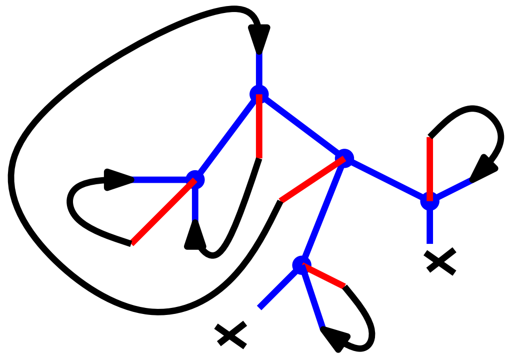

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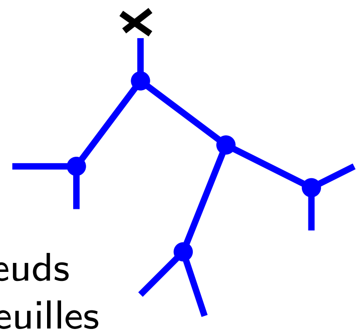
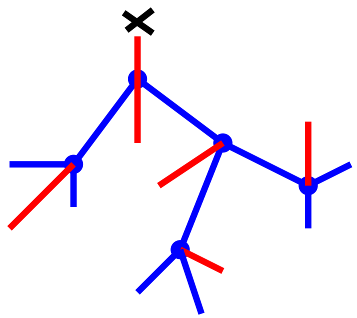


n nœuds
$n+2$ feuilles

$$\frac{1}{n+1}\binom{2n}{n}$$



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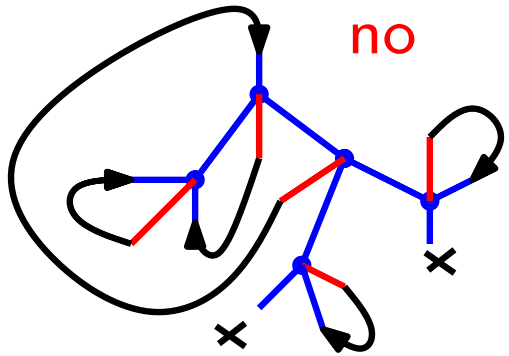

The example of eulerian maps

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Example. 4-regular maps: all vertices have degree 4

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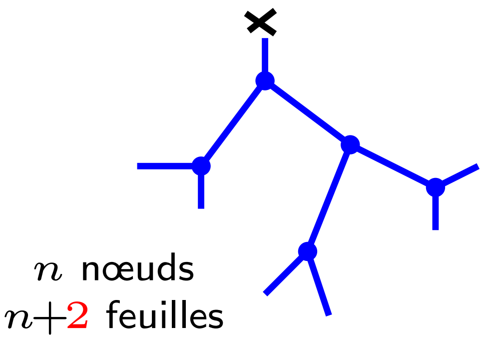

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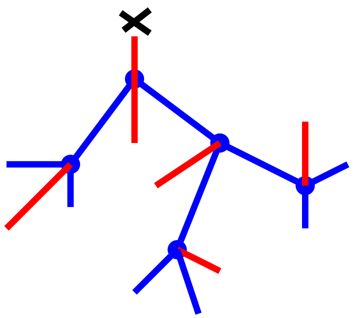

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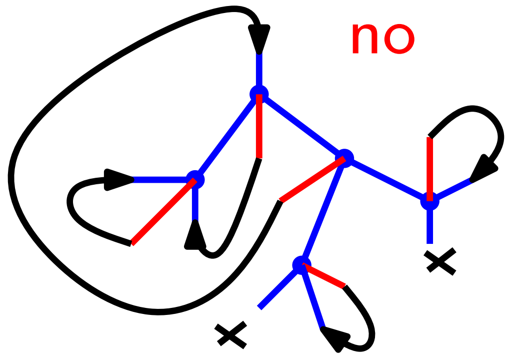

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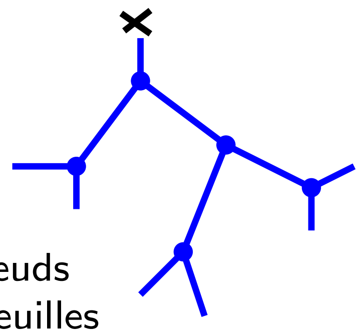

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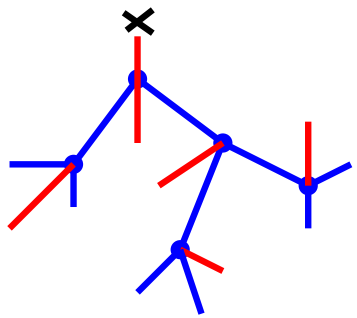

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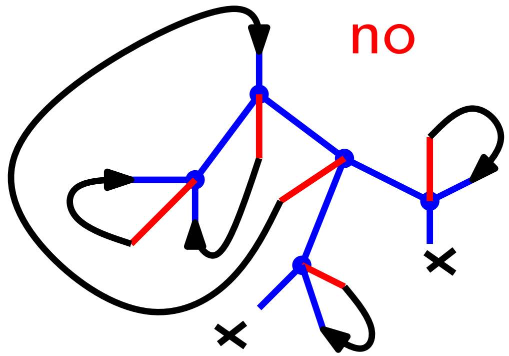

However, 2 among $n+2$ are balanced: $\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$

(Tutte 1964, S. 1997)

Summary of the blossoming tree strategy

To enumerate maps admitting $\alpha_{\mathcal{F}}$-orientations:

– endow them with their minimal $\alpha_{\mathcal{F}}$-orientation (hope it is accessible)

– construct the associated canonical spanning trees (Bernardi)

– open the resulting tree-rooted maps (Mullin)

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In Bernardi original bijection, the basepoint must be in the outer face. But in some cases the orientation is not outerface accessible.

This approach was further extended in (Albenque, Poulalhon 2012) to cover essentially all known blossoming bijections, including Bernardi-Fusy's fractional orientations.

Third lecture

Applications to factorization problems

Which factorisations, which maps?

m-eulerian maps

Hurwitz problem

Conclusion

What do we want to enumerate?

Recall. There is a bijection between

- Labelled ramified covering of $\mathbb{S}$ of type $\Lambda = (\lambda_0, \ldots, \lambda_m)$
- Factorizations $(\sigma_1 \cdots \sigma_m = \sigma_0)$ of type Λ
- labelled m-star-constellations of type Λ.

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m arbitray factors

$\Rightarrow \sum_{i=1}^{m} (n - \ell_i) = 2n - 2$

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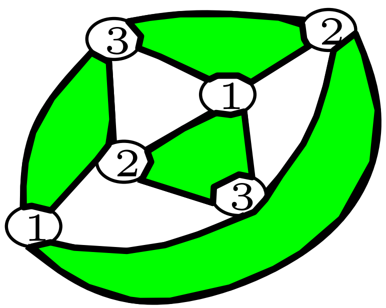
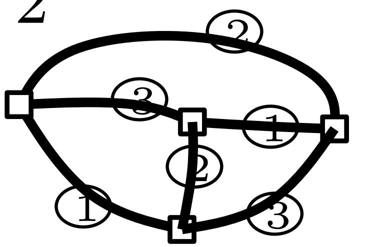


transpositions

$\Rightarrow m = 2\ell - 2$

increasing maps



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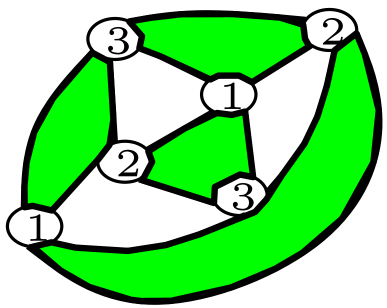

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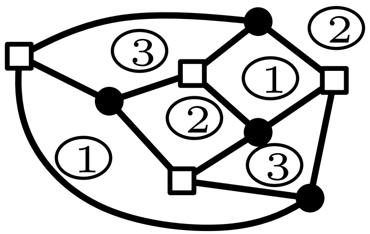
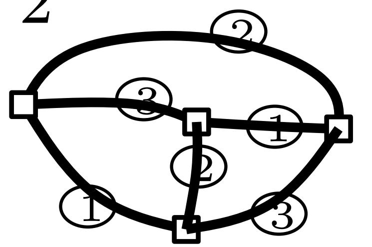


$m = 3$

transpositions

$\Rightarrow m = 2\ell - 2$

increasing maps



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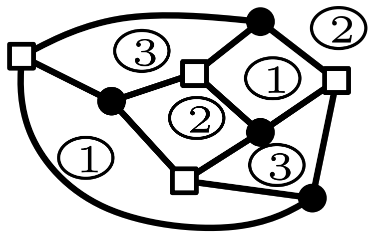

From maps to trees: constellations

α-orientations for m-eulerian maps

Bipartite map with black and white vertices of degree m such that:

– faces with labels in $\{1, \ldots, m\}$

– around black vertices, face labels read $1, \ldots, m$ in cw order

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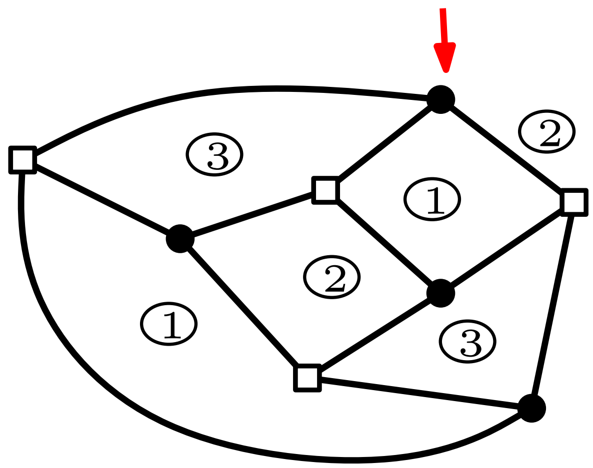

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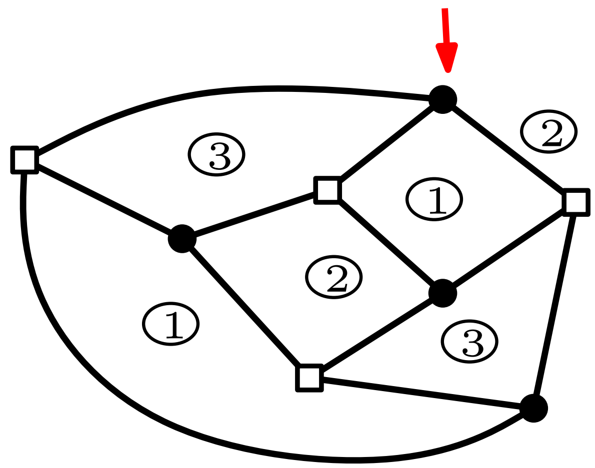

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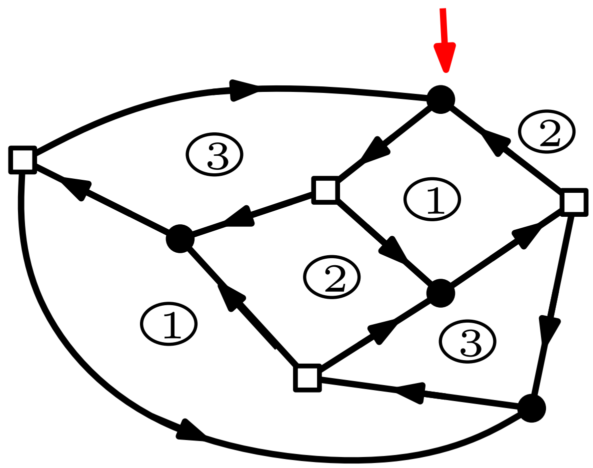

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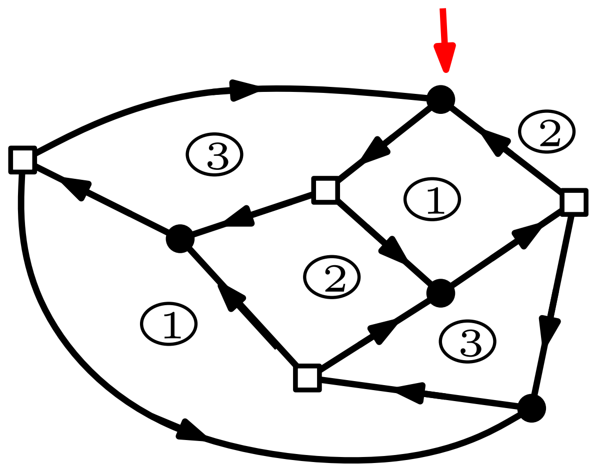

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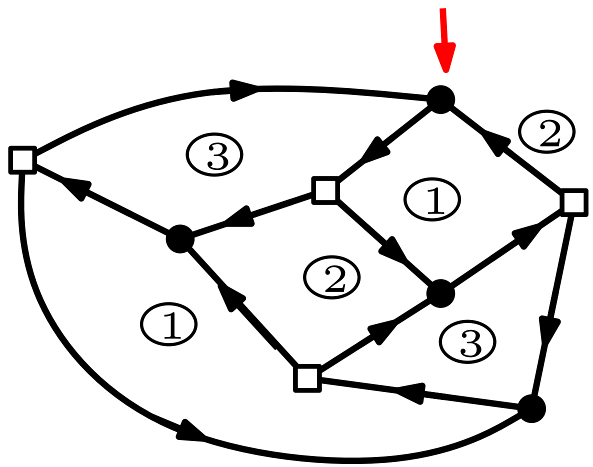

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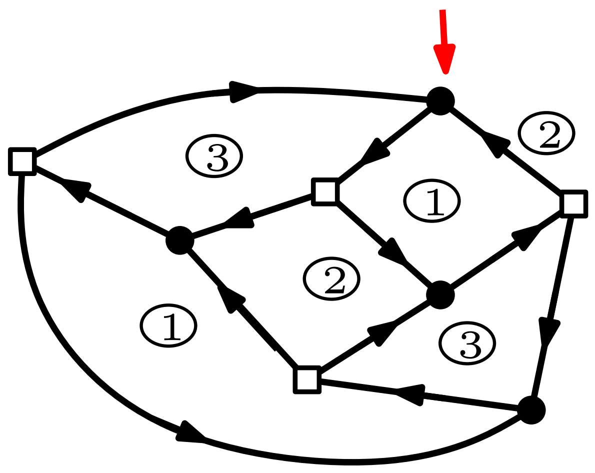
This orientation is accessible, in fact strongly connected.

We can apply our strategy!

Openning a m-eulerian map



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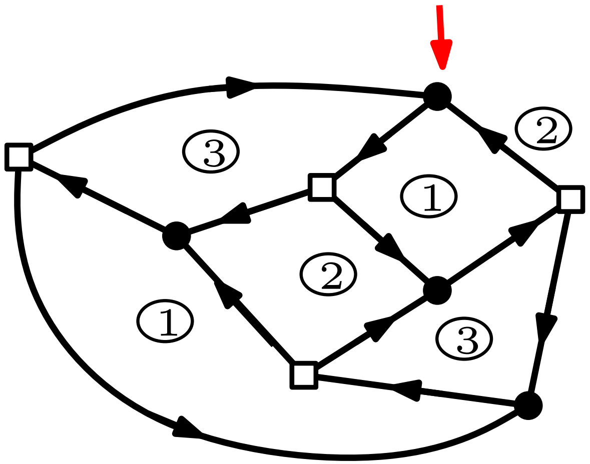

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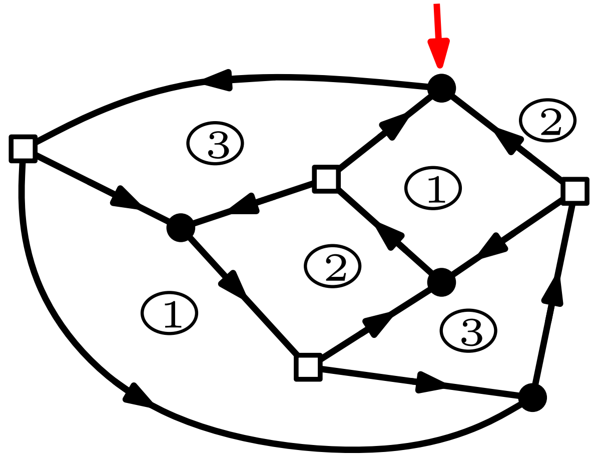

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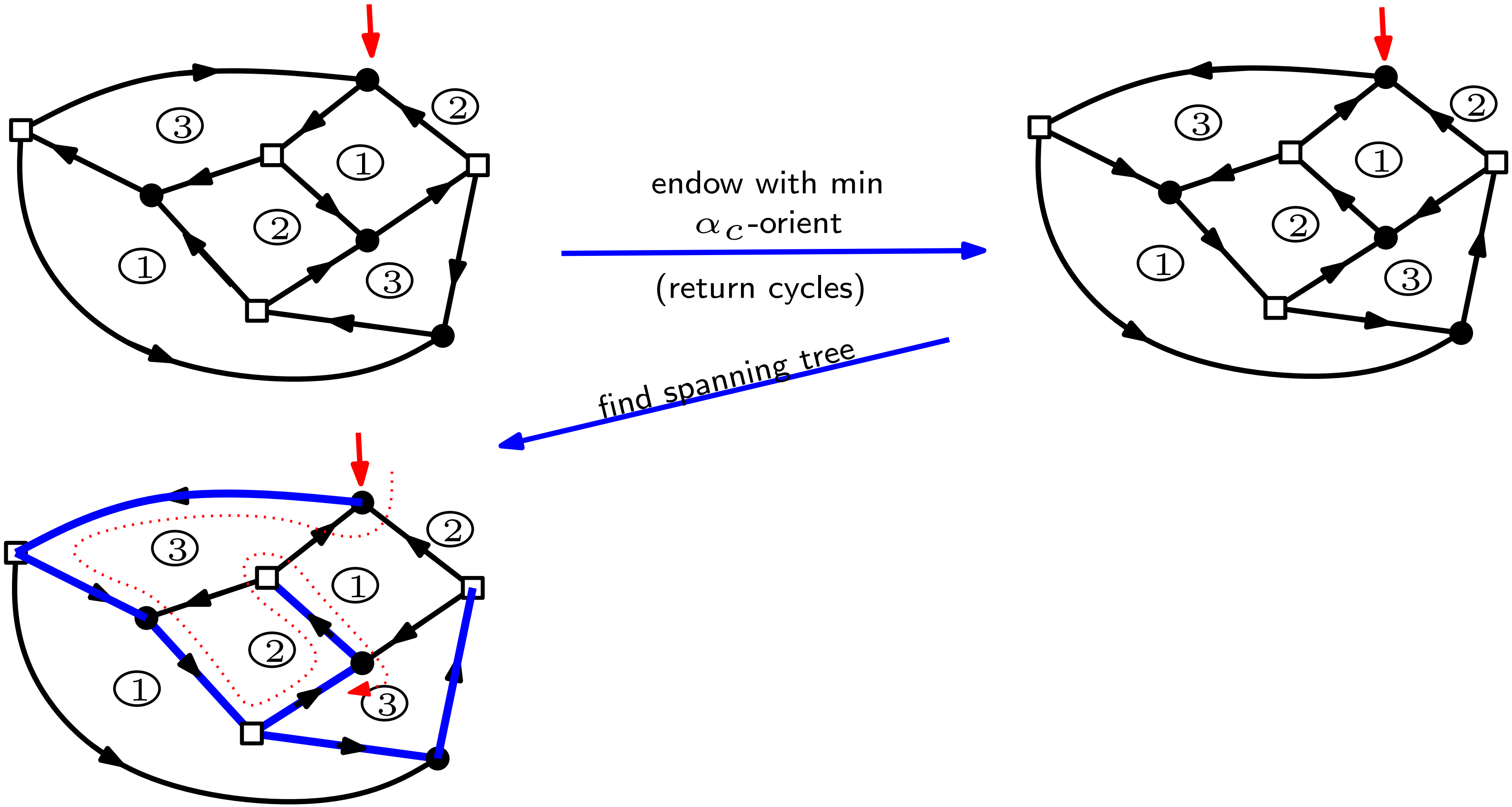

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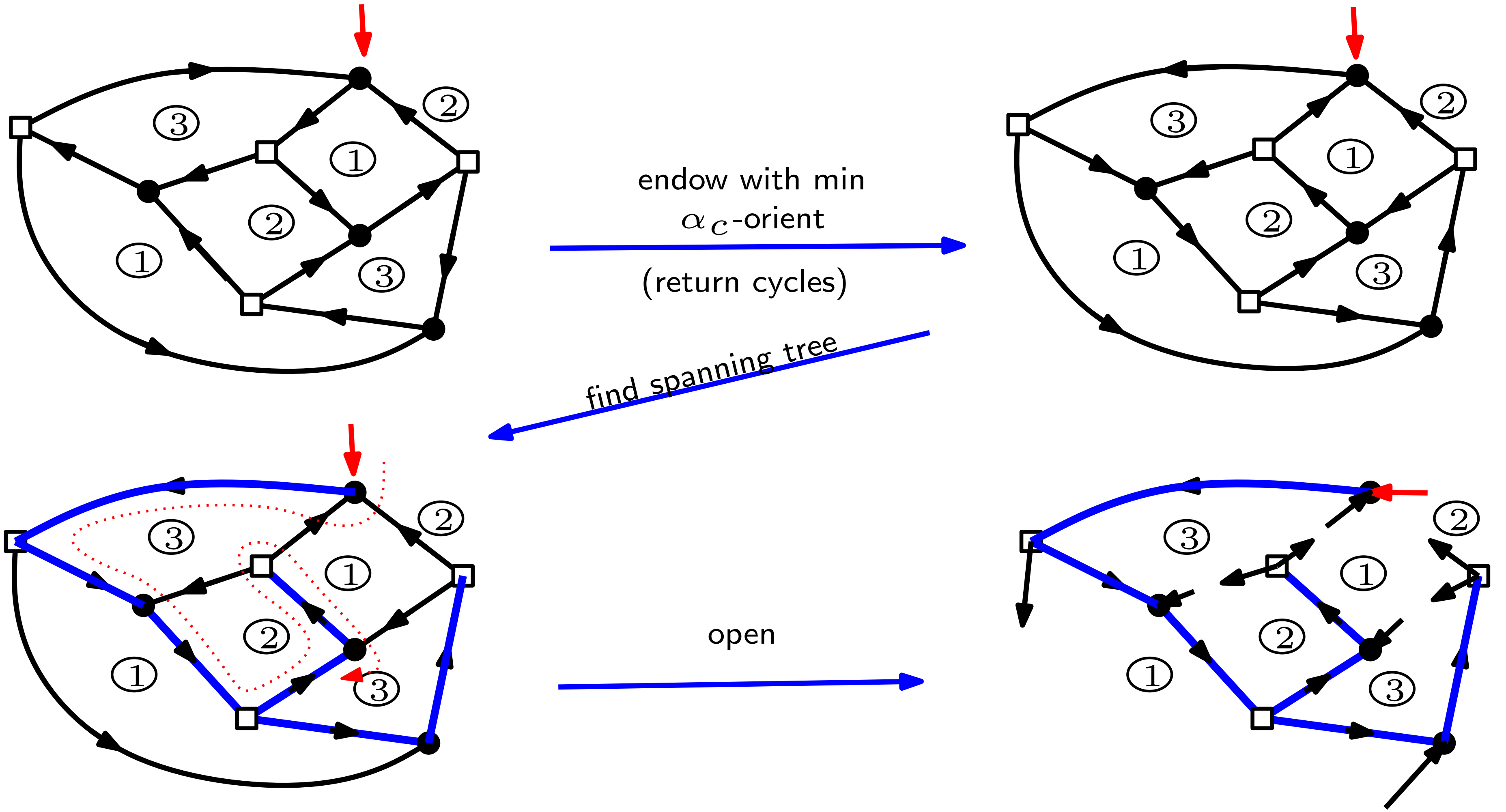

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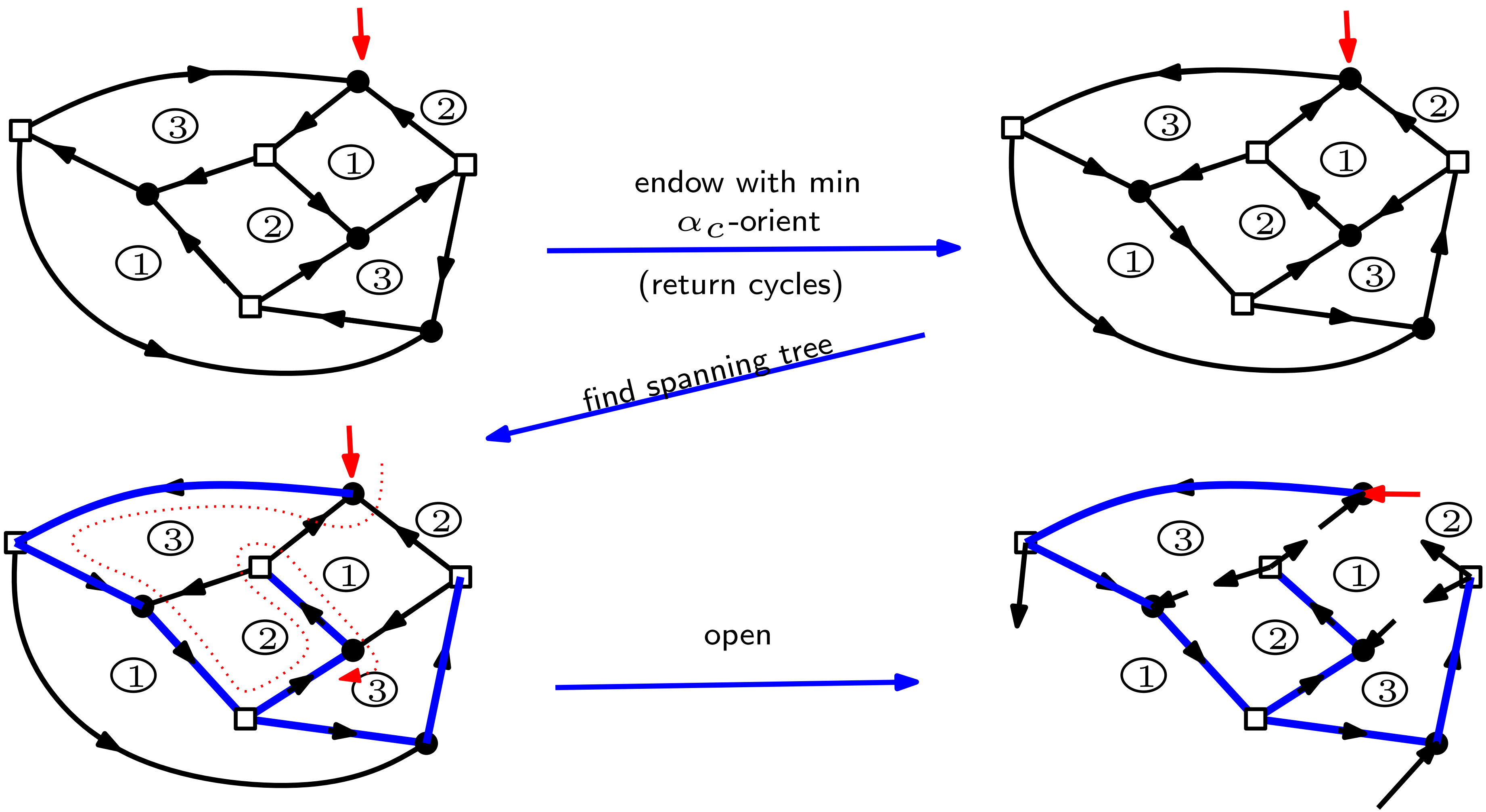

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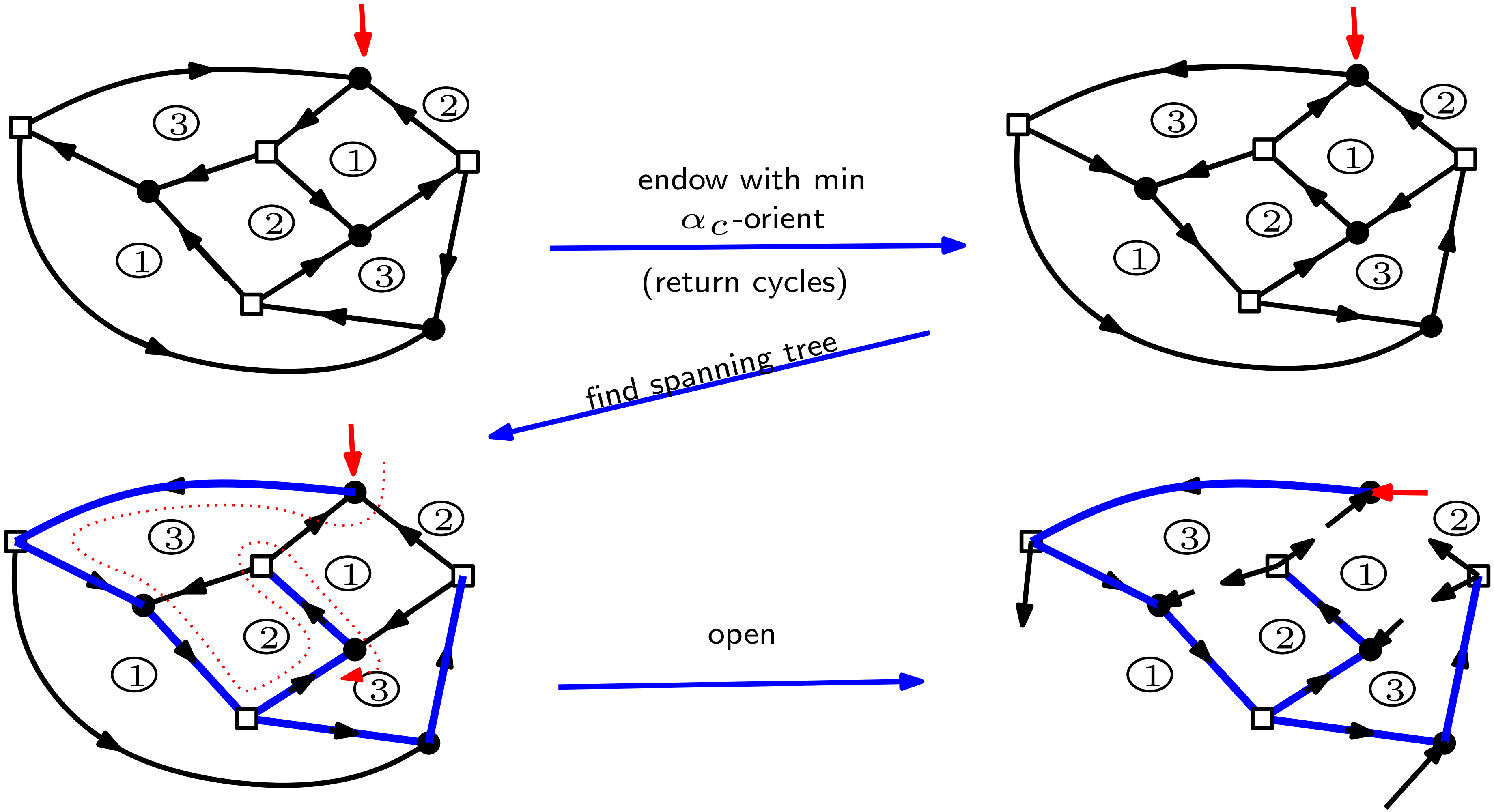

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- half-edges are incoming at black, outgoing at white, the tree is balanced

The enumeration of constellations

Theorem:[Bousquet-Mélou–S. 2000] m-eulerian maps are in bijection* with trees such that:

– white vertices carry $m-1$ sibblings
(black vertices or half-edges)

– black vertices carry $m-2$ half-edges
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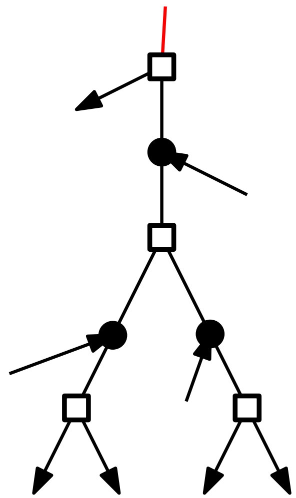

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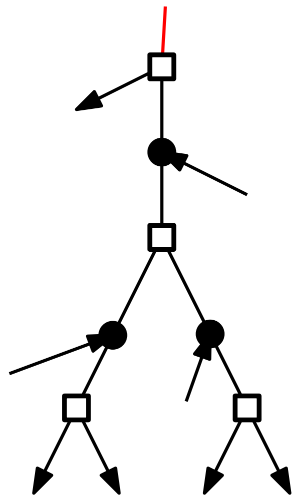

Counting the trees: this is a familly of simple tree

$$A_{-\square}(t) = (1 + A_{-\bullet}(t))^{m-1}, \quad A_{-\bullet}(t) = (m-1) \cdot A_{-\square}(t)$$

or observe directly that they are $(m-1)$-ary trees with $(m-1)$ types of edges

$$\Rightarrow \qquad \frac{1}{(m-2)n+1}\binom{(m-1)n}{n} \cdot (m-1)^{n-1}$$

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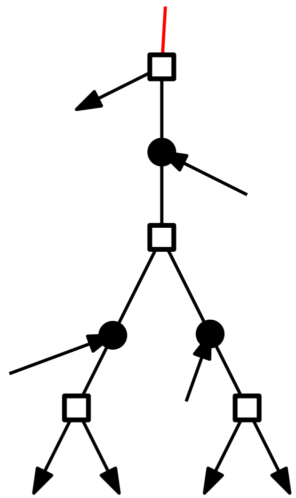

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$$\Rightarrow \frac{m}{(m-2)n+2} \frac{1}{(m-2)n+1} \binom{(m-1)n}{n} \cdot (m-1)^{n-1}$$

A formula for general factorizations [BMS00]

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition of n, and $\ell = \sum_i \ell_i$.
The number of m-uple of permutations $(\sigma_1, \ldots, \sigma_m)$ such that

- (factorization) $\sigma_1 \cdots \sigma_m = \sigma$ with cycle type λ
- (transitivity) $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$
- (minimality) the total rank of factors is $\sum_i r(\sigma_i) = n + \ell - 2$

is

$$m \frac{((m-1)n-1)!}{(mn-(n+\ell-2))!} \cdot n! \cdot \prod_i \frac{1}{\ell_i!} \binom{mi-1}{i}^{\ell_i}$$

Proofs:

(Bousquet-Mélou–Schaeffer 2000) (Goulden–?? 2009)

(bijection + inclusion/exclusion)(gfs and differential eqns)

$\lambda = n$, factorizations of n-cycles: $\frac{1}{(mn+1)} \binom{mn+1}{n} \cdot (n-1)!$

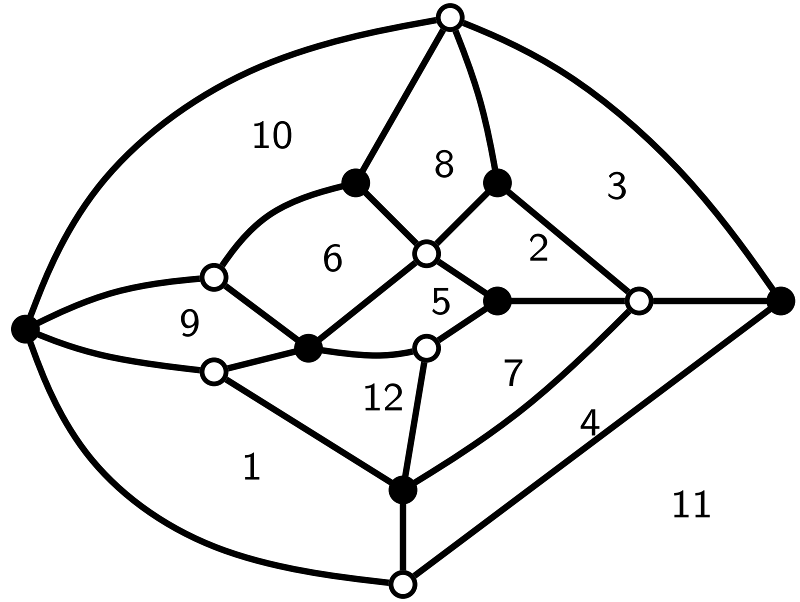
$\lambda = 1^n$, identity factorizations: $\frac{m}{(m-2)n+2} \frac{(m-1)^{n-1}}{(m-2)n+1} \binom{(m-1)n}{n} \cdot (n-1)!$

From maps to trees: Hurwitz formula

α-orientations for increasing quadrangulations

Planar quadrangulations (faces are 4-gons) such that:

- faces have labels in $\{1, \ldots, 2n-2\}$
- around labeled vertices, face labels increase in ccw order
- around white vertices, face labels increase in cw order



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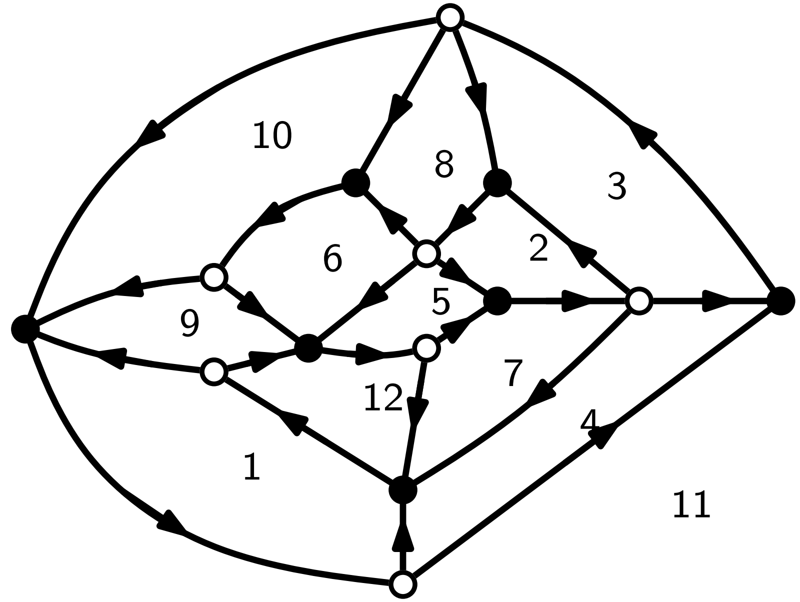

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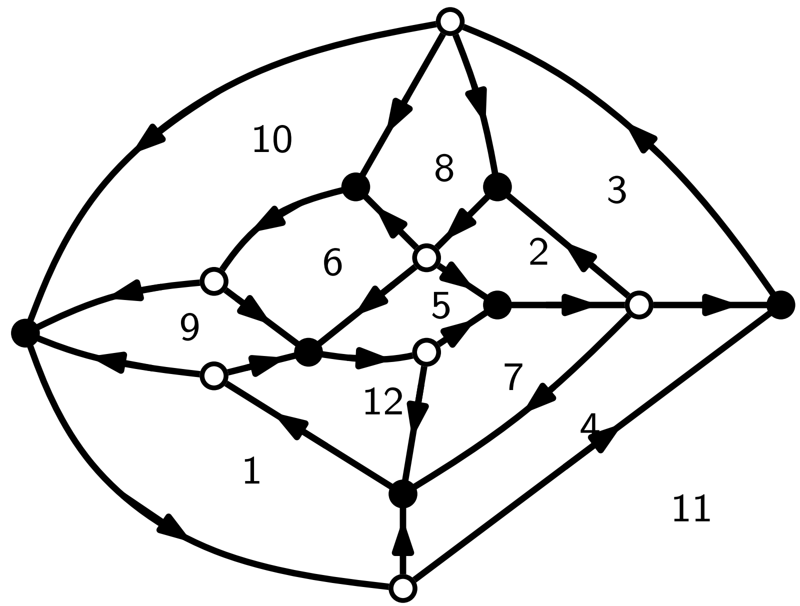

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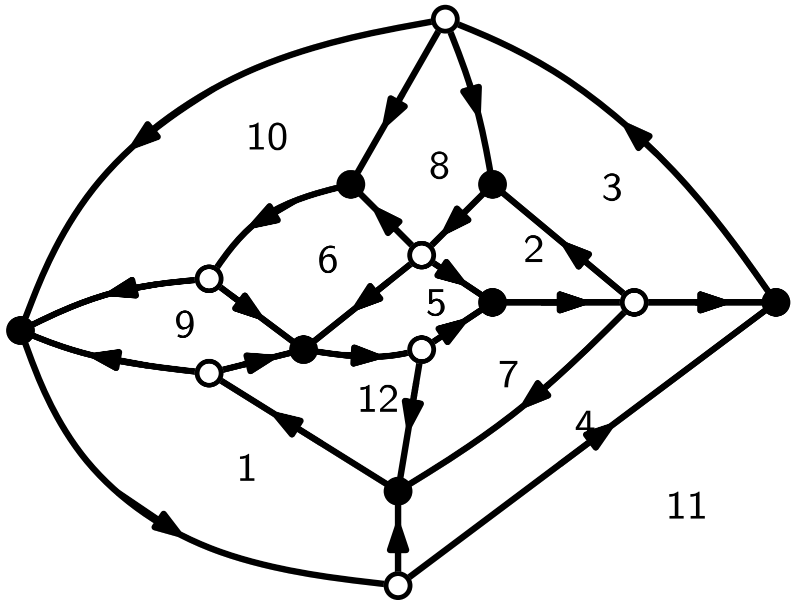
opening of an increasing quadrangulation



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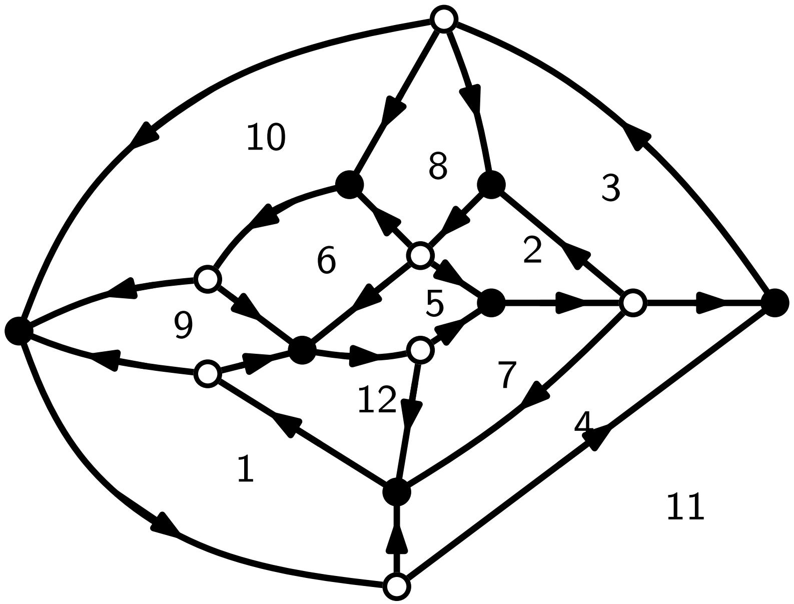

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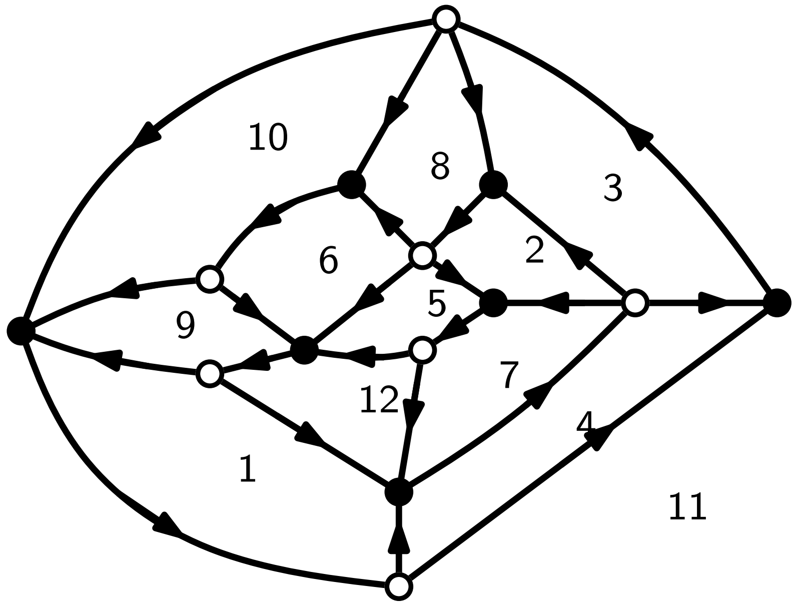

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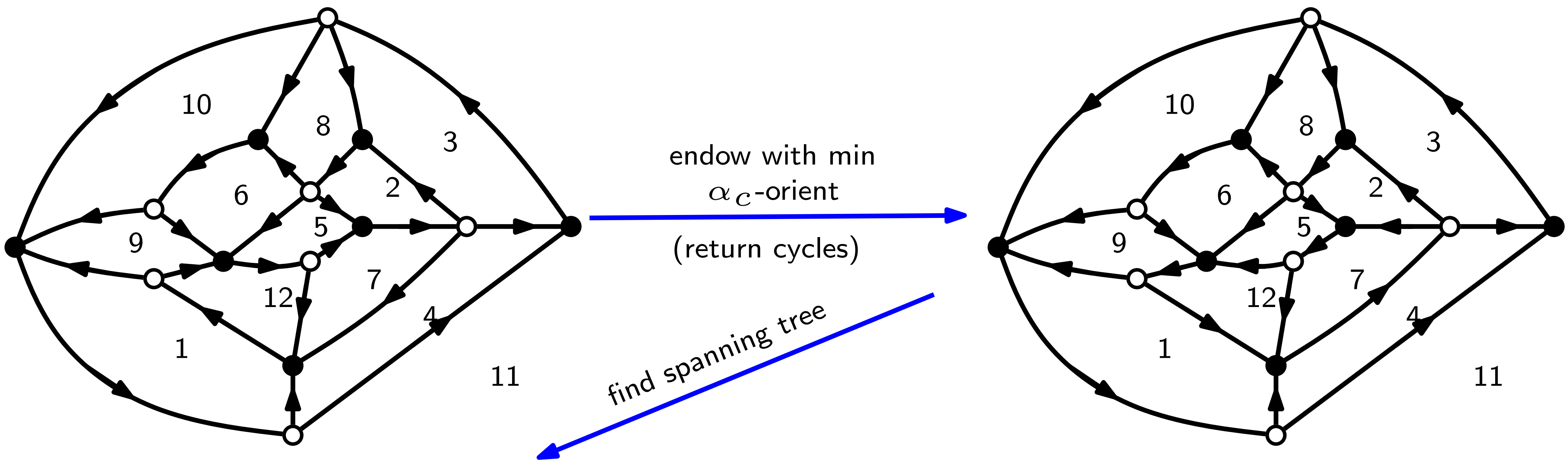

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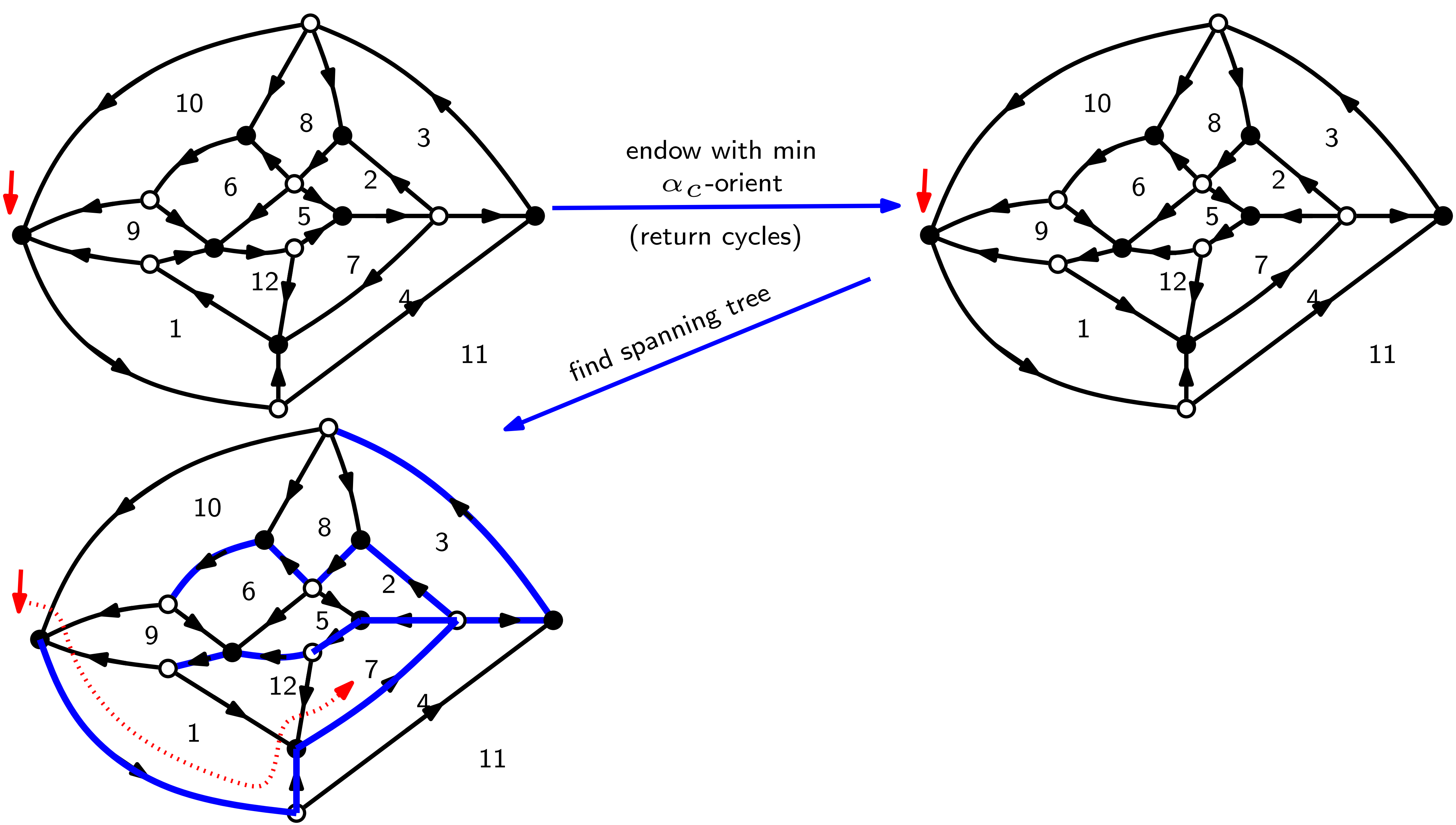

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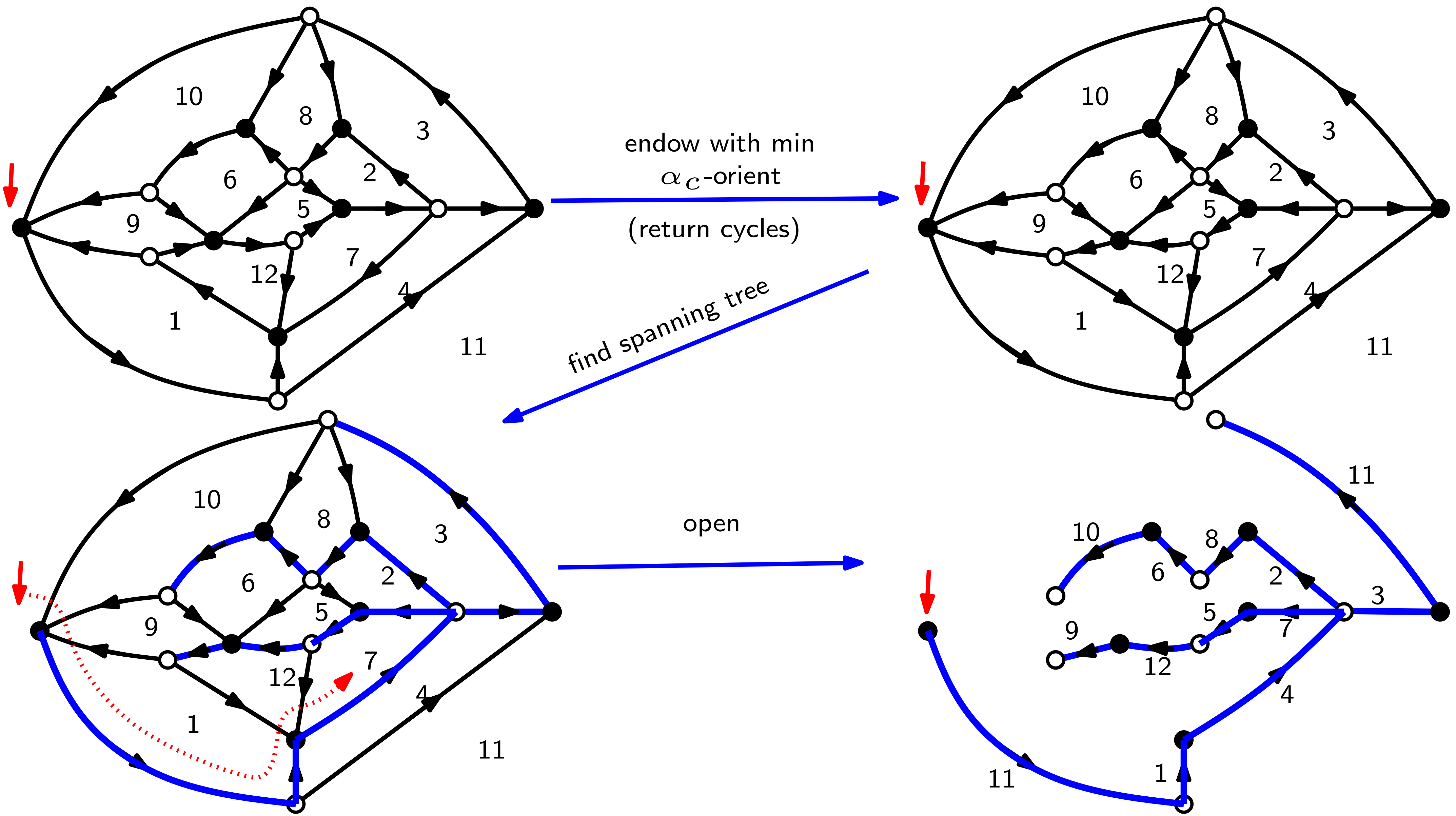

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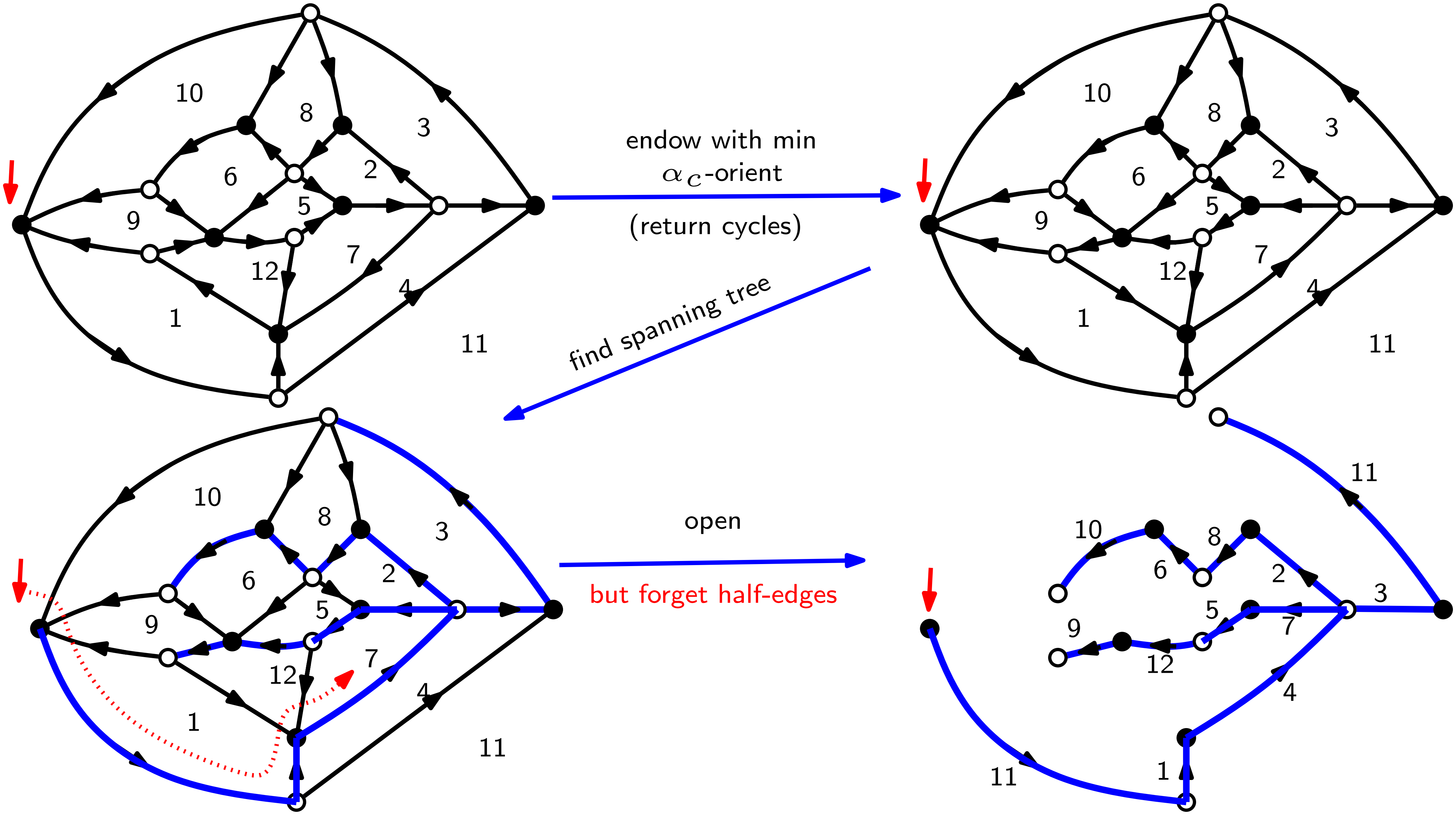

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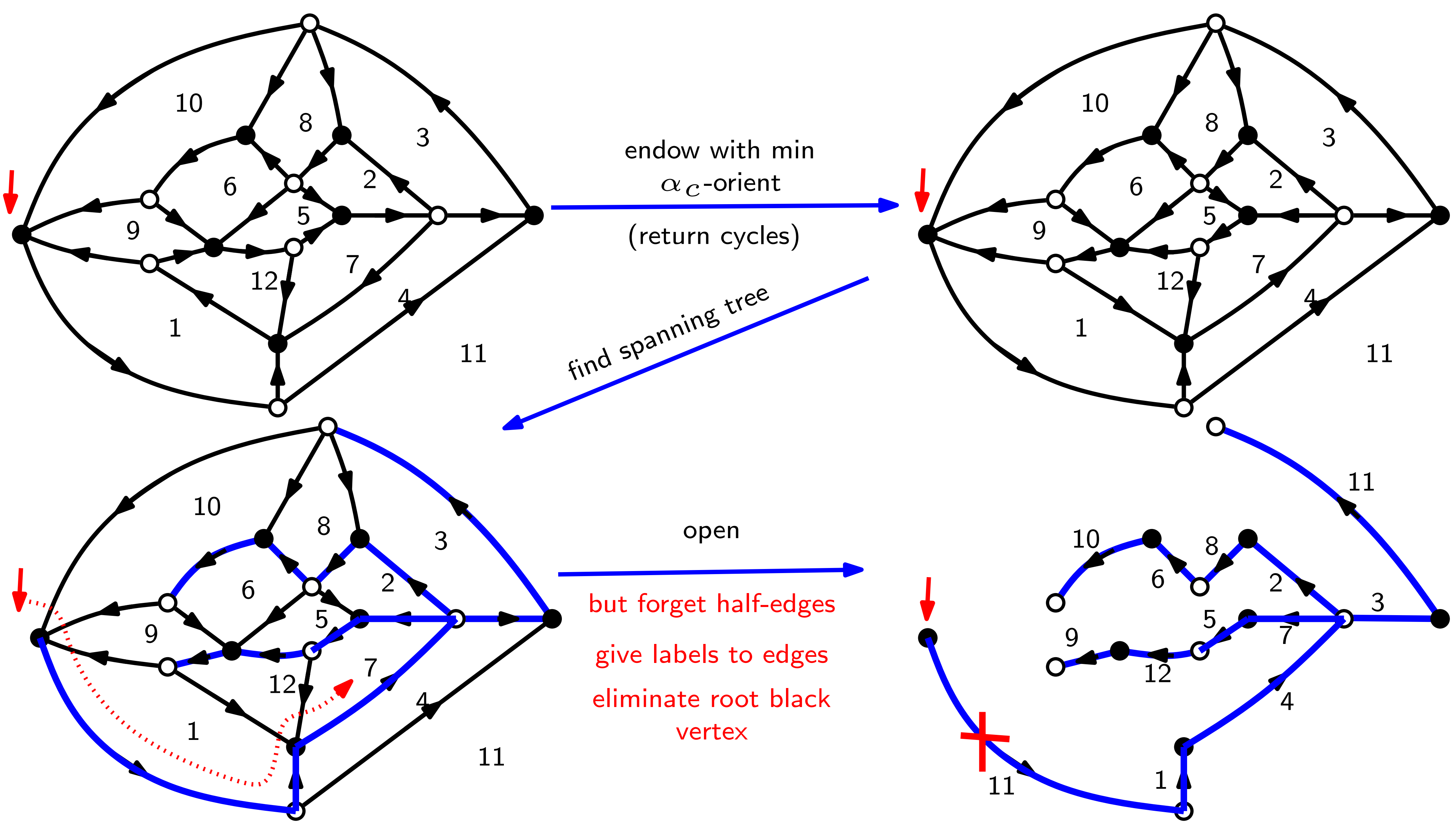
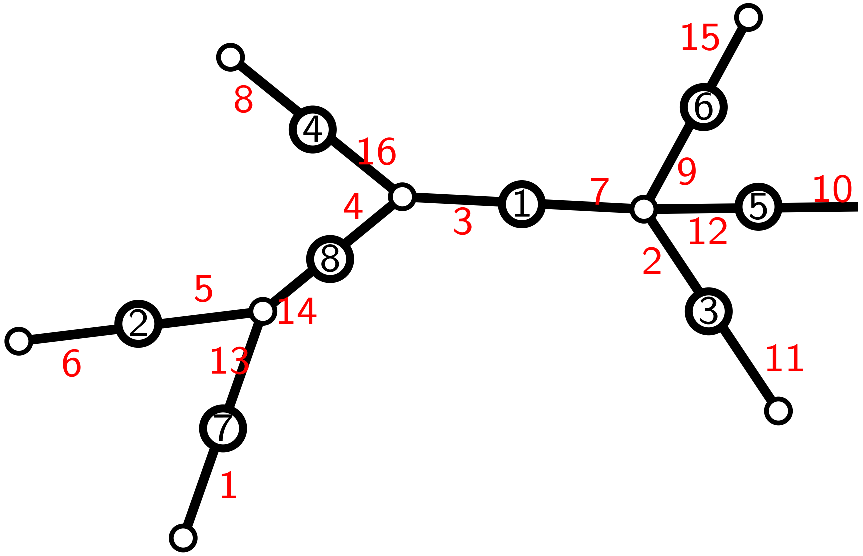


Hurwitz formula for increasing quadrangulations

Theorem[Duchi-Poulalhon-S. 2012] Increasing quadrangulations (size n) are in bijection with simple Hurwitz trees having n unlabelled vertices, $n-1$ labeled vertices of degree 2, $2n-2$ edges that increase ccw around labeled vertices.



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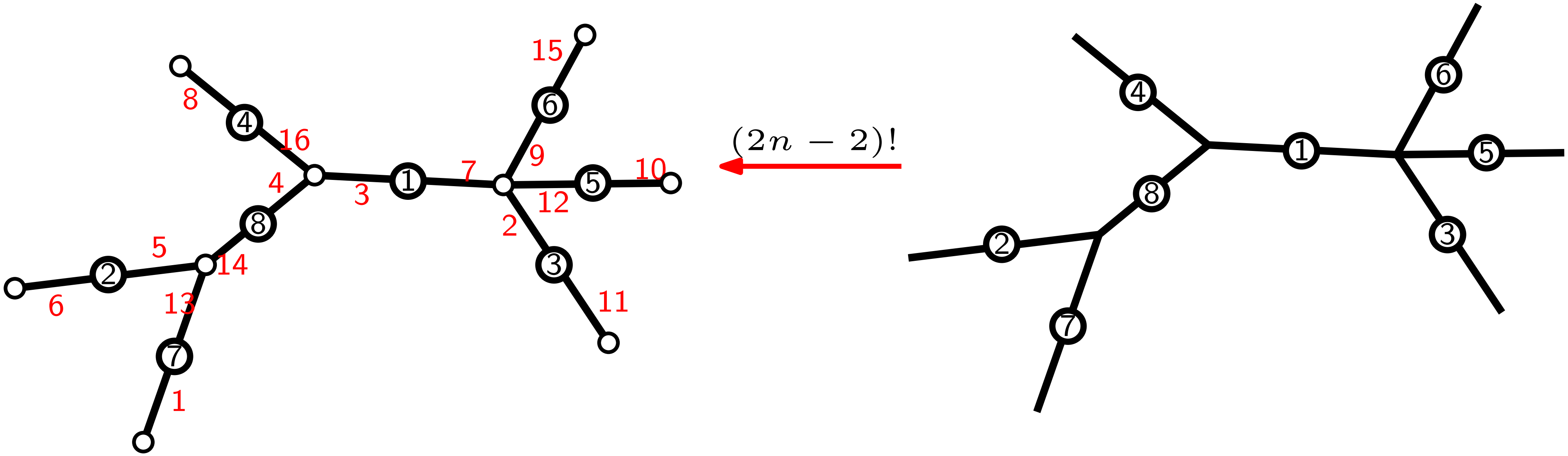

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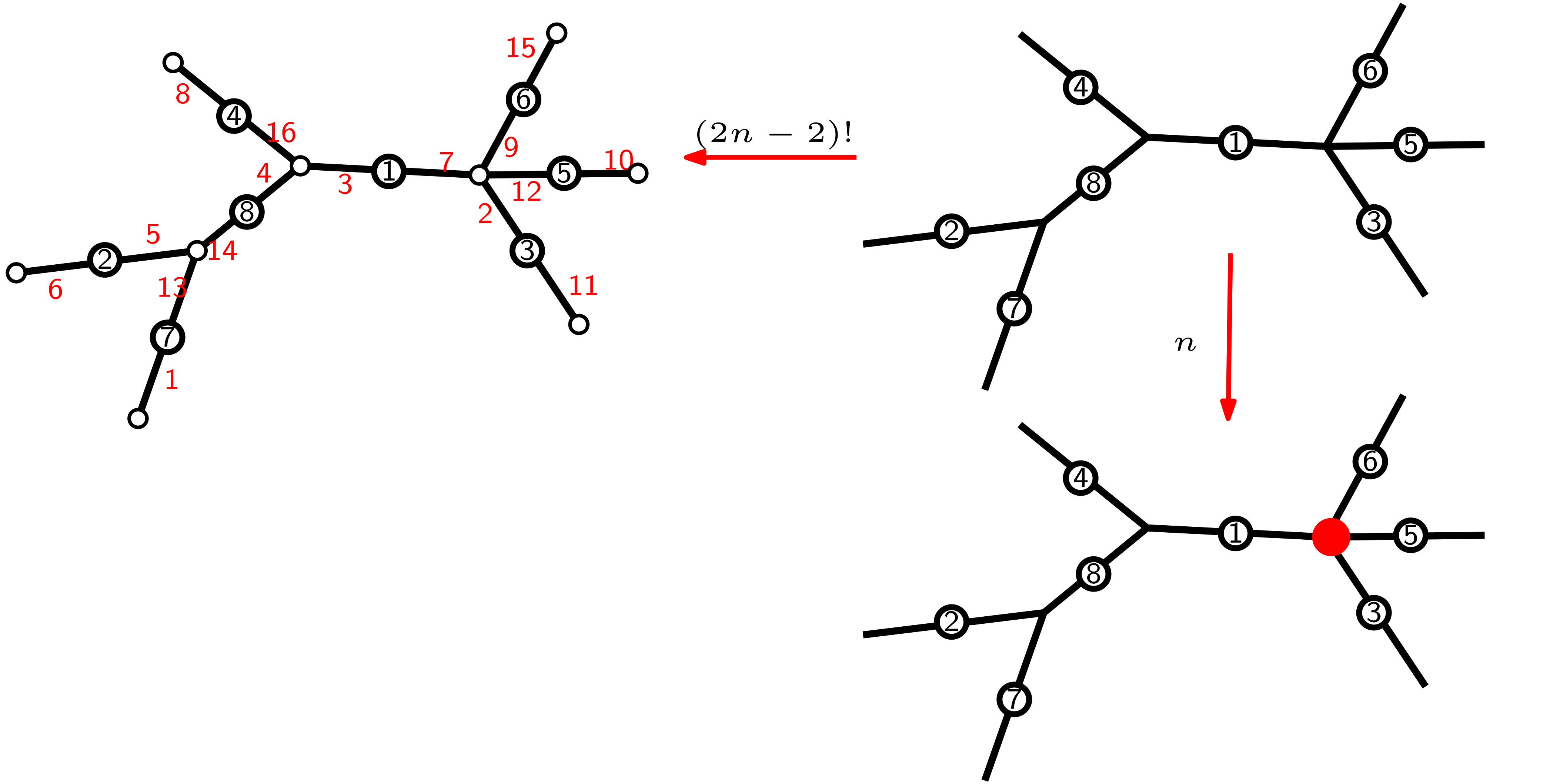
Theorem[Duchi-Poulalhon-S. 2012] Increasing quadrangulations (size n) are in bijection with simple Hurwitz trees having n unlabelled vertices, $n-1$ labeled vertices of degree 2, $2n-2$ edges that increase ccw around labeled vertices.



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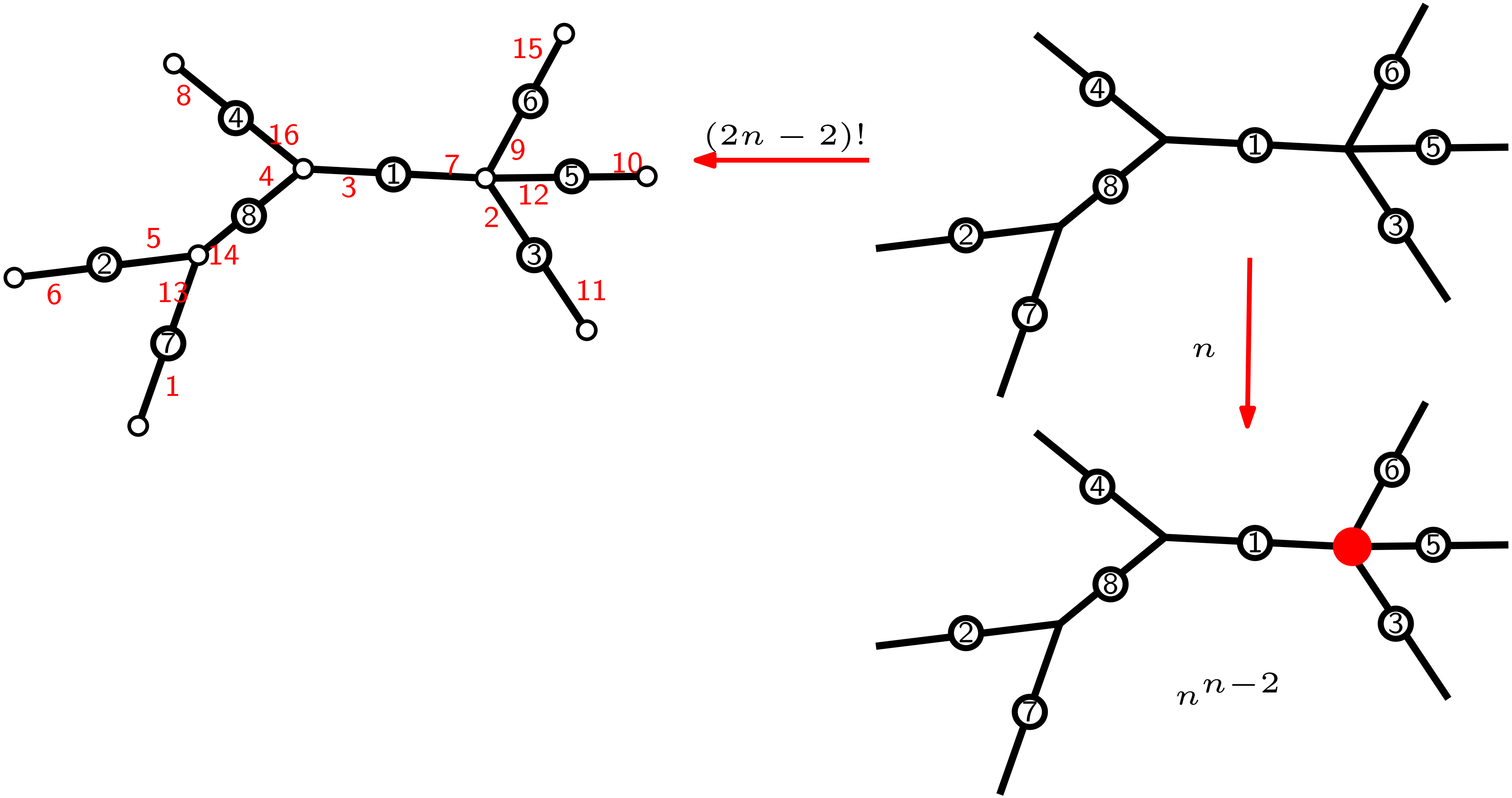

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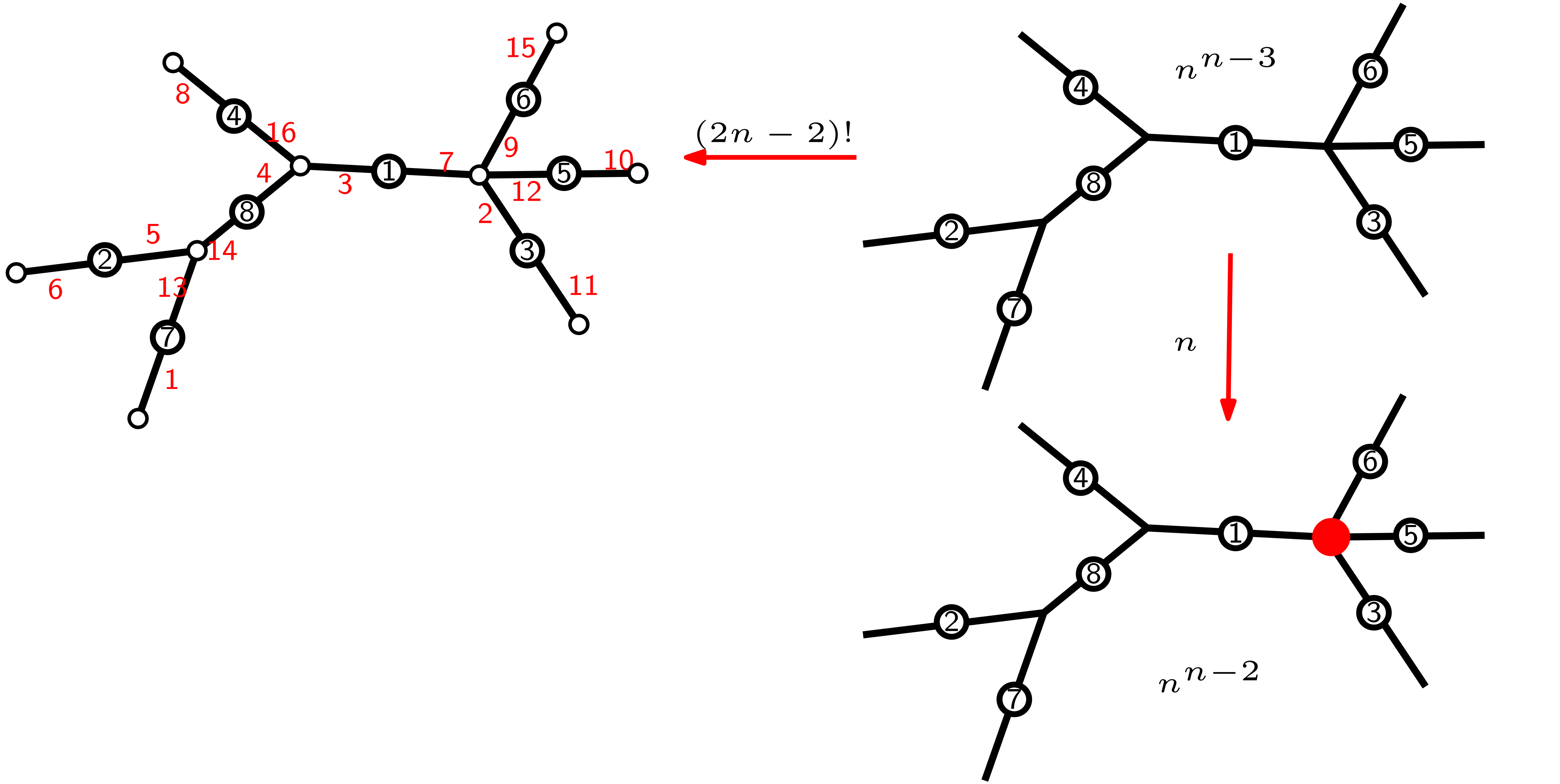

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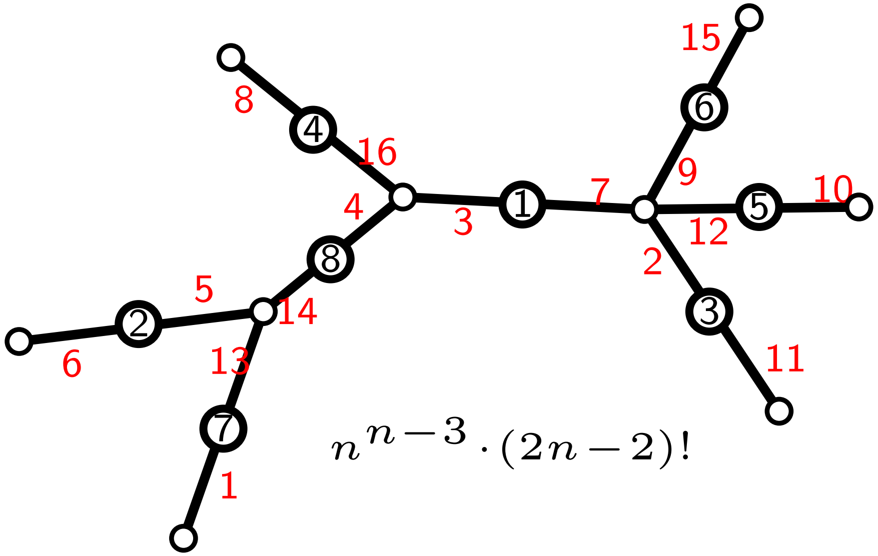
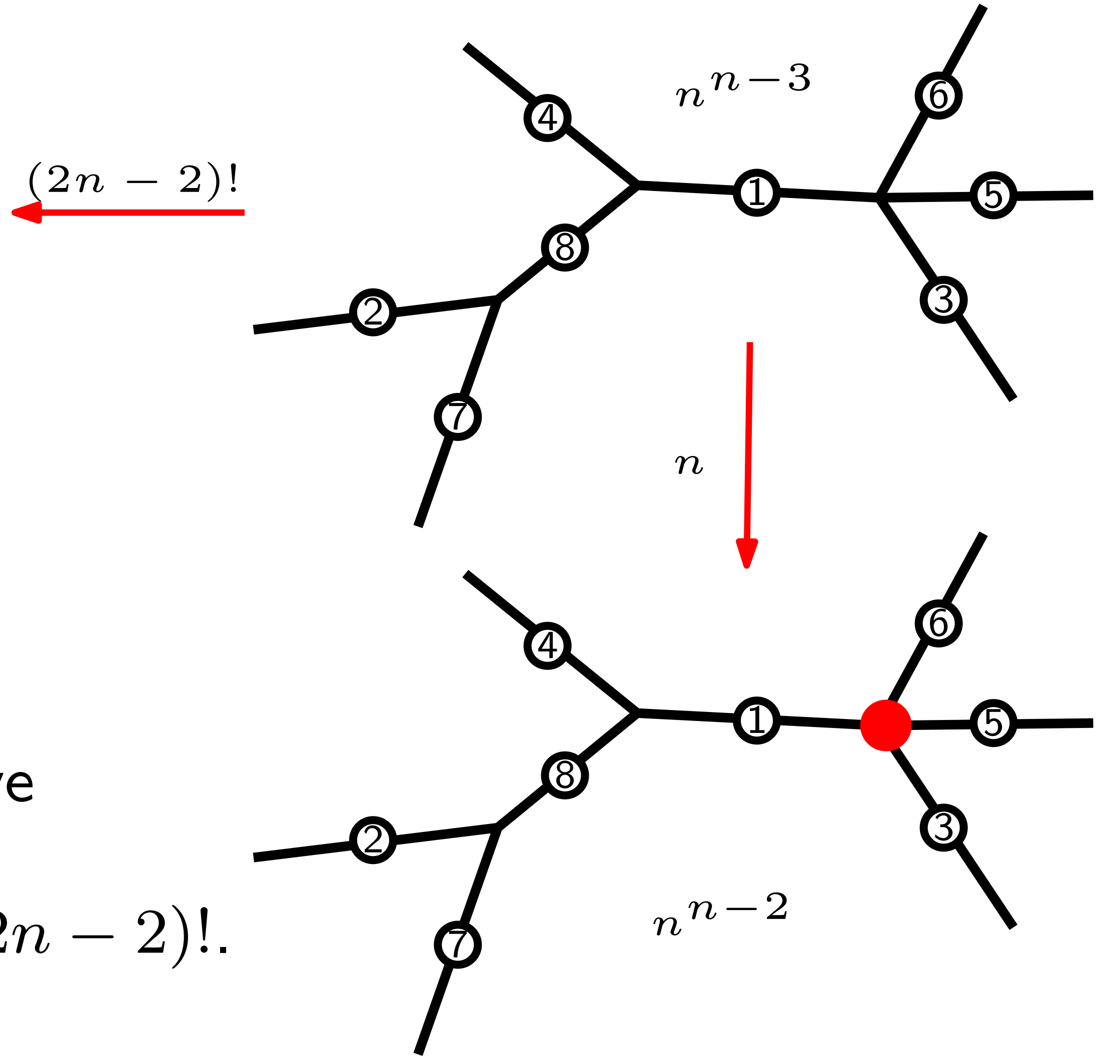
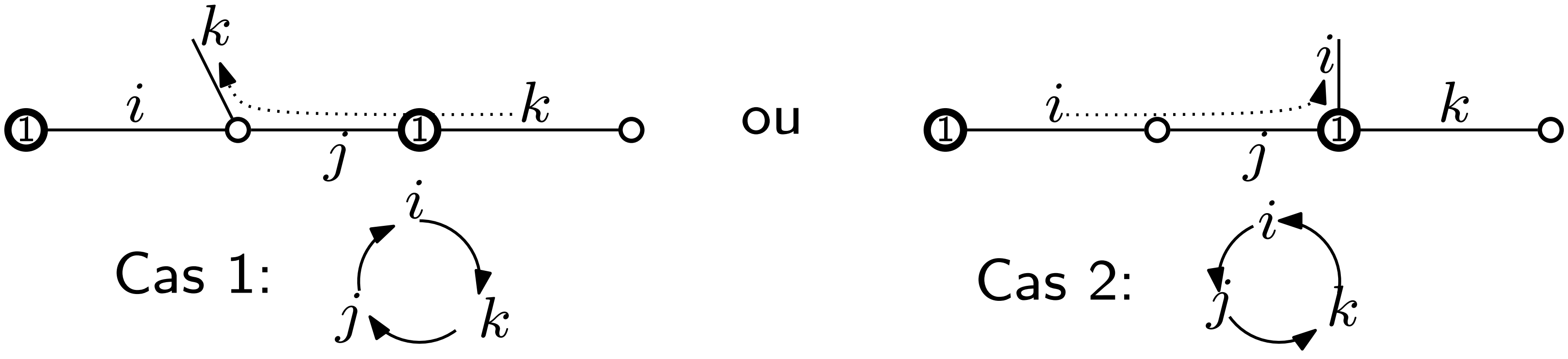




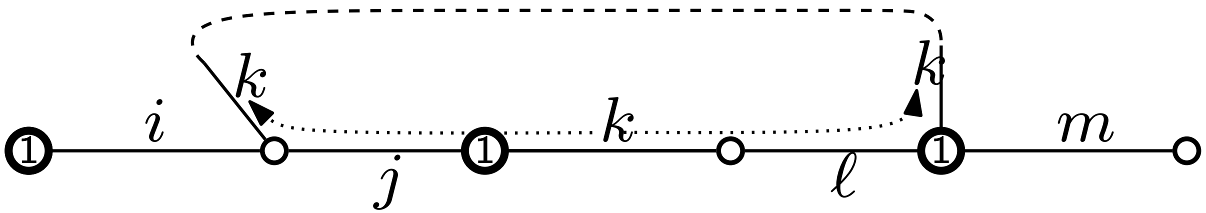
Corollary. The number of transitive factorisations of the identity of S_n in $2n-2$ transpositions is $n^{n-3}(2n-2)!$.

From simple Hurwitz trees to factorizations

A local rule to create increasing half edges

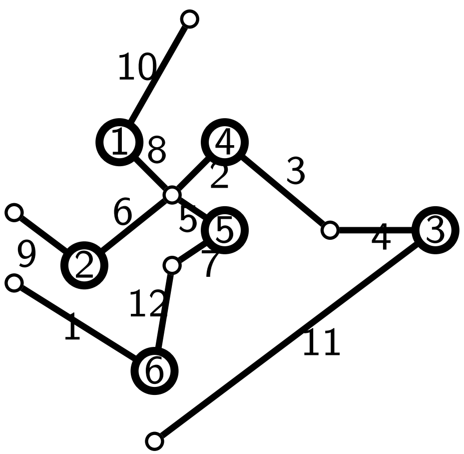


Two half-edges with same label ⇒ edge and face of degree 4

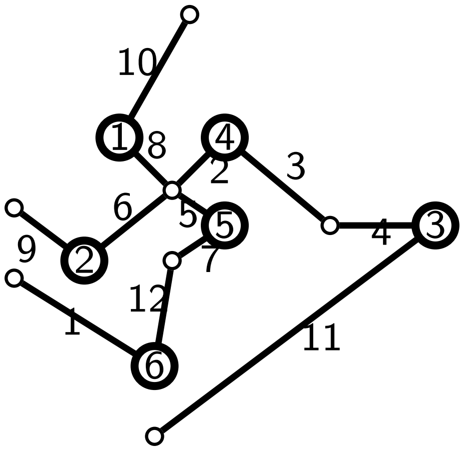


Iterate the local rules as long as possible...

From simple Hurwitz trees to factorizations



From simple Hurwitz trees to factorizations



vertex label are useless
for the bijection

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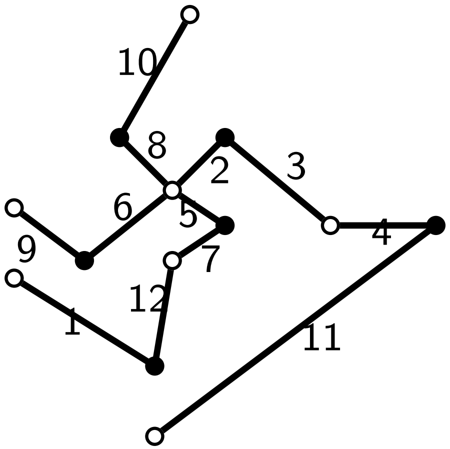

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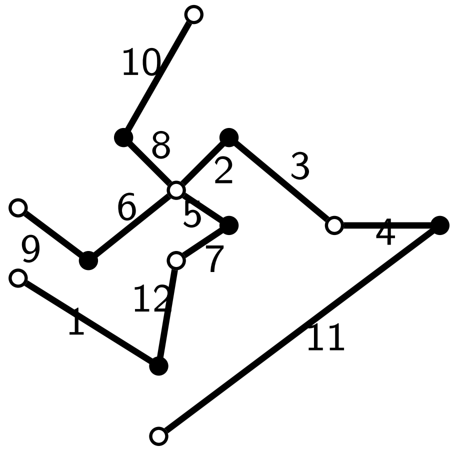

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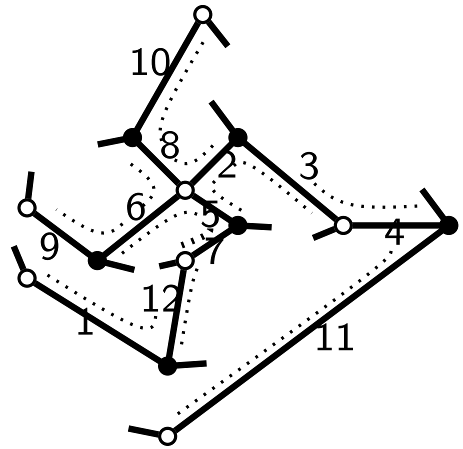
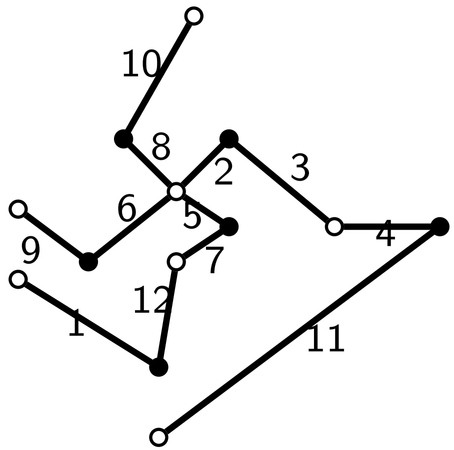


adding buds

From simple Hurwitz trees to factorizations



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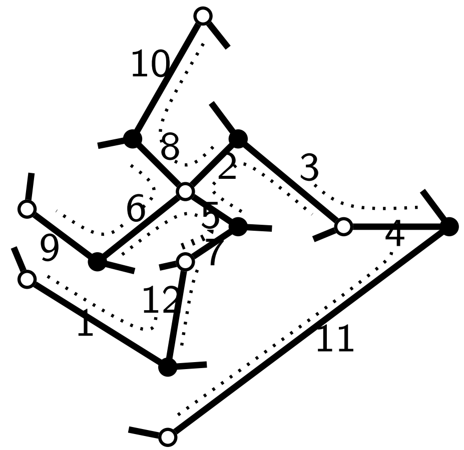

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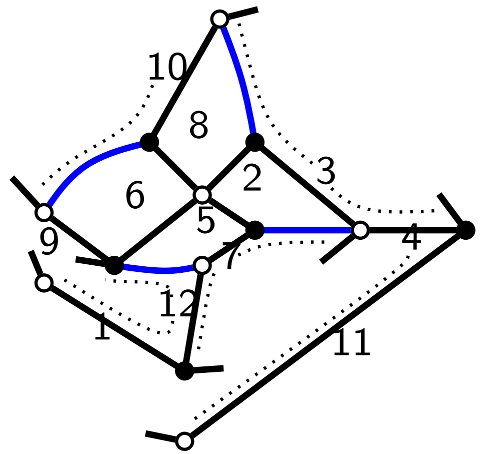
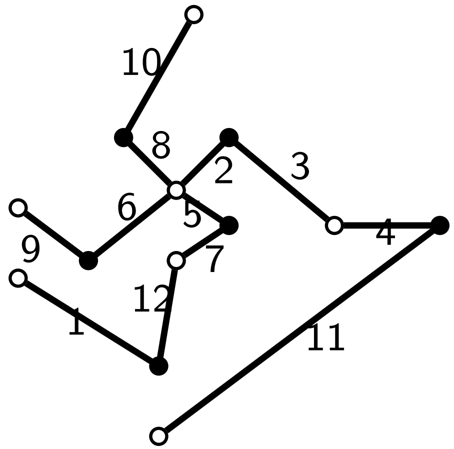


Parings and adding buds again

From simple Hurwitz trees to factorizations



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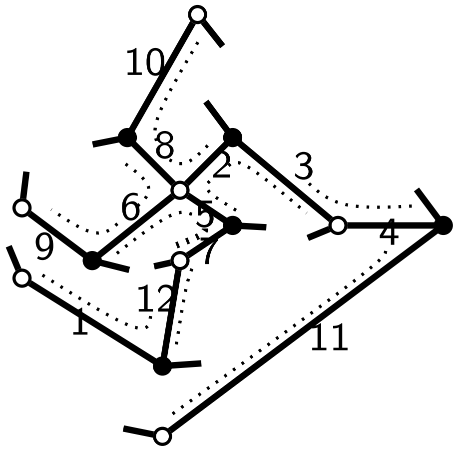

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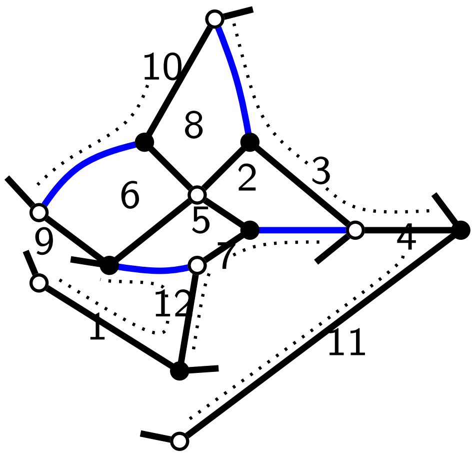

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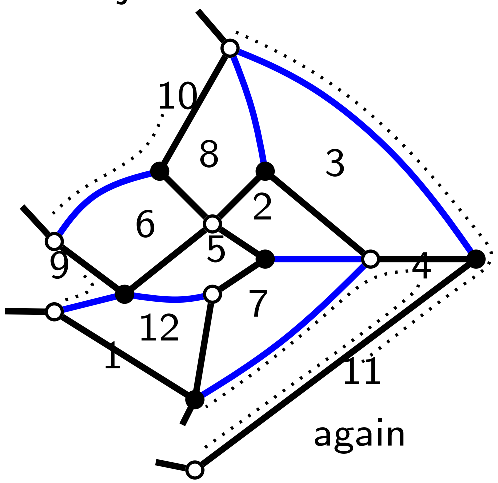

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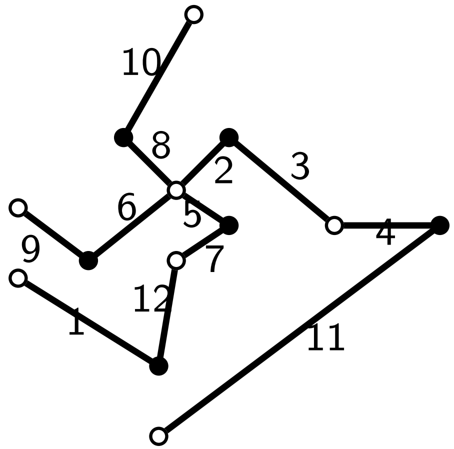

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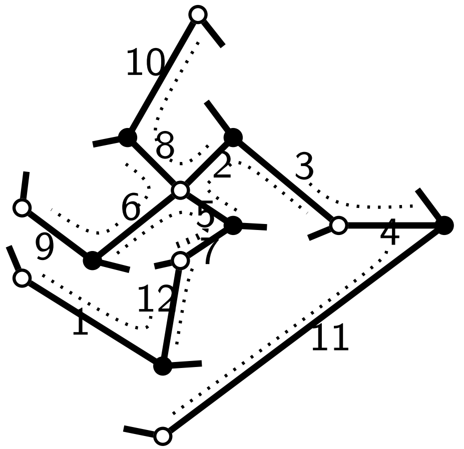

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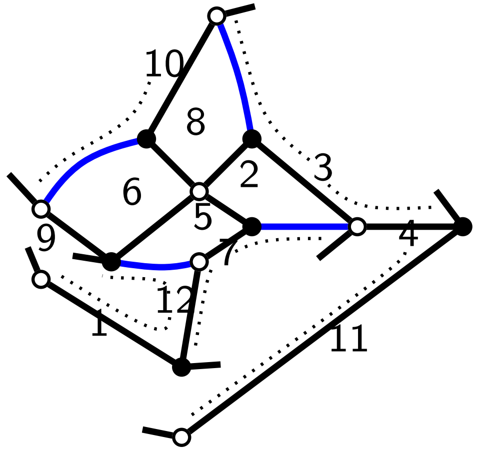

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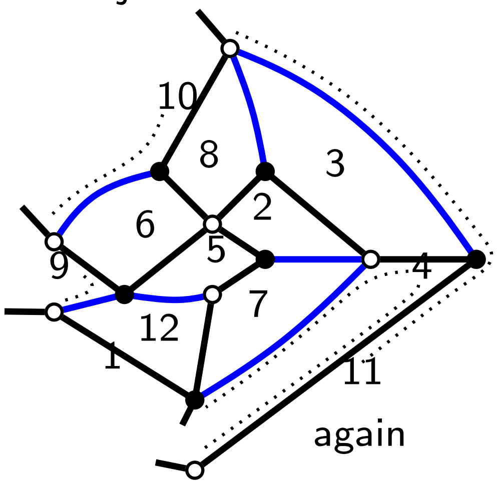

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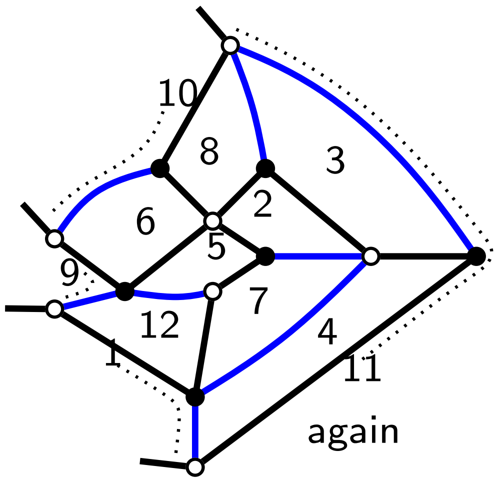

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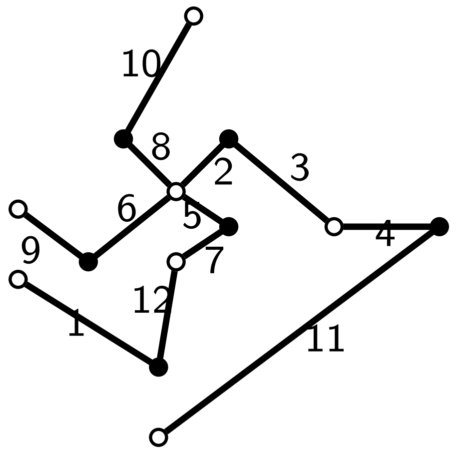

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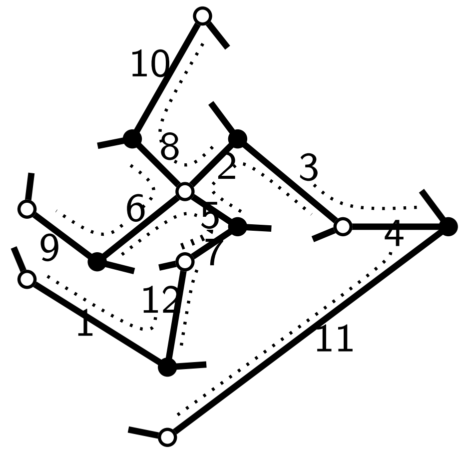

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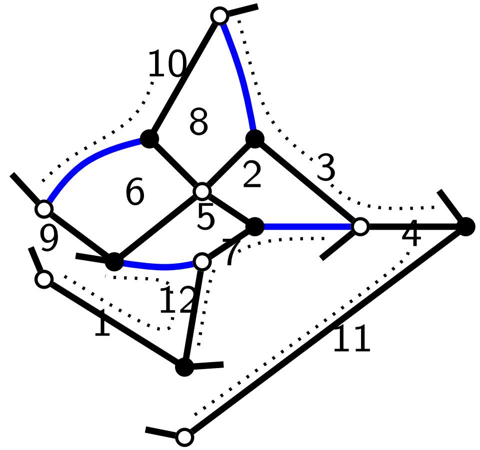

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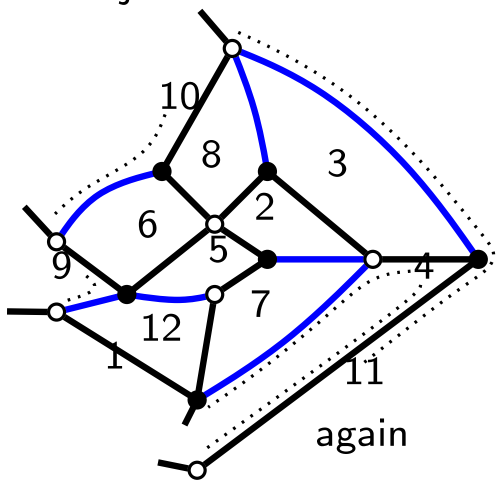

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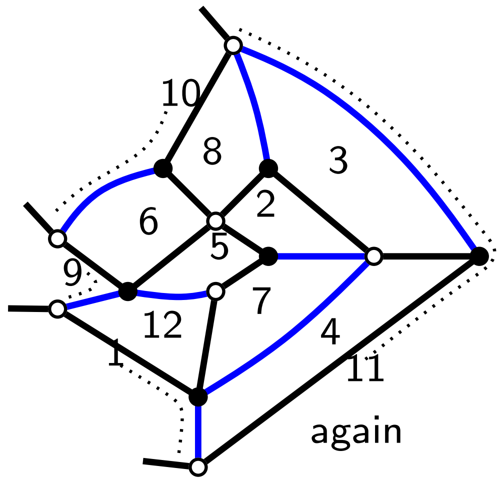
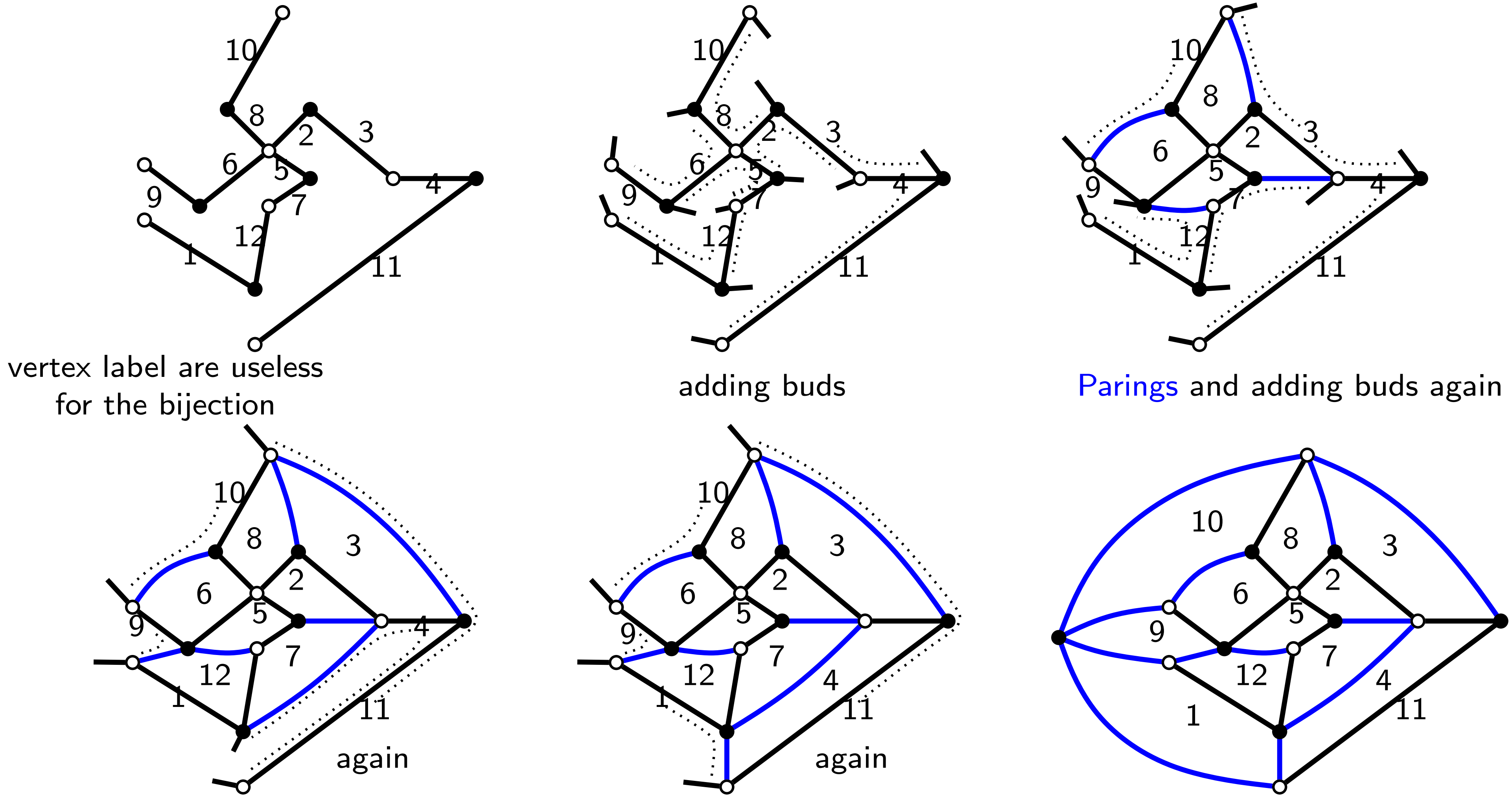


again

Lemma. When it stops, there are only white half-edges left.

From simple Hurwitz trees to factorizations



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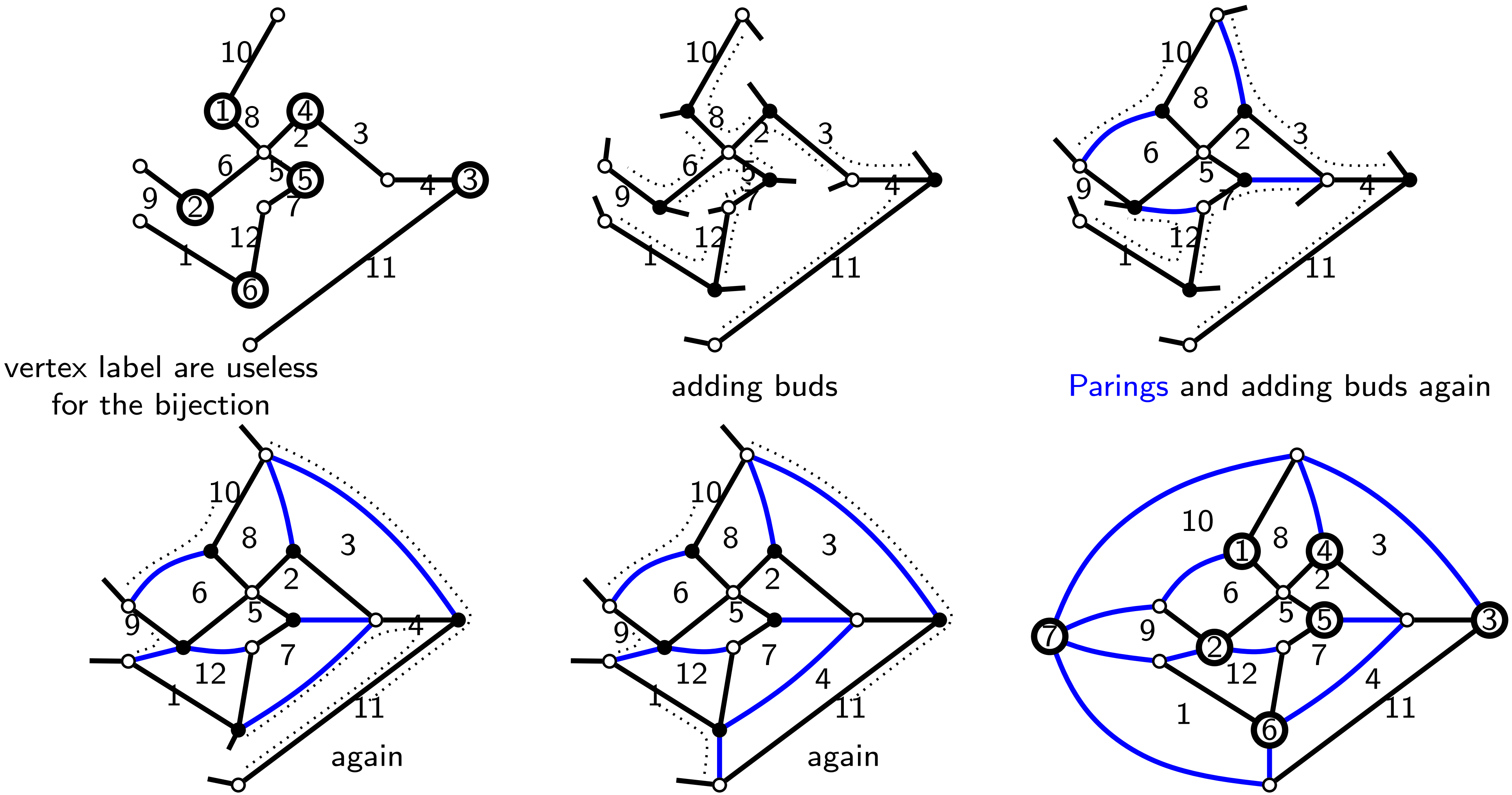
Parings and adding buds again

Lemma. When it stops, there are only white half-edges left.

We connect them to a new black vertex and reload labels.

From simple Hurwitz trees to factorizations



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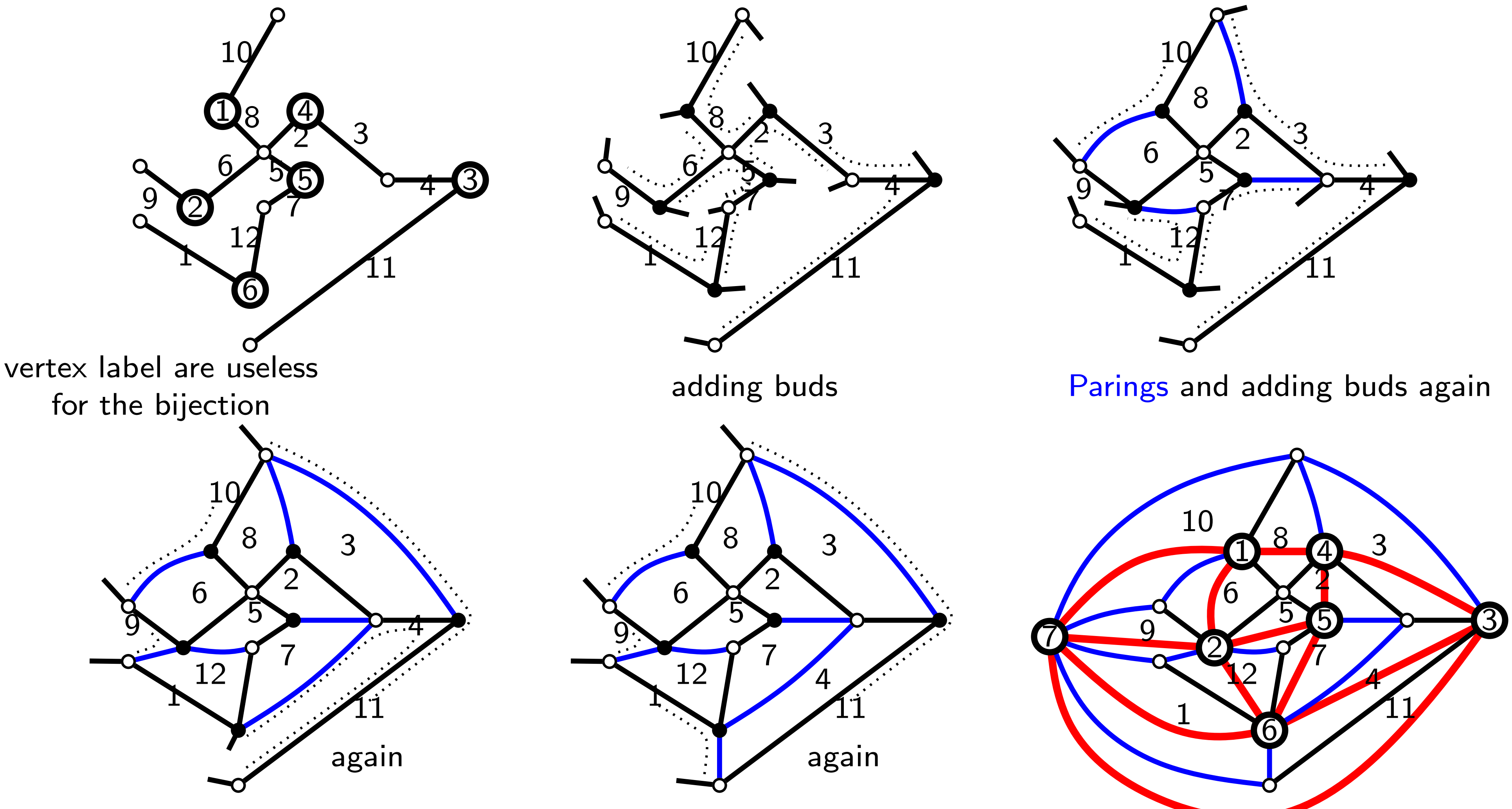

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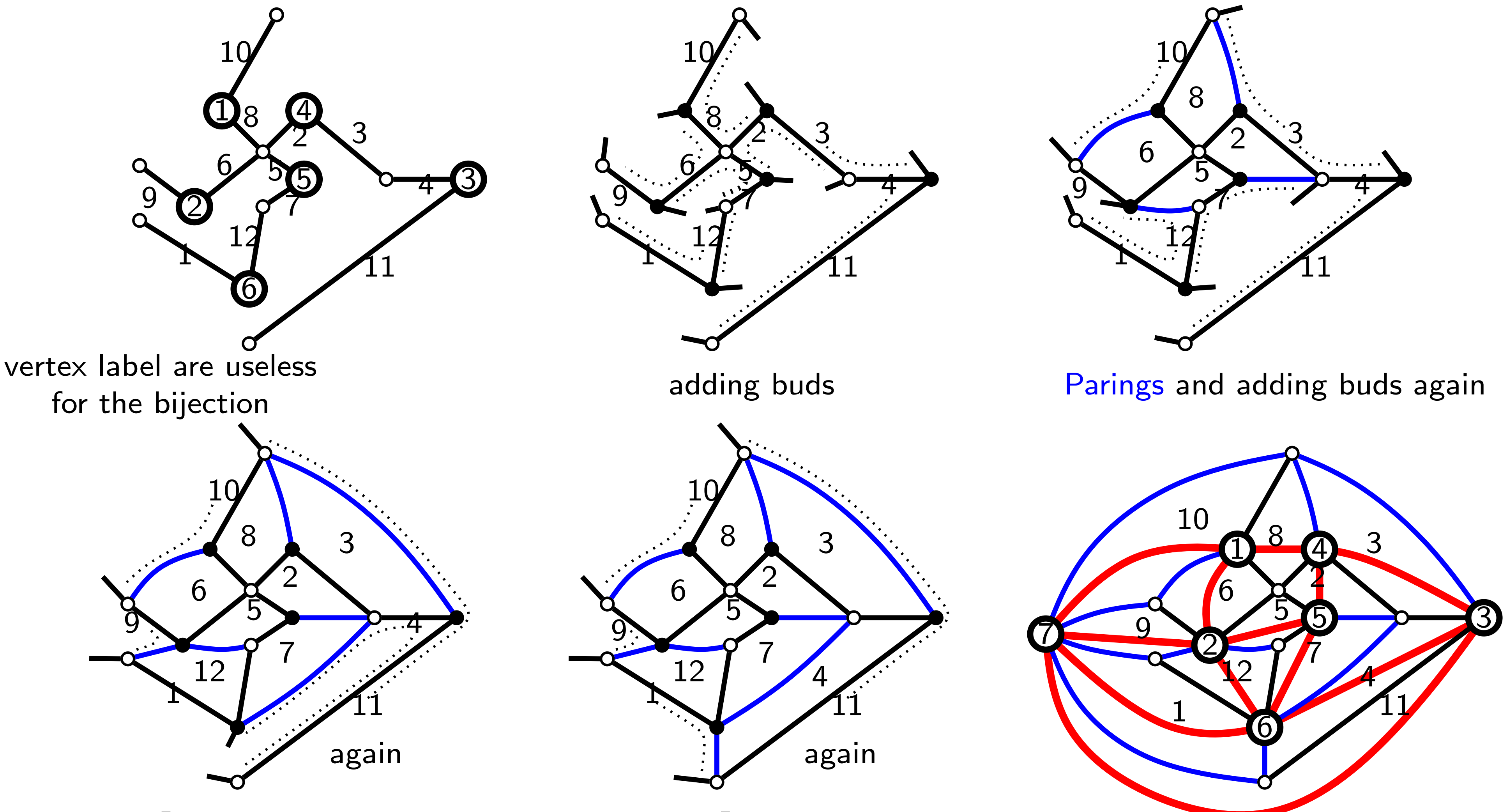
Face number i defines transposition τ_i. **Lemma:** the product is the identity permutation.

$$(6,7)(4,5)(3,4)(3,6)(2,5)(1,2)(5,6)(1,4)(2,7)(1,7)(3,7)(2,6)=(1)(2)(3)(4)(5)(6)(7)$$

From simple Hurwitz trees to factorizations

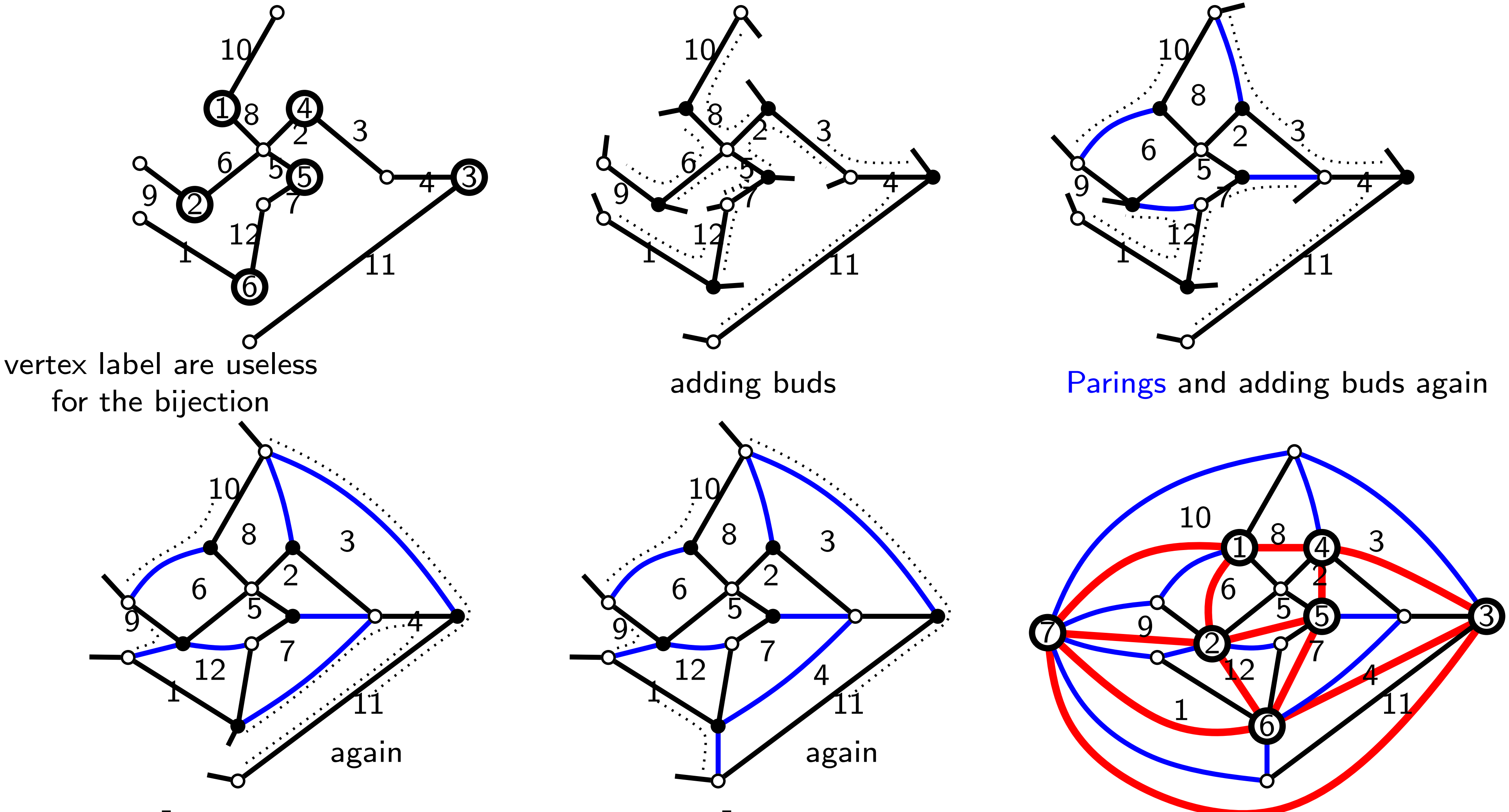


Theorem[Duchi-Poulalhon-S. 2012] Closure is the reverse bijection between

– simple Hurwitz trees of size n, and

– minimal transitive factorizations of the identity in S_n.

From simple Hurwitz trees to factorizations



Theorem[Duchi-Poulalhon-S. 2012] Closure is the reverse bijection between
– ~~simple~~ Hurwitz trees of size n, and
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type λ

Hurwitz formula for the number of minimal transitive factorizations in transpositions

Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition n, and $\ell = \sum_i \ell_i$.
The number of m-uples of transpositions $(\tau_1, \ldots, \tau_m)$ such that

- (product cycle type) $\tau_1 \cdots \tau_m = \sigma$ has cycle type λ
- (transitivity) the associated graph is connected
- (minimality) the number of factors is $m = n + \ell - 2$

is

$$n^{\ell-3} \cdot m! \cdot n! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left(\frac{i^i}{i!}\right)^{\ell_i}$$

Proofs:

(Hurwitz 1891, Strehl 1996) (recurrences, Abel identities)
(Goulden–Jackson 1997) (gfs and differential eqns)
(Lando–Zvonkine 1999) (geometry of LL mapping)
(Bousquet-Mélou–Schaeffer 2000) (bijection + inclusion/exclusion)

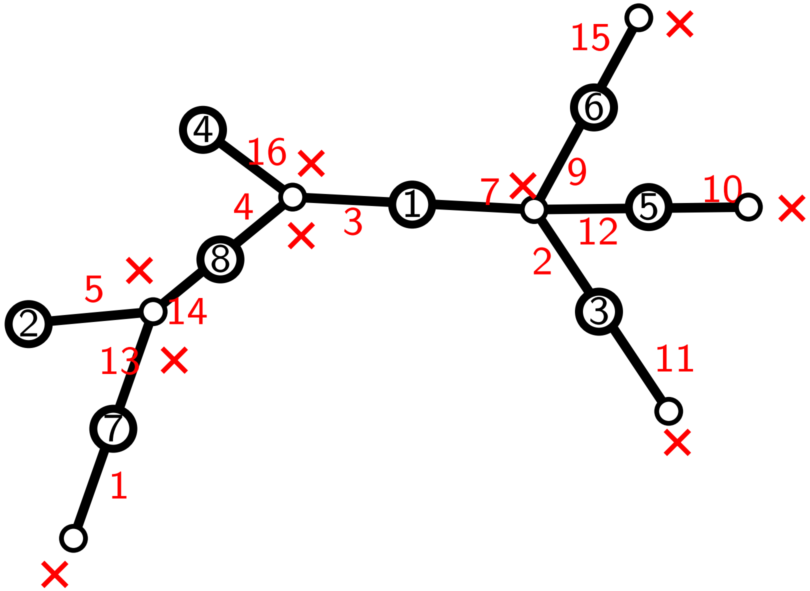
$\lambda = n$, factorizations of n-cycles: $n^{n-2} \cdot (n-1)!$

$\lambda = 1^n$, factorizations of the identity: $n^{n-3} \cdot (2n-2)!$

Arbres de Hurwitz de type λ et formule d'Hurwitz

Pour traiter le cas général de la formule il faut définir des arbres de Hurwitz de type λ: ce sont des arbres plans avec

- $n-1$ sommets noirs de degré 2 ou 1, étiqueté avec $\{1,\ldots,n-1\}$
- ℓ sommets blancs dont ℓ_i portent i séparateurs et $i-1$ feuilles noires
- $m=n+\ell-2$ arêtes avec étiquettes distintes dans $\{1,\ldots,m\}$
- les arêtes sont croissantes en sens direct entre 2 séparateurs



$H_n = n^{n-2}(n-1)!$ $\quad H_{1n} = n^{n-3}(2n-2)!$ $\quad H_\lambda = n^{\ell-3} m! n! \prod_{i\geq 1} \frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

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Lemme. Le nb d'arbres d'Hurwitz de type λ est $n^{\ell-3}m!n!\prod_{i\geq 1}\frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

Théorème La clôture s'étend en une bijection des arbres de Hurwitz de type λ avec les factorisations minimales transitives en transpositions de permutations de type cyclique λ.

Corollare: La formule d'Hurwitz.

$H_n = n^{n-2}(n-1)!$ $\quad H_{1n} = n^{n-3}(2n-2)!$ $\quad H_\lambda = n^{\ell-3}m!n!\prod_{i\geq 1}\frac{1}{\ell_i!}\left(\frac{i^i}{i!}\right)^{\ell_i}$

Conclusion

— Cayley trees are plane trees and

Hurwitz formula counts variant of Cayley trees

— A second strategy (and proof) using Hurwitz mobiles

also extends to higher genus

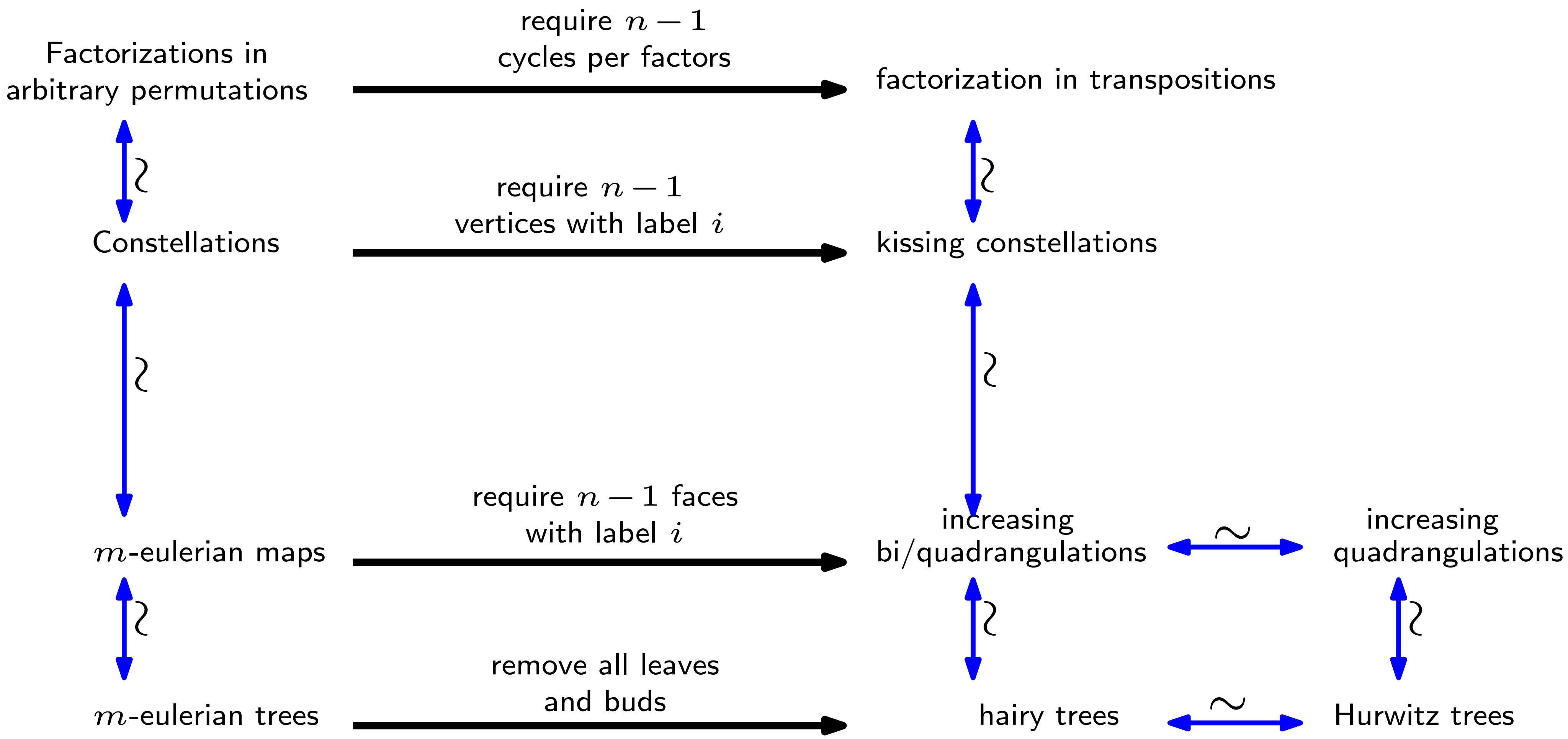
— Open problems:

double Hurwitz numbers

inequivalent factorisations in transpositions

Post Scriptum

Lately we realized that we should have found this much earlier...



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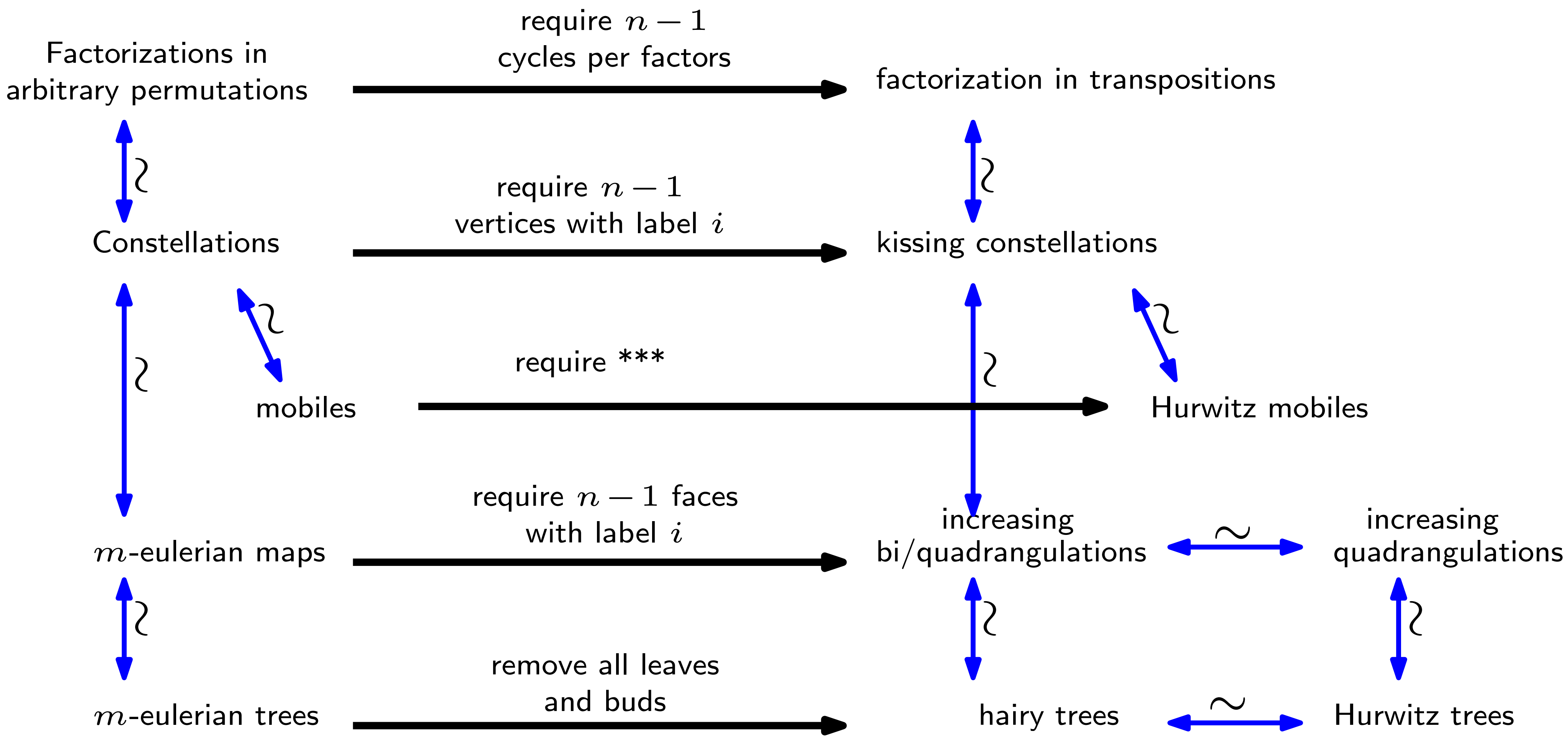

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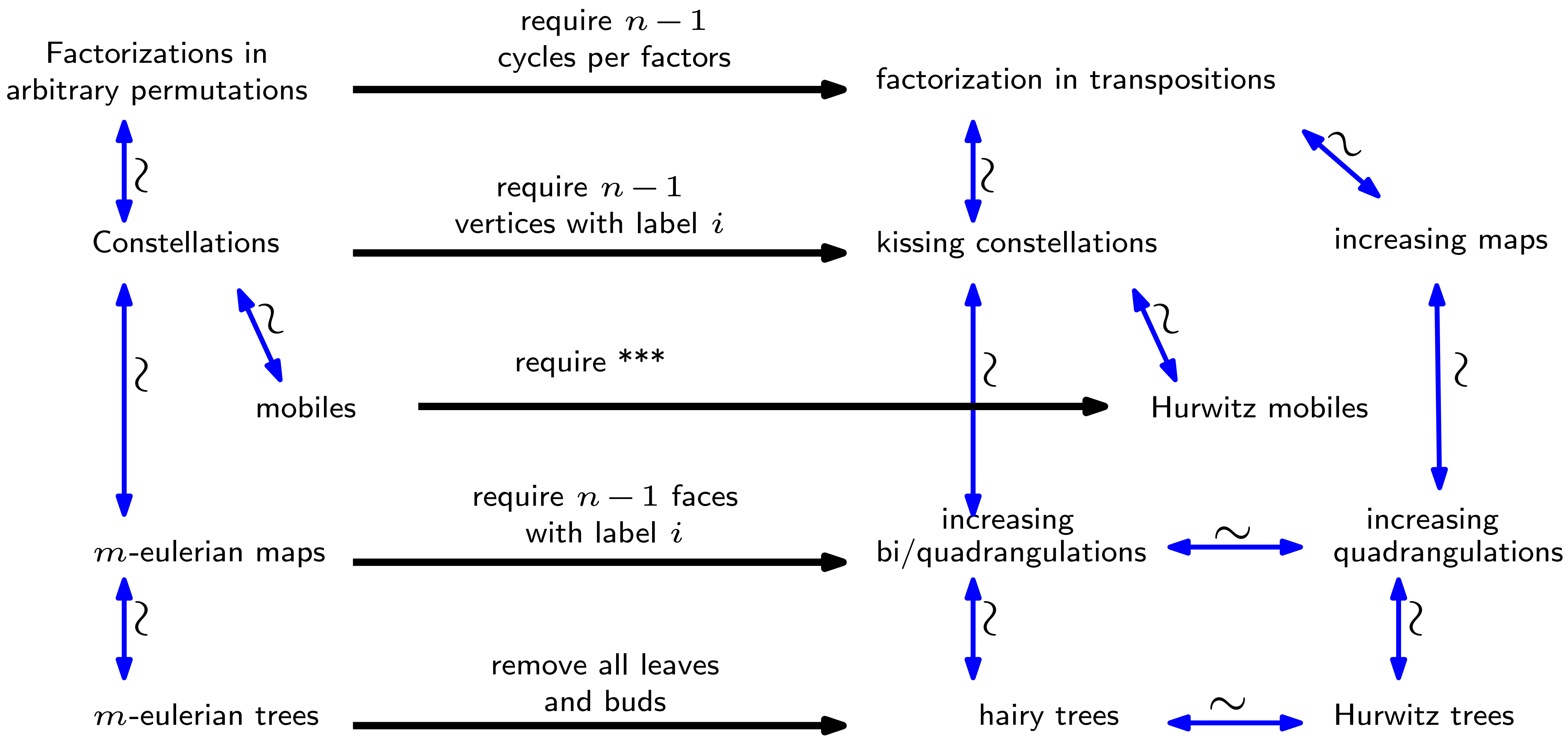

That's all!