Math 8680 Apr. 16, 2021

Vectors, matroids and log-concavity conjectures

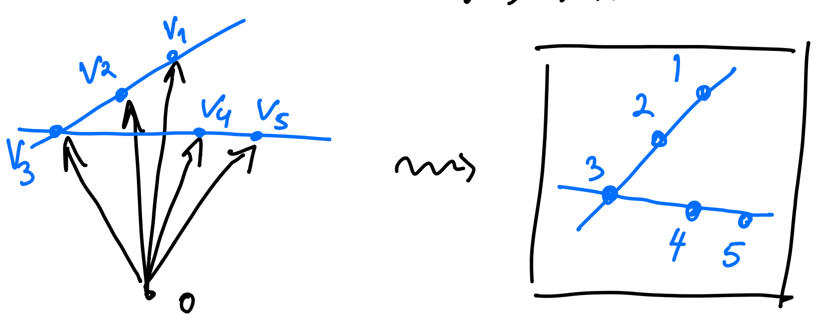
Recall from our overview in 1st week or so ...

a list of vectors $\{v_1, v_2, \ldots, v_n\} = (v_i)_{i \in E := \{1,2,\ldots,n\}}$

$\subseteq K^r$ for some field r

give rise to various kinds of combinatorial data and sequences.

EXAMPLE: These $\{v_1, v_2, v_3, v_4, v_5\} \subset \mathbb{R}^3$



affine picture

give rise to their collection of (linearly) independent subsets

$$\mathcal{I} = \mathcal{I}(v_i) = \{\varnothing$$

$\varnothing$	1	12	~~123~~
	2	13	124
	3	14	125
	4	15	134
	5	23	135
		24	145
		25	234
		34	235
		35	245
		45	~~345~~ $\}$

$$\begin{array}{cccc} (1, & 5, & 10, & 8) \\ i_0 & i_1 & i_2 & i_3 \end{array}$$

with $(i_0, i_1, \ldots, i_r)$

$i_k = \#\left\{\begin{array}{l} \text{lin. indep} \\ \text{subsets } I \in \mathcal{I} \text{ of} \\ \text{with } \# I = k \end{array}\right\}$

$\mathcal{I}$ will satisfy certain axioms:

(I1) $\emptyset \in \mathcal{I}$

(I2) $J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$

} $\mathcal{I}$ is an abstract simplicial complex on vertex set $E = \{1,2,\ldots,n\}$

(I3) If $I, J \in \mathcal{I}$ with $\#I < \#J$ then $\exists\, j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}$

} (Maclane-Steinitz) exchange axiom

DEF'N: A matroid M on ground set $E = \{1,2,\ldots,n\}$ (specified by indep. sets) is a collection $\mathcal{I} \subset 2^E$ satisfying (I1, I2, I3) above.

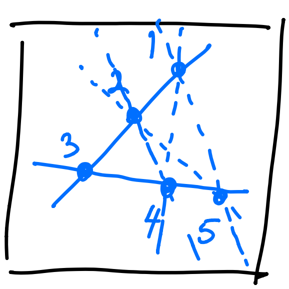
M is realized by $(v_i)_{i \in E} \subseteq K^r$ if it comes from such vectors.

(But some matroids are not realizable at all, and some only realizable over certain fields K)

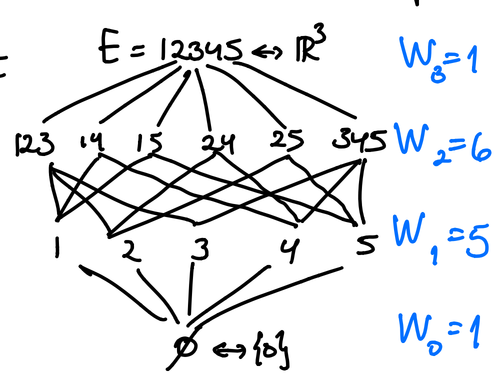
Everything we will say about realized matroids $(v_i)_{i\in E}$ will only use the data in $\mathcal{I}$, and only use the axioms (I1, I2, I3), not the coordinates of $(v_i)_{i\in E}$.

More matroid data we discussed...

The flats $\mathcal{F}$ of $(v_i)_{i\in E}$ are the subsets $F \subseteq E$ closed under K-linear span.



The poset L_M of flats $\mathcal{F}$ (lattice) ordered by inclusion



W_k = k^{th} Whitney number of 2^{nd} kind = # flats F of rank k

$W_0\ W_1\ W_2\ W_3$
$(1, 5, 6, 1)$

Flat axioms: (F1) $E \in \mathcal{F}$

(F2) $F, G \in \mathcal{F} \Rightarrow F \cap G \in \mathcal{F}$

(F3) $\forall F \in \mathcal{F}$ and $e \in E \setminus F$, $\exists$ a unique $G \in \mathcal{F}$ with $G \supset F \cup \{e\}$ and G covers F (i.e. $\nexists$ $H \in \mathcal{F}$ with $G \subsetneq H \subsetneq F$)

i.e. the elements of $E \setminus F$ are a disjoint union of $G \setminus F$ for flats G covering F

One can also characterize M by the rank function for $A \subseteq E$

$$A \longmapsto r_M(A) := \max\{\#I : A \supseteq I \in \mathcal{I}\}$$

NOTATION: The rank of M $r(M) := r_M(E)$

satisfying rank axioms:

(R1) $r_M(A) \in \{0, 1, 2, \ldots\}$

(R2) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (submodular inequality)

(R3) $\forall e \in E, A \subseteq E$ $\quad r(A) \leq r(A \cup \{e\}) \leq r(A) + 1$

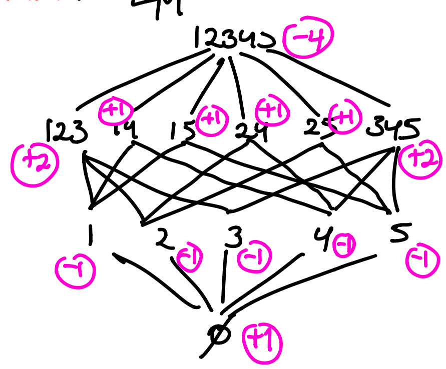
EXERCISE: How to pass between matroids M specified by $\mathcal{I}$ indep sets, $\mathcal{F}$ flats, rank r_M ?

$\omega_k :=$ k^{th} (signless) Whitney number of 1^{st} kind for L_M = lattice of flats

$$:= \sum_{\substack{k\text{-dim'l} \\ \text{flats } F}} |\mu(\emptyset, F)|$$

Möbius function of L

$$\begin{cases} \mu(\emptyset,\emptyset) = +1 \\ \mu(\emptyset, F) = -\sum_{G : G < F} \mu(\emptyset, G) \end{cases}$$



$\omega_3 = 4$

$\omega_2 = 8$

$\omega_1 = 5$

$\omega_0 = 1$

$$\begin{matrix} \omega_0 & \omega_1 & \omega_2 & \omega_3 \\ (1, & 5, & 8, & 4) \end{matrix}$$

We stated these two classical matroid conjectures:

CONJ (Welsh 1971, Mason 1972) $\mathcal{I}$ has $(i_0, i_1, \ldots, i_r)$ not only unimodal but also log-concave: $i_k^2 \geq i_{k-1} \cdot i_{k+1}$ for $2 \leq k \leq r-1$

e.g. $\begin{array}{cccc} i_0 & i_1 & i_2 & i_3 \\ (1, & 5, & 10, & 8) \end{array}$

$5^2 \geq 1 \cdot 10$, $10^2 \geq 5 \cdot 8$

CONJ (Read 1968, Hoggar 1974, Rota 1971, Heron 1972, Welsh 1976) $(w_0, w_1, \ldots, w_r)$ is log-concave.

e.g. $\begin{array}{cccc} w_0 & w_1 & w_2 & w_3 \\ (1, & 5, & 8, & 4) \end{array}$

$5^2 \geq 1 \cdot 8$, $8^2 \geq 5 \cdot 4$

... and we said that they would both follow from a more basic result, that begins by factoring this polynomial

$$w_0 + w_1 t + w_2 t^2 + \ldots + w_r t^r = (1+t)(\bar{w}_0 + \bar{w}_1 t + \bar{w}_2 t^2 + \ldots + \bar{w}_{r-1} t^{r-1})$$

reduced (signless) Whitney numbers of the 1st kind for M

e.g. $(w_0, w_1, w_2, w_3) = (1, 5, 8, 4) \rightsquigarrow 1 + 5t + 8t^2 + 4t^3 = (1+t)(1 + 4t + 4t^2)$

$\Rightarrow (\bar{w}_0, \bar{w}_1, \bar{w}_2) = (1, 4, 4)$

THEOREM

(2010 Huh for M realized with char(K)=0
2011 Huh-Katz for M realized over any K
2015 Adiprasito-Huh-Katz for any matroid M)

$(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{r-1})$ is log-concave.

$\Downarrow$ EXERCISE #2(b) on 2nd part HW list

$(w_0, w_1, \ldots, w_r)$ is log-concave

We claimed that this also implies $(i_0, i_1, \dots, i_r)$ is log-concave

because there is a matroid construction $M \rightsquigarrow M \times e$ such that

(Brylawski, Lenz
1977 2011)

$$(i_0, i_1, \dots, i_r) \text{ for } M$$
$$= (\bar{\omega}_0, \bar{\omega}_1, \dots, \bar{\omega}_r) \text{ for } M \times e$$

So let's learn this matroid construction, along with the basic ones,

and tie things together with the Tutte polynomial $T_M(x,y)$.

Matroid constructions

Some are easier to describe

using the matroid M's bases $\mathcal{B} = \{ I \in \mathcal{I} : I \text{ maximal under inclusion} \}$

which satisfy basis axioms:

(B1) $\mathcal{B} \neq \emptyset$

(B2) $\forall B, B' \in \mathcal{B}$ with $B \neq B'$,

$\forall b \in B - B'$ $\exists$ $b' \in B' - B$ such that

both $B - \{b\} \cup \{b'\}$, $B' - \{b'\} \cup \{b\} \in \mathcal{B}$

(symmetric basis exchange)

- Deletion $M \setminus e$ has ground set $E \setminus e$

  and $\mathcal{I}(M \setminus e) = \{ I \in \mathcal{I}(M) : e \notin I \}$

  $\mathcal{B}(M \setminus e) = \{ B \in \mathcal{B}(M) : e \notin B \}$

  models removing vector v_e from $(v_i)_{i \in E}$

Warning: Don't define $M \setminus e$ this way if e is an isthmus/coloop

that is, e lies in all bases $B \in \mathcal{B}$

- Contraction M/e also has ground set $E \setminus e$

  and $\mathcal{I}(M/e) = \{ I - \{e\} \in \mathcal{I}(M) : e \in I \}$

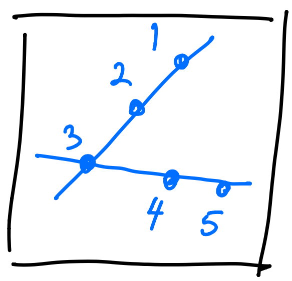
  $\mathcal{B}(M/e) = \{ B - \{e\} \in \mathcal{B}(M) : e \in B \}$

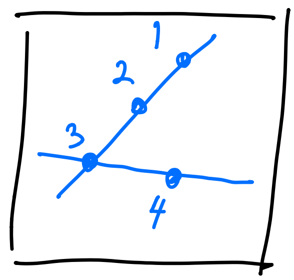
  models applying $K^r \xrightarrow{\pi} K^r / K v_e$

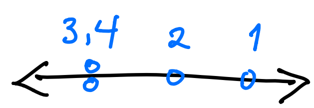
  $(v_i)_{i \in E \setminus e} \mapsto (\pi(v_i))_{i \in E \setminus e}$

Warning: Don't define M/e this way if e is a loop

that is, e lies in no bases $B \in \mathcal{B}$

M 

delete v_5 ⇝ $M \setminus 5$ 

contract on v_5 ⇝ $M/5$ 

- Dual / Orthogonal

  $M^* = M^\perp$ has same ground set E as M

  and $\mathcal{B}(M^*) = \{ E \setminus B : B \in \mathcal{B}(M) \}$

This is a remarkable operation that models

· duality of Plücker coordinates in $Gr(r, K^n)$ versus $Gr(n-r, K^n)$:

EXERCISE

When matrices

$$M := \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \Big\} r$$

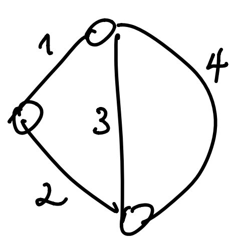
$$M^\perp := \begin{bmatrix} | & | & & | \\ v_1^* & v_2^* & \cdots & v_n^* \\ | & | & & | \end{bmatrix} \Big\} n-r$$

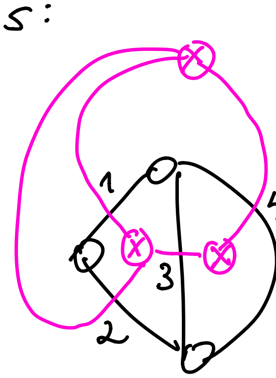
have an orthogonal decomposition

$$K^n = \mathrm{RowSpace}(M) \oplus \mathrm{RowSpace}(M^\perp)$$

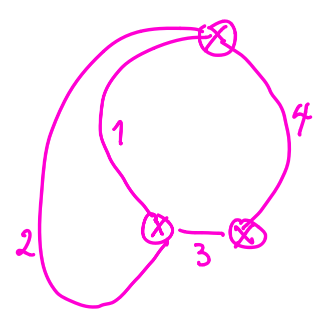
then a minor of M corr. to column set B is $r \times r$ invertible $\iff$ the complementary minor $E \setminus B$ of $M^\perp$ is $(n-r) \times (n-r)$ invertible

... and it also models duality of spanning trees in planar dual graphs:

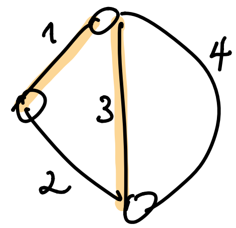
$G =$ 



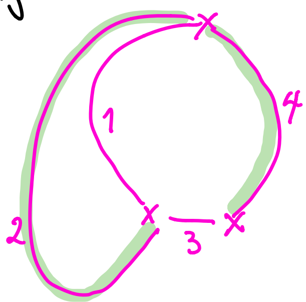
G^* = planar dual



spanning tree edge subsets B in G $=$ complements $E \setminus B$ of spanning tree edge subsets in G^*



$B = \{1,3\}$



$E \setminus B = \{2,4\}$

FACT: $(M^*)^* = M$ and $(M \setminus e)^* = M^* / e$ if e is neither loop nor coloop

---

Math 8680 Apr. 19, 2021

- Principal extension $M+e$

$M+e$ has ground set $E \cup \{e\}$

and this rank function:

for $A \subseteq E$, $r_{M+e}(A) = r_M(A)$

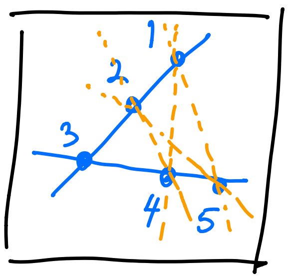
while $r_{M+e}(A \cup \{e\}) = \begin{cases} r_M(A)+1 & \text{if } r_M(A) < r_M(E) \\ r_M(A) & \text{if } r_M(A) = r_M(E) \end{cases}$

It models adding a new vector v_e to $(v_i)_{i \in E} \subseteq K^r$

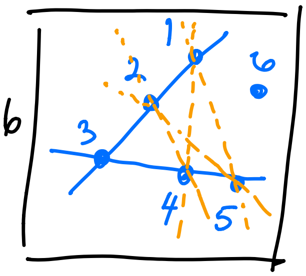
in general position, i.e., on none of the proper flats $F \subsetneq E$

Realizing $M+e$ might require passing to a field extension $K \subset \overline{K}$

M



$M+6$



We'd like to show that the construction

$$M \rightsquigarrow M \times e := (M^* + e)^*$$

has the desired property that $(i_0, i_1, \ldots, i_r)$ for M
 $=$
$(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_r)$ for $M \times e$

Tutte polynomial (1947)

This is an amazing 2-variate polynomial $T_M(x,y)$ that specializes to generating functions for $(i_0, i_1, \ldots, i_r)$
and $(\omega_0, \omega_1, \ldots, \omega_r)$,
and many other interesting invariants.

DEF'N: For a matroid M with rank function $r(A) = r_M(A)$,

$$T_M(x,y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{\#A - r(A)}$$

Equivalently, (Crapo 1967) $T_M(x+1,y+1)=\sum_{A\subseteq E} x^{r(E)-r(A)}\,y^{\#A-r(A)}$ Whitney's "corank-nullity polynomial"

$\swarrow\ y=0$ $\qquad\qquad\qquad\qquad$ $x=0\ \searrow$

$$T_M(x+1,1)=\sum_{I\in\mathcal{I}\text{ independent}} x^{r(E)-\#I}$$

$$=i_0x^r+i_1x^{r-1}+\dots+i_{r-1}x+i_r$$

$$T_M(1,y+1)=\sum_{\substack{S\subseteq E\\ \text{spanning},\\ \text{i.e. } r(S)=r(E)}} y^{\#S-r(E)}$$

$x=0\ \searrow$ $\qquad\qquad$ $\swarrow\ y=0$

$$T_M(1,1)=\#\mathcal{B}=\text{number of bases}$$

PROPOSITION: The Tutte polynomial $T_M(x,y)$ satisfies

(T1) $\quad T_M = T_{M\setminus e}+T_{M/e}$ if e is neither a loop nor coloop of M

(T2) $\quad T_M=\begin{cases} x\,T_{M/e} & \text{if } e \text{ is a coloop}\\ y\,T_{M\setminus e} & \text{if } e \text{ is a loop}\end{cases}$

(T3) $\quad T_M=1$ if $E=\emptyset$

and is the unique invariant of matroids in $\mathbb{Z}[x,y]$ with properties (T1, T2, T3).

Furthermore,

(T4) $\quad T_{M^*}(x,y)=T_M(y,x)$

(T5) $\quad T_M(x,y)\in\mathbb{N}[x,y]$, i.e. it has nonnegative coefficients

<u>proof sketch</u>: It's not hard to show $T_M(x,y)$ satisfies (T1, T2, T3), with (T1) coming from classifying $A \subseteq E$ according to $e \notin A$ versus $e \in A$.

It's then easy to see (T1, T2, T3) characterize it using induction on $\#E$. And then it's also easy to show (T4, T5) follow by induction on $\#E$. ∎

A key fact for us is that $T_M(x,y)$ also captures $(w_0, w_1, \ldots, w_r)$:

<u>PROP</u>: The characteristic polynomial

$$\chi_M(t) \overset{\text{DEFN}}{:=} \sum_{\text{flats } F \in \mathcal{F}} \mu(\emptyset, F)\, t^{r(M)-r(F)}$$

$$= (-1)^{r(M)}\, T_M(1-t, 0)$$

and has coefficients that alternate in sign, that is

$$\chi_M(t) = w_0 t^r - w_1 t^{r-1} + w_2 t^{r-2} - \cdots \pm (-1)^r w_r \quad \text{with } w_0 = 1 \text{ and}$$

nonnegative $(w_0, w_1, \ldots, w_r)$.

In particular, $\boxed{w_0 t^r + w_1 t^{r-1} + \cdots + w_r = T_M(1+t, 0)}$

proof: Compute

$$(-1)^{r(M)} T_M(\underset{\substack{\| \\ x}}{1-t},\underset{\substack{\| \\ y}}{0}) = (-1)^{r(M)} \sum_{A \subseteq E} (\underbrace{-t}_{=x-1})^{r(M)-r(A)} (\underbrace{-1}_{=y-1})^{\#A-r(A)}$$

$$= \sum_{A \subseteq E} (-1)^{\#A}\, t^{r(M)-r(A)}$$

$$= \sum_{\text{flats } F \in \mathcal{F}} t^{r(M)-r(F)} \underbrace{\sum_{\substack{A \subseteq F: \\ r(A)=r(F)}} (-1)^{\#A}}_{\text{Call this } m(F)}$$

We claim $m(F) = \mu(\emptyset, F)$, since it satisfies its defining recursion for this Möbius function:

$$\forall G \in \mathcal{F}, \quad \sum_{\substack{F \in \mathcal{F}: \\ \emptyset \subseteq F \subseteq G}} m(F) = \sum_{\substack{F \in \mathcal{F}: \\ \emptyset \subseteq F \subseteq G}} \sum_{\substack{A \subseteq F: \\ r(A)=r(F)}} (-1)^{\#A}$$

$$= \sum_{A \subseteq G} (-1)^{\#A} = \begin{cases} 1 & \text{if } G = \emptyset \\ 0 & \text{else} \end{cases}$$

Hence $(-1)^{r(M)} T_M(1-t, 0) = \sum_{F \in \mathcal{F}} \mu(\emptyset, F)\, t^{r(M)-r(F)} = \chi_M(t)$.

This then implies by property (T1) of $T_M(x,y)$ that

$$\chi_M(t) = \chi_{M \setminus e}(t) - \chi_{M/e}(t) \quad \text{for nonloop, non coloop } e$$

and lets one check by induction on $\#E$ that the coefficients of $\chi_M(t)$ will alternate in sign. $\blacksquare$

The final piece is this fact (which was news to me!):

LEMMA: $T_{M+e}(0,1+t) = (1+t)\,T_M(1,1+t)$

proof:

$$\text{LHS} = \sum_{A \subseteq E \cup \{e\}} (-1)^{r(M+e)-r(A)}\, t^{\#A - r_{M+e}(A)}$$

$$= \sum_{A \subseteq E} \Bigg(\underbrace{(-1)^{r(M)-r(A)}\, t^{\#A - r_M(A)}}_{\text{Call this } X} + \underbrace{(-1)^{r(M+e)-r(A\cup\{e\})}\, t^{\#A+1-r_{M+e}(A\cup\{e\})}}_{\substack{\text{Since } r_M(A) < r(M) \text{ implies} \\ r_{M+e}(A\cup\{e\}) = r_M(A)+1, \\ \text{this term is} \\ \begin{cases} -X & \text{if } r_M(A) < r(M) \\ t\cdot X & \text{if } r_M(A) = r(M) \end{cases}}} \Bigg)$$

$$= (1+t) \sum_{\substack{A \subseteq E: \\ r_M(A) = r(M)}} t^{\#A - r(M)}$$

← i.e. A is spanning for M

$$= (1+t)\,T_M(1,1+t) \qquad \blacksquare$$

COROLLARY: Defining $M \times e := (M^* + e)^*$, one has

$$\begin{array}{l}(i_0, i_1, \ldots, i_r) \text{ for } M \\ \qquad \| \\ (\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_r) \text{ for } M\times e\end{array}$$

<u>proof</u>: Let $r = r(M)$, so that $r(M^*) = n - r = r(M^* + e)$

and $r(M\times e) = (n+1) - (n-r) = r+1$

Also let $(\omega_0, \omega_1, \ldots, \omega_r, \omega_{r+1})$ be the signless Whitney numbers for $M\times e$

Then

$$\left.\begin{aligned} &(t+1)\left(\bar{\omega}_0 t^r + \bar{\omega}_1 t^{r-1} + \ldots + \bar{\omega}_{r-1} t + \bar{\omega}_r\right) \\ &= \omega_0 t^{r+1} + \omega_1 t^r + \ldots + \omega_r t + \omega_{r+1}\end{aligned}\right\} \text{ for } M\times e$$

$$\begin{aligned}
&= \chi_{M\times e}(t) = T_{M\times e}(1+t, 0) \\
&\qquad \text{DEF'N of } M\times e \\
&= T_{(M^*+e)^*}(1+t, 0) \\
&\qquad \text{DUALITY} \\
&= T_{M^*+e}(0, 1+t) \\
&\qquad \text{LEMMA} \\
&= (1+t)\, T_{M^*}(1, 1+t) \\
&\qquad \text{DUALITY} \\
&= (1+t)\, T_M(1+t, 1) \\
&= (1+t)\left(i_0 t^r + i_1 t^{r-1} + \ldots + i_{r-1} t + i_r\right) \text{ for } M
\end{aligned}$$

Hence

$$\underbrace{\sum_{k=0}^{r} \bar{\omega}_k \cdot t^{r-k}}_{\text{for } M\times e} = \underbrace{\sum_{k=0}^{r} i_k \cdot t^{r-k}}_{\text{for } M} \qquad \blacksquare$$

DIGRESSION: What do $\chi_M(t) = \omega_0 t^r - \omega_1 t^{r-1} + \cdots + (-1)^r \omega_r$

and $\overline{\chi}_M(t) = \overline{\omega}_0 t^{r-1} - \overline{\omega}_1 t^{r-2} + \cdots + (-1)^{r-1} \overline{\omega}_{r-1}$ mean?

Roughly, $\chi_M(t)$ counts points in the hyperplane arrangement complements

$$\mathcal{M}_{\mathbb{F}_p} := \mathbb{F}_p^r \setminus \underbrace{\bigcup_{i \in E} v_i^\perp}_{\substack{\text{the arrangement of}\\ \text{hyperplanes normal to } v_i}} \qquad \text{when } t \text{ is a large prime } p$$

and $\overline{\chi}_M(t)$ does same in the projectivized complement $\mathcal{M}_{\mathbb{F}_p} / \mathbb{F}_p^\times$

or the deconed affine arrangements where we send a hyperplane to ∞

Topologically, their signless and rescaled versions

$$\pi_M(t) = \omega_0 + \omega_1 t + \omega_2 t^2 + \cdots + \omega_r t^r \qquad (= t^r \chi_M(t^{-1}))$$

$$\overline{\pi}_M(t) = \overline{\omega}_0 + \overline{\omega}_1 t + \cdots + \overline{\omega}_{r-1} t^{r-1} \qquad (= t^{r-1} \overline{\chi}_M(t^{-1}))$$

give the Poincaré polynomials (= Betti # generating functions)

for the complex hyperplane complement

$$\mathcal{M}_{\mathbb{C}} := \mathbb{C}^r \setminus \bigcup_{i \in E} v_i^\perp, \qquad \pi_M(t) = \sum_{i=0}^{r} t^i \cdot \operatorname{rank}_{\mathbb{Z}} H^i(\mathcal{M}_{\mathbb{C}}, \mathbb{Z})$$

and the projectivized complement

$$\mathcal{M}_{\mathbb{C}} / \mathbb{C}^\times, \qquad \overline{\pi}_M(t) = \sum_{i=0}^{r} t^i \cdot \operatorname{rank}_{\mathbb{Z}} H^i(\mathcal{M}_{\mathbb{C}} / \mathbb{C}^\times, \mathbb{Z})$$

(or for the affine deconed arrangements)

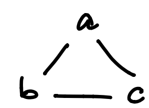
This is closely related to the original occurrence of

$\chi_M(t)$ for graphic matroids M of a graph G

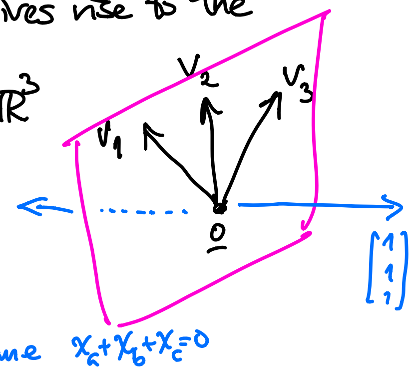
in the work of Birkhoff and Lewis (1912) on the graph chromatic polynomial

$$\chi_G(t) = \#\text{ proper vertex-colorings of } G \text{ with } t \text{ colors}$$

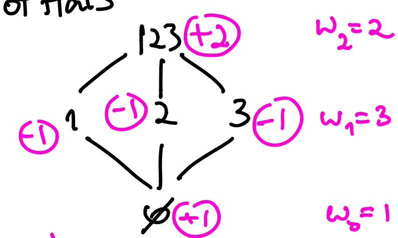
$$= t^{\#\text{conn. components of } G} \cdot \chi_M(t)$$

EXAMPLE: This graph $G=$  gives rise to the

realized graphic matroid $M=\{v_1=e_a-e_b,\ v_2=e_a-e_c,\ v_3=e_b-e_c\}$ in $\mathbb{R}^3$



having poset of flats

$L_M=$ 

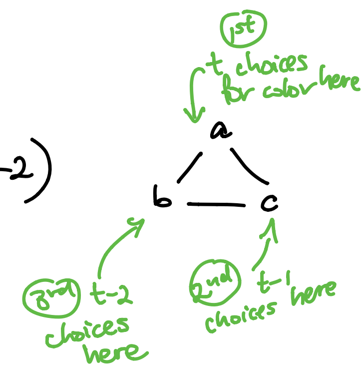
$\mu(\emptyset, X)$ circled

$$\chi_M(t)=t^2-3t+2=(t-1)(t-2)$$

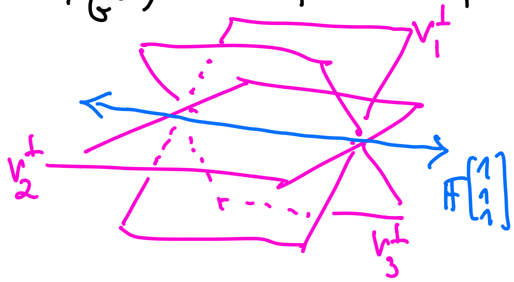
$$\pi_M(t)=1+3t+2t^2=(1+t)(1+2t)$$

The chromatic polynomial is $\chi_G(t)=t(t-1)(t-2)$

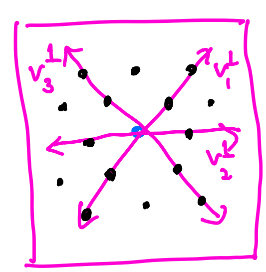
$$=t\cdot\chi_M(t)$$



$\chi_G(t)$ counts points in $\mathbb{F}_p^3 \setminus \bigcup_{i=1,2,3} v_i^\perp$ for p large primes

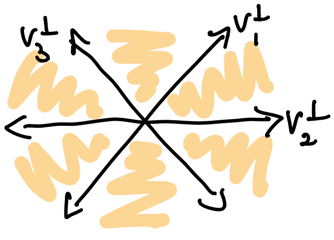


essentialize by $\rightsquigarrow$ modding out $\mathbb{F}_p\begin{bmatrix}1\\1\\1\end{bmatrix}$



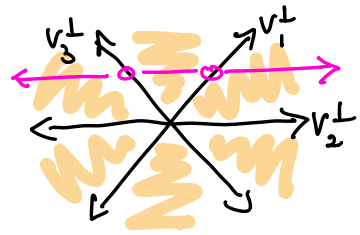
$\chi_M(t)$ counts points in $\underbrace{\left(\mathbb{F}_p^3/\mathbb{F}_p\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\setminus \bigcup_{i=1,2,3} v_i^\perp}_{M_{\mathbb{F}_p}:=}$ for p large primes

Working instead over $\mathbb{C}$ (but drawing pictures over $\mathbb{R}$)
if we start with the complement $\mathcal{M}_{\mathbb{C}} := \mathbb{C}^r \setminus \bigcup_{i=1,2,3} v_i^{\perp}$ and
pick one of its hyperplanes $v_2^{\perp}$ to make the "hyperplane at ∞"



$\mathcal{M}_{\mathbb{C}}$

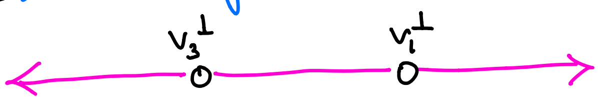
decone the arrangement using $v_2^{\perp}$ $\rightsquigarrow$ = intersect with affine hyperplane $v_2 \cdot \underline{x} = 1$



$\mathcal{M}_{\mathbb{C}}$

one can understand the points of the projectivized complement $\mathcal{M}_{\mathbb{C}}/\mathbb{C}^{\times}$ as represented by the points in the complement of this deconed affine arrangement inside the hyperplane $v_2 \cdot \underline{x} = 1$:

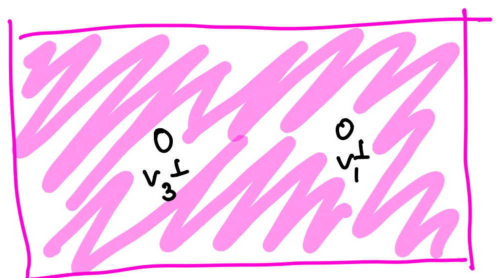
$\mathcal{M}_{\mathbb{R}}/\mathbb{R}^{\times}$ $\cong$



inside $v_2 \cdot \underline{x} = 1$ over $\mathbb{R}$, a copy of $\mathbb{R}^1$

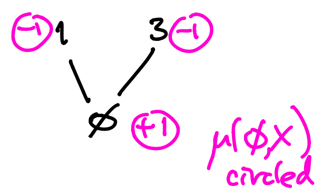
homeomorphic

$\mathcal{M}_{\mathbb{C}}/\mathbb{C}^{\times}$ $\cong$



inside $v_2 \cdot \underline{x} = 1$ over $\mathbb{C}$, a copy of $\mathbb{C}^1$

The affine arrangement has a poset of flats with characteristic polynomial, Poincaré polynomial



$$t-2 = \overline{\chi}_M(t) = \frac{\chi_M(t)}{t-1}$$

$$1+2t = \overline{\pi}_M(t) = \frac{\pi_M(t)}{1+t} = \sum_i t^i \cdot \operatorname{rank}_{\mathbb{Z}} H^i(\mathcal{M}_{\mathbb{C}}/\mathbb{C}^{\times}, \mathbb{Z})$$