How do A-H-K (2015) prove Kähler package for $A(M) = H(\Sigma(M))$, and Braden-Huh-Matherne-Proudfoot-Wang (2020) re-prove it?

Using variations on the ideas from McMullen/Fleming-Karu for $H(\Sigma)$ with $\Sigma = \mathcal{N}(P) = \mathcal{F}(P^\Delta)$

simple polytope

its simplicial polar dual

IDEA 1: The same (easy) base case where

$$A(M) = \mathbb{R}[x]/(x^r) = H\left(\text{normal fan of } (r-1)\text{-simplex}\right)$$

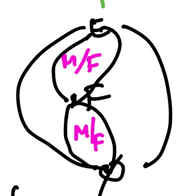
appears in multiple locations within inductions.

IDEA 2: The Local-to-global lemma appears in both proofs when using induction on rank(M).

HRM HL

The BHMPW proof uses it together with a local product decomposition

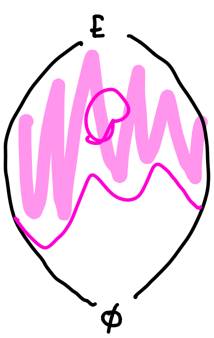
$$\mathrm{star}_{\Sigma(M)}(e_F) \cong \underbrace{\Sigma_{M|_F}}_{\text{restriction to } F} \times \underbrace{\Sigma_{M/F}}_{\text{contraction on } F}$$



In the AHK proof, they use a more complicated inductive structure, relying on more general fans consider by Teichner-Yuzvinsky for each order filter $\mathcal{P} \subseteq L_M - \{\emptyset\}$:

a subset closed under going up, i.e. $Y > X \in \mathcal{P} \Rightarrow Y \in \mathcal{P}$

$\Sigma_{(M,\mathcal{P})}$ has rays $e_1, \ldots, e_n$ and $\{e_F\}_{F \in \mathcal{P}}$



with cones spanned by $\{e_i\}_{i \in I} \cup \{e_{F_1}, \ldots, e_{F_k}\}$ for

$I \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ with F_i flats in $\mathcal{P} - \{E\}$

and $I \notin \mathcal{P}$

(still living in $\mathbb{R}^E / \mathbb{R}e_E = \mathbb{R}^n / \mathbb{R}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$)

Their proof shows the Kähler package for $A_{(M,\mathcal{P})} = H(\Sigma_{(M,\mathcal{P})})$

via double induction, 1st on rank(M), then on $\#\mathcal{P}$.

When using Local-to global LEMMA (HRM, HL), they also use local product decompositions

$$\mathrm{star}_{\Sigma_{(M,\mathcal{P})}}(e_F) \cong \Sigma_{(M|_F,\mathcal{P}|_F)} \times \Sigma_{M/F}$$

$$\mathrm{star}_{\Sigma_{(M,\mathcal{P})}}(e_i) \cong \Sigma_{(M/i,\mathcal{P}/i)}$$

In both proofs, one also applies an easy tensor product lemma

that uses $H(\Sigma \times \Sigma') \cong H(\Sigma) \otimes H(\Sigma')$

to show HL, HRM for $H(\Sigma), H(\Sigma')$

imply the same for $H(\Sigma \times \Sigma')$.

IDEA 3: Once one knows HL holds in a fixed rank(M),

then HRM persists when Σ and hence $H(\Sigma)$ are fixed,

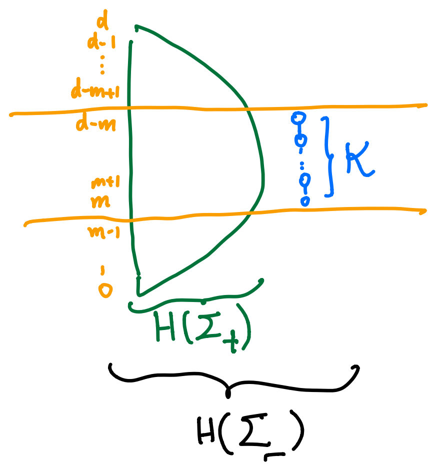
and only $\ell = \ell_t$ varies continuously in t.

This plays a role in particular when they use

the last (and trickiest) idea...

IDEA 4: There are direct sum decompositions, which are orthogonal with respect to the quadratic forms $Q_\ell(-)$, analogous to the one from the McMullen/Fleming-Karu flips $\Sigma_- \rightsquigarrow \Sigma_+$, where recall we had

$$H(\Sigma_-) = H(\Sigma_+) \oplus K$$



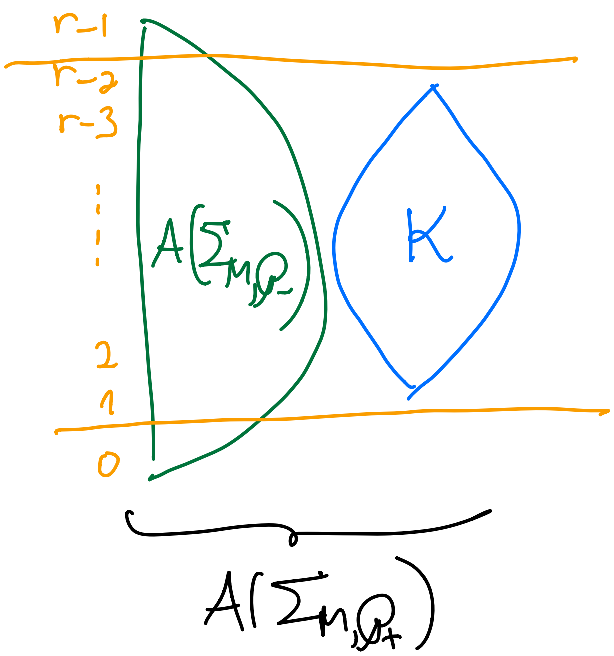
In fact, in both the AHK and BHMPW proofs,

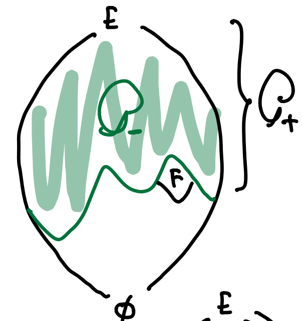
- they rely on AHK's proof of the existence of the degree/evaluation iso. $A^{r-1} \xrightarrow{\sim} \mathbb{R}$ (which either uses some alg. geometry, or the Feichtner-Yuzvinsky Gröbner basis)
- they 1st prove, using their same inductive structures, the Poincaré duality part of the package simultaneously with these decomposition theorems:

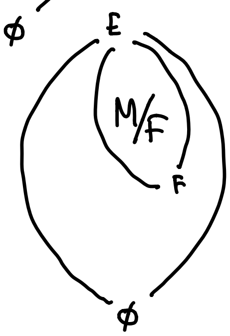
THM (A-H-K) Thm 6.18 When $\mathcal{P}_+ = \mathcal{P}_- \cup \{F\}$, one has

$$A(\Sigma_{M,\mathcal{P}_+}) = A(\Sigma_{M,\mathcal{P}_-}) \oplus K$$

where $K \cong \mathbb{Z}[x]/(x^{r(F)-1}) \otimes A(M/F)$







THM (BHMPW) Thm 1.1 For a non-coloop $i \in E$,

"semismall decomposition"

$$A(M) = A(M\setminus i) \oplus \bigoplus_{\substack{\emptyset \neq F \neq E \\ F,\, F\cup\{i\} \text{ flats of } M}} x_{F\cup\{i\}} A(M\setminus i)$$

where one includes $A(M\setminus i) \hookrightarrow A(M)$

via $x_F \longmapsto x_F + x_{F\cup\{i\}}$

$\forall$ nonempty proper flats F of $M\setminus\{i\}$

interpreted as 0 if they are not flats of M

REMARK: BHMPW also prove, via the same inductive structure, that the Kähler package holds for the augmented Chow ring of M

$$CH(M) := \underbrace{\mathbb{R}\left[\{x_F\}_{\substack{\text{flats} \\ \emptyset \subseteq F \neq E}}, y_1, y_2, \dots, y_n\right] \Big/ \left((x_F x_G)_{\substack{F,G \\ \text{incomparable}}} + (y_i x_F)_{i \notin F} \right)}_{\mathbb{R}[\Delta_\Sigma]} \Big/ \underbrace{\left(y_i - \sum_{F : F \not\ni i} x_F \right)_{i=1,2,\dots,n}}_{(\Theta_\Sigma)}$$

where Σ is the augmented Bergman fan for M

$$\Sigma \subset \mathbb{R}^n = \mathbb{R}^E$$

having cones σ spanned by $\{e_i\}_{i \in I} \cup \{-e_{E \setminus F_1}, \dots, -e_{E \setminus F_k}\}$

for $I \subseteq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \; (\neq E)$

with I independent in M

which then plays an important role in their 2^{nd} paper proving the Dowling-Wilson Top-Heavy Conjecture ...

Math 8680 May 3, 2021 – Wrap-up!

We mentioned last time that BHMPW1 re-proved the Kähler package for the Chow ring $A(M)$ of matroid M, but also for the augmented Chow ring of M

$$CH(M) := \underbrace{\mathbb{R}\left[\{x_F\}_{\substack{\text{flats} \\ \emptyset\subseteq F\neq E}}, y_1, y_2, \dots, y_n\right] \Big/ \underset{\substack{F,G \\ \text{incomparable}}}{(x_F x_G)} + \underset{i\notin F}{(y_i x_F)}}_{\mathbb{R}[\Delta_\Sigma]} \Big/ \underbrace{\left(y_i - \sum_{F:F\not\ni i} x_F\right)_{i=1,2,\dots,n}}_{(\Theta_\Sigma)}$$

where $\Sigma \subset \mathbb{R}^n$ is the augmented Bergman fan for M

with cones σ spanned by $\{e_i\}_{i\in I} \cup \{-e_{E\setminus F_1}, \dots, -e_{E\setminus F_k}\}$
for $I \subseteq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \ (\neq E)$ with I independent in M

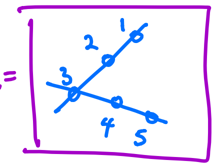
Note the special variables $y_1, y_2, \dots, y_n$ in $CH(M)$, which play several interesting roles ...

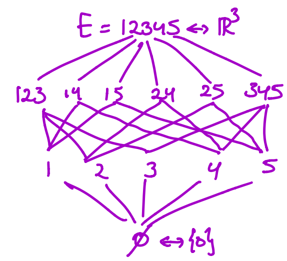
FIRST ROLE

The $y_1,\dots,y_n$ generate a subalgebra of $CH(M)$ isomorphic to the graded Möbius algebra $H(M)$ (from our overview), whose Hilbert series models Whitney numbers of the 2^{nd} kind $(W_0,W_1,W_2,\dots,W_r)$ for the lattice L_M of flats:

$y_F \qquad H(M) :=$ $\mathbb{R}$-vector space on basis $\{y_F\}_{\text{flats } F}$

with $y_F \cdot y_G := \begin{cases} y_{F\vee G} & \text{if } r(F\vee G)=r(F)+r(G) \\ 0 & \text{otherwise} \end{cases}$

$M=$ 

$L_M=$ 

$W_3=1$

$W_2=6$

$W_1=5$

$W_0=1$

$$y_F \mapsto \prod_{i\in I} y_i \qquad H(M) \hookrightarrow CH(M)$$

for any indep. set I with $F=\overline{I}$

$$CH(M) := \mathbb{R}\left[\{x_F\}_{\substack{\text{flats}\\ \emptyset\subseteq F\neq E}}, y_1,y_2,\dots,y_n\right] \Big/ \left(x_F x_G\right)_{\substack{F,G\\ \text{incomparable}}} + \left(y_i x_F\right)_{i\notin F} + \Big(y_i - \sum_{F:F\not\ni i} x_F\Big)_{i=1,2,\dots}$$

SECOND ROLE (less important)

It's not hard to check (2nd part HW #3(c)) that if we let

$$\mathfrak{m} := (y_1,\dots,y_n) \quad \left[= H(M)_+ = \bigoplus_{d\geq 1} H(M)_d\right]$$

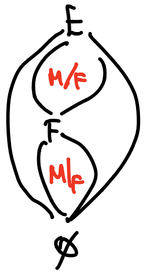
then $\underbrace{A(M)}_{\text{Chow ring of } M} \cong \underbrace{CH(M)}_{\text{Augmented Chow ring of } M} \big/ \mathfrak{m}\, CH(M)$

THIRD (most crucial) ROLE

In BHMPW2, they introduce a crucial $H(M)$-submodule $IH(M) \subseteq CH(M)$, containing $H(M)$,

intersection homology of M

via a recursive definition using the lattice of flats L_M.



But then they prove it has several amazing properties, that prove the Dowling–Wilson Top Heavy Conjecture

$$\left(W_k \leq W_m \text{ if } k \leq m \leq r-k \right)$$

and also give properties of the

matroid Kazhdan Lusztig polynomials $P_M(t)$

matroid Z-polynomials $Z_M(t)$

$\Bigg\} \in \mathbb{Z}[t]$

defined recursively by

- $P_\emptyset(t) = 1$
- $P_M(t)$ has degree $< \frac{r(M)}{2}$
- $Z_M(t) := \sum_{\substack{\text{flats} \\ F}} t^{r(F)} P_{M/F}(t)$ satisfies $t^{r(M)} Z_M\left(\frac{1}{t}\right) = Z_M(t)$

force $Z_M(t)$ to have symmetric coefficient sequence

for which it had been conjectured that

Elias-Proudfoot-Wakefield 2016

$P_M(t)$ has nonnegative coefficients,

$Z_M(t)$ has nonnegative, (symmetric) unimodal coefficients

BHMPW2 prove all of these via a "grand induction"

THEOREM:

$IH(M)$ has a (nondegenerate) Poincaré duality pairing

$$IH^k \times IH^{r-k} \longrightarrow IH^r \cong \mathbb{R}$$

with Lefschetz elements $\underset{\substack{\text{flats } F: \\ \dim F = 1}}{\ell := \sum c_F\, y_F} \in H(M)$ acting on $IH(M)$, $c_F > 0$

satisfying the Kähler package HL, HRM.

In particular, if $k \le m \le r-k$ then

$$IH^k \xrightarrow[\sim]{\cdot \ell^{r-2k}} IH^{r-k} \quad \text{is an isomorphism}$$

so $\quad IH^k \xrightarrow{\cdot \ell^{m-k}} IH^m \quad$ is injective

so $\quad H(M)_k \xrightarrow{\cdot \ell^{m-k}} H(M)_m \quad$ is injective

so $\quad W_k \le W_m \quad$ (top-heaviness)

repeated from overview

THEOREM:

$IH(M)$ has $Z_M(t) = \sum \dim_{\mathbb{R}} IH^k(M) \cdot t^k$

as its Hilbert series, so it has (symmetric) unimodal coefficients, via HL.

THEOREM:

$IH(M) / \mathfrak{m}\, IH(M)$ has $P_M(t)$ as its Hilbert series,

so it has nonnegative coefficients.

$H(M)_+ = (y_1, \ldots, y_n)$

Here's why we didn't do the proof in BHMPW2, and didn't assign it as HW:

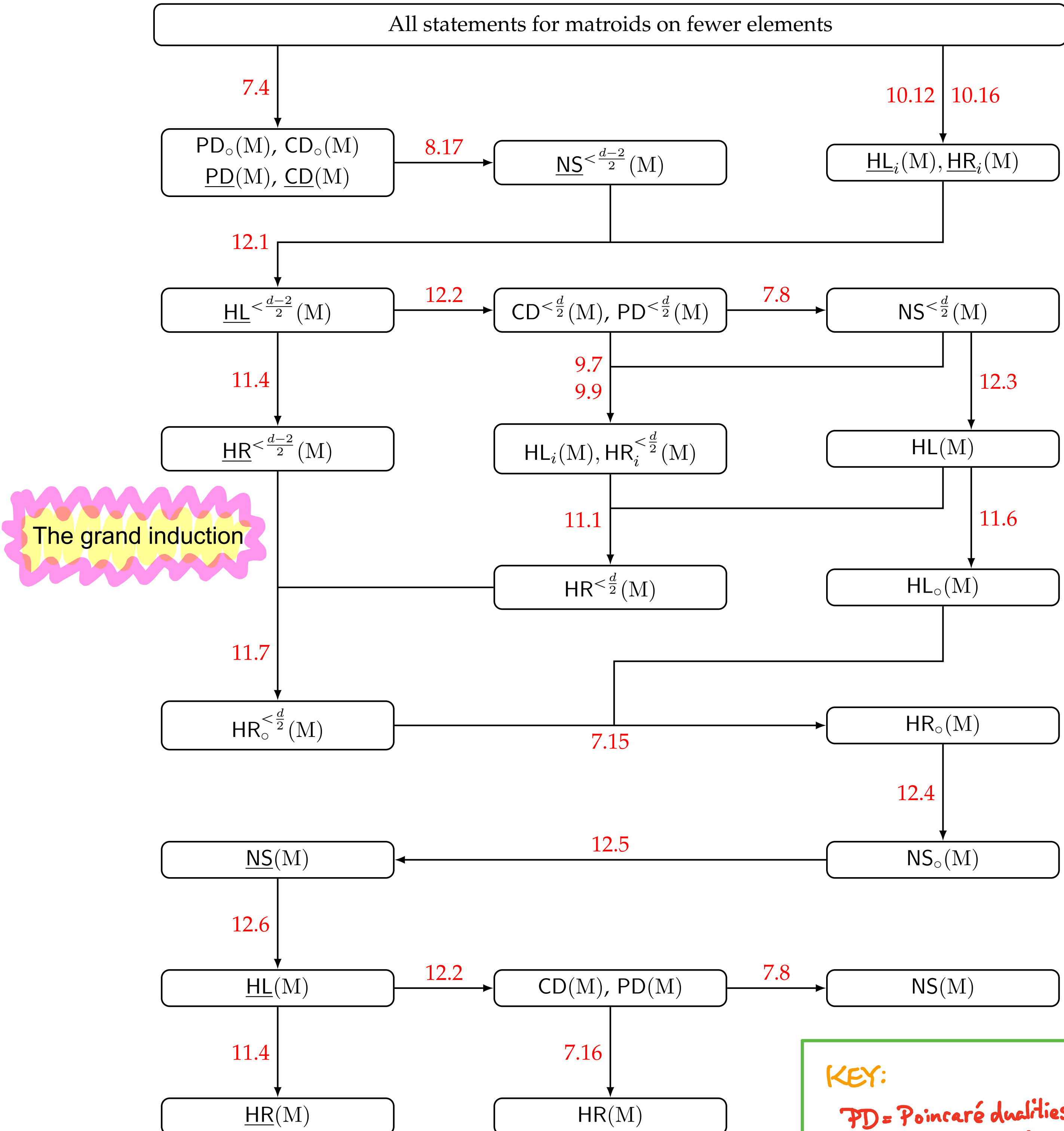


FIGURE 1. Diagram of the proof

KEY:

PD = Poincaré dualities
HL = Hard Lefschetzes
HR = Hodge-Riemann ineqs.
CD = canonical decompositions
NS = no socles in certain degrees

(Our original) PLAN:

- Simplicial complexes & Stanley-Reisner rings
- Simplicial fans & piecewise polynomials
- Fleming-Karu (2018) proof of Kähler package for simplicial polytopes
- Matroids
- Bergman fans & Chow rings
- Sketch of Adiprasito-Huh-Katz proof of Kähler package

- Augmented Bergman fans, Chow rings graded Möbius algebra

Not enough? What to read/watch next?

Videos linked on the syllabus by

- Huh
- Braden
- Eur
- Ardila

more algebro-geometric motivation!

Lorentzian polynomials!

- Adiprasito ← results that give Lefschetz elements satisfying HL (but not HRM) avoiding convexity, for (homology) spheres

Particularly informative intro sections in papers on syllabus:

- A-H-K (5 pages)
- BHMPW2 (13 pages)
- Huh-Wang (4 pages)
- Ardila-Denham-Huh (12 pages)
- Brändén-Huh (6 pages)

Thanks
for hanging
in there!