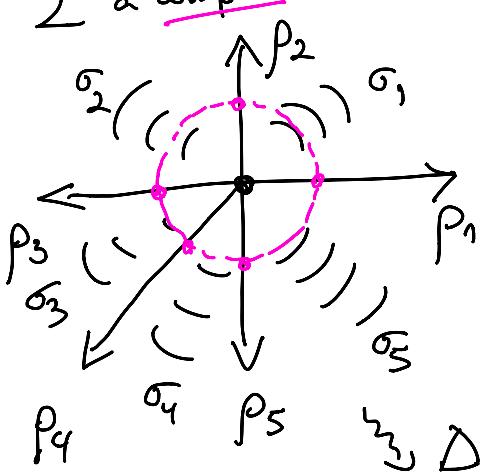
Math 8680 March 8, 2021

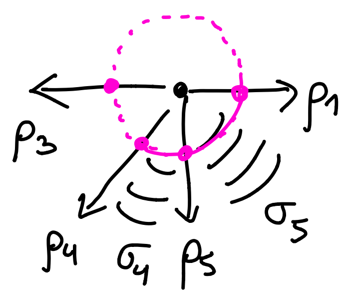
Some loose ends regarding
- fans
- systems of parameters
- Cohen-Macaulay-ness
- Gorenstein-ness

Fans — where have our secret choices of l.s.o.p.'s come from?

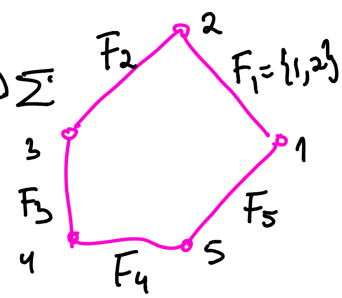
DEF'N: Given a simplicial fan $\Sigma \subset \mathbb{R}^d$ having rays $\rho_1, \rho_2, \ldots, \rho_n$ one can associate a simplicial complex Δ_Σ on vertex set $\{1, 2, \ldots, n\}$ with a face $F \in \Delta$ whenever $\{\rho_i\}_{i \in F}$ span a cone $\sigma = \mathbb{R}_{\geq 0}\rho_{i_1} + \cdots + \mathbb{R}_{\geq 0}\rho_{i_k}$, $F = \{i_1, \ldots, i_k\}$

EXAMPLE: Σ a complete fan in $\mathbb{R}^2$ $\supset$ Σ' a sub-fan of Σ

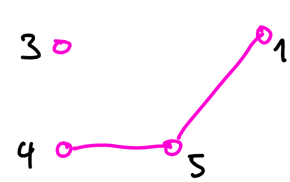




$\rightsquigarrow \Delta_\Sigma$:



$\supset$ $\rightsquigarrow \Delta_{\Sigma'}$ a subcomplex



REMARK: It's not hard to show that

$$\underbrace{\|\Delta_\Sigma\|}_{\substack{\text{geometric}\\ \text{realization}\\ \text{of } \Delta_\Sigma}} \underset{\substack{\text{homeo-}\\ \text{morphic}}}{\cong} \Big(\underbrace{\cup \Sigma}_{\substack{\text{union}\\ \text{of cones}\\ \sigma \text{ in } \Sigma\\ \text{inside } \mathbb{R}^d}}\Big) \cap \underbrace{\mathbb{S}^{d-1}}_{\substack{\text{unitsphere}\\ \text{in } \mathbb{R}^d\\ \text{about } \underline{0}}}$$

In particular, $\|\Delta_\Sigma\| = \mathbb{S}^{d-1}$

$\iff \Sigma$ is a complete fan in $\mathbb{R}^d$

Suppose the rays $\rho_1, \ldots, \rho_n$ in Σ have natural spanning vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$ with $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$

(e.g. $\Sigma = \mathcal{F}(P)$ for P a simplicial polytope with $v_1, \ldots, v_n$ as vertices)

Then for any functional $f \in (\mathbb{R}^d)^*$ get an element $\Theta_f \in K[\Delta_\Sigma]_1$ defined via

$$\Theta_f := \sum_{j=1}^{n} \underbrace{f(v_j)}_{\in \mathbb{R}} \cdot x_j$$

in $K[\Delta_\Sigma] = K[x_1, \ldots, x_n] / I_{\Delta_\Sigma}$

PROP/DEF'N: In the above setting of $\Sigma \subset \mathbb{R}^d$

define $(\underline{\Theta}_\Sigma) := (\Theta_1, \ldots, \Theta_d) \in K[\Delta_\Sigma]_1$ $\boxed{\text{with } K = \mathbb{R}}$

$\Theta_1 = \Theta_{f_1}, \quad \Theta_d = \Theta_{f_d}$

by picking any $\mathbb{R}$-basis $f_1, \ldots, f_d$ for $(\mathbb{R}^d)^*$.

Then $K[\Delta_\Sigma]/(\underline{\Theta}_\Sigma)$ is fin. dim'l over K,

or equivalently, $K[\Delta_\Sigma]$ is a fin. gen'd $K[\Theta_1, \ldots, \Theta_d]$-module

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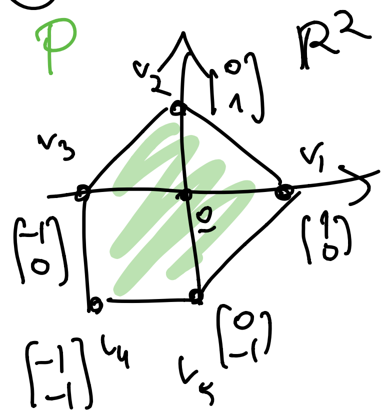
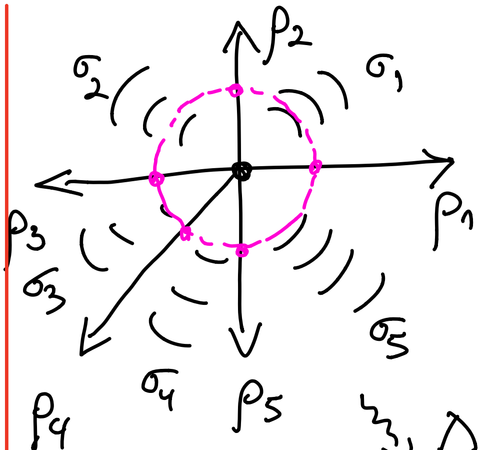
EXAMPLE: P a simplicial polytope

$v_1, \ldots, v_n$ its vertices

$$\Sigma = \mathcal{F}(P) = \text{face fan of } P$$

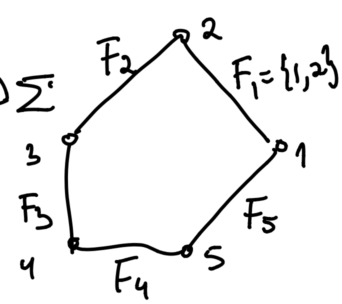
①





$\mathcal{F}(P) = \Sigma$

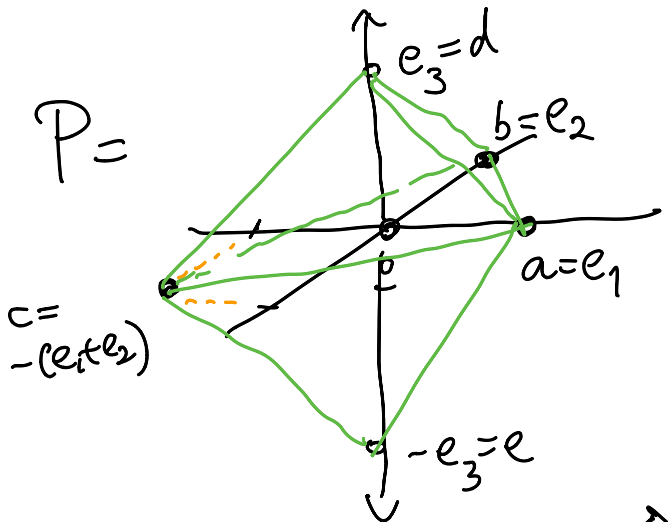
$\leadsto \Delta_\Sigma$



$$\begin{array}{ll} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{array}{l} \Theta_1 = x_1 - x_3 - x_4 \\ \Theta_2 = x_2 - x_4 - x_5 \end{array} & \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

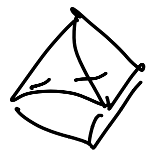
so $K[\Delta_{\Sigma_1}]/(\theta_1,\theta_2)$ is finite-dim'l over K

② $P =$



$\Sigma = \mathcal{F}(P)$

$\Delta_\Sigma =$ bipyramid



$$K[\Delta_\Sigma] = K[a,b,c,d,e]/(de, abc)$$

$$\begin{array}{l} \theta_1 = a-c \\ \theta_2 = b-c \\ \theta_3 = d-e \end{array} \qquad \begin{array}{ccccc} a & b & c & d & e \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$K[\Delta_\Sigma]/(\underline{\theta}_\Sigma) \cong K[a,b,c,d,e]/(de, abc, a-c, b-c, d-e)$$
$$\cong K[c,e]/(c^3, e^2)$$

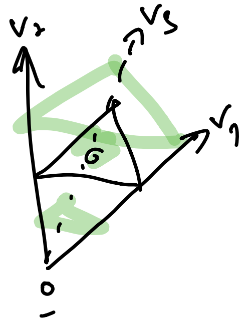
<u>proof of PROP/DEF'N:</u> Recall $(\theta_1, \ldots, \theta_d) \in K[\Delta]_1$

have $K[\Delta]/(\underline{\theta})$ fin. dim'l over K

$\iff$ for every face F of Δ, $(\theta_1|_F, \ldots, \theta_d|_F)$ have K-span of rank $\#F$

$\iff$ for every cone $\sigma \in \Sigma$, we have $\left(f_i(v_j) \right)_{\substack{i=1,\ldots,d \\ j=1,\ldots,s}}$ is a matrix of rank $\underline{\underline{s = \#F}}$

$\sigma = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_s$ (i.e. maximal rank)



$\iff$ $v_1, v_2, \ldots, v_s \in \mathbb{R}^d$ are $\mathbb{R}$-linearly indep.

(since $f_1, f_2, \ldots, f_d$ are a <u>basis</u> for $(\mathbb{R}^d)^*$) $\boxed{\Xi}$

<u>REMARK</u>: The choice of $f_1, \ldots, f_d$ as a basis of $(\mathbb{R}^d)^*$, makes no difference in the K-span of $\theta_1, \ldots, \theta_d \in K[\Delta]_1$ or the ideal $(\underline{\theta}_\Sigma) = (\theta_1, \ldots, \theta_d)$ within $K[\Delta]$ or subalgebra

Math 8680 March 10, 2021

Systems of parameter & Krull dimension

DEF'N: Given R a fin. gen'd (commutative) K-algebra the Krull dimension

$$\dim(R) := \max\{d : \exists\ \theta_1, \theta_2, \ldots, \theta_d \in R \text{ algebraically indep.}\}$$

i.e. $\nexists\ f(y_1, \ldots, y_d) \in K[y_1, \ldots, y_d]$ with $f(\theta_1, \ldots, \theta_d) = 0$ in R

i.e. $K[y_1, \ldots, y_d] \longrightarrow R$ is injective.
$y_i \longmapsto \theta_i$

(easy)
PROP: If in addition $R = \bigoplus_{k=0}^{\infty} R_k$ is an $\mathbb{N}$-graded K-algebra,

then $\dim(R) = \max\{d : \exists\ \theta_1, \theta_2, \ldots, \theta_d \in R \text{ homogeneous and algebraically indep.}\}$

$=$ order of the pole at $t=1$ in $\mathrm{Hilb}(R,t) \in \mathbb{Z}[[t]]$

$=$ smallest d such that $\lim_{t \to 1} (1-t)^d\, \mathrm{Hilb}(R,t)$ exists.

EXAMPLE: If Δ is a $(d-1)$-dim'l simplicial complex

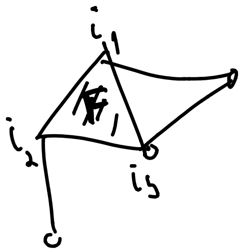
then $\dim(K[\Delta]) = d = \dim(\Delta)+1$

since $$\mathrm{Hilb}(K[\Delta], t) = \frac{h_0 + h_1 t + \ldots + h_d t^d}{(1-t)^d}$$

with $\lim_{t \to 1} \sum_{k=0}^{d} h_k t^k = h_0 + h_1 + \ldots + h_d = f_{d-1} > 0.$

Also, $\exists\ \underset{\underset{x_{i_1}}{\shortparallel}}{\Theta_1}, \ldots, \underset{\underset{x_{i_d}}{\shortparallel}}{\Theta_d}$ homogeneous linear and alg. indep in $K[\Delta]$

where $F = \{i_1, \ldots, i_d\}$ is any $(d-1)$-dim'l face $F \in \Delta$



<u>DEF'N:</u> If $d = \dim(R)$, then call homog. elements $\Theta_1, \ldots, \Theta_d$ a <u>homogeneous system of parameters</u> (h.s.o.p.)

if $R/(\underline{\Theta}) = R/(\Theta_1, \ldots, \Theta_d)$ is of Krull dimension 0, i.e. is fin. dim'l over K

$\iff$ R is a fin. gen'd $K[\underline{\Theta}] = K[\Theta_1, \ldots, \Theta_d]$-module.

<u>REMARK:</u> One can show that this forces $\Theta_1, \ldots, \Theta_d$ to be algebraically indep. inside R, but this isn't sufficient to give an h.s.o.p.

<u>EXAMPLES:</u> $R = K[x,y]$

$(\Theta_1, \Theta_2) = (x, y)$ are a l.s.o.p. for R (linear) h.s.o.p.

$(\Theta_1, \Theta_2) = (x^n, y^m)$ are an h.s.o.p.

since $R/(x^n, y^m)$ has K-basis $\{x^i y^j\}_{\substack{0 \le i < n \\ 0 \le j < m}}$

but $(\theta_1, \theta_2) = (x, xy)$ are alg. indep.,
and <u>not</u> an h.s.o.p.
since $R/(\theta_1, \theta_2) = K[x,y]/(x, xy)$
$\cong K[x,y]/(x)$
$\cong K[y]$

(not so hard)
THEOREM (Noether's normalization lemma)
(a) Every fin'ly gen'd (comm.) K-algebra R <u>has</u> an h.s.o.p. $\theta_1, \ldots, \theta_d$
graded

(b) If R is gen'd over K by R_1 (sometimes called a <u>standard graded K-algebra</u>)

then one can pick an l.s.o.p. $\theta_1, \ldots, \theta_d \in R_1$
linear

assuming $\#K$ is sufficiently large (e.g. $\#K = \infty$ is big enough)

<u>EXAMPLE</u>: When $R = K[\Delta]$, we've seen that
$(\underline{\theta}) = (\theta_1, \ldots, \theta_d) \in R_1$ are an l.s.o.p.
$d = \dim \Delta + 1$

$\iff (\underline{\theta}|_F)$ has K-span of dimension same as $\#F$
$\forall$ faces $F \in \Delta$
(or $\forall$ facets $F \in \Delta$)

Some h.s.o.p.'s and rings are better than others...

DEF'N: For an R-module M, say $(\underline{\Theta}) = (\Theta_1, \ldots, \Theta_k) \in R$ is an <u>M-regular sequence</u>

depends on the ordering $\Theta_1, \Theta_2, \ldots$

if Θ_1 is a <u>non zero divisor</u> (NZD) on M

i.e. $\mathrm{Ann}_M(\Theta_1) = 0$

$:= \{ m \in M : \Theta_1 \cdot m = 0 \text{ in } M \}$

and Θ_2 is an NZD on $M/\Theta_1 M$

Θ_3 — " — $M/(\Theta_1 M + \Theta_2 M)$

$\vdots$

Θ_k — " — $M/(\Theta_1 M + \cdots + \Theta_{k-1} M)$

PROP: When $M \neq 0$ is an $\mathbb{N}$-graded R-module for R an $\mathbb{N}$-graded (comm.) K-algebra,

then $(\underline{\Theta}) = \underset{\in R}{\overset{\text{homogeneous}}{(\Theta_1, \ldots, \Theta_k)}}$ is an M-regular sequence

Do not depend on the order $\Theta_1, \Theta_2, \ldots$

$$\iff \mathrm{Hilb}(M,t) = \prod_{i=1}^{k} \frac{1}{(1-t^{\deg \Theta_i})} \, \mathrm{Hilb}(M/(\Theta_1 M + \cdots + \Theta_k M), t)$$

$\iff$ $\Theta_1, \ldots, \Theta_k$ are alg. indep. in R and M is a <u>free</u> $K[\Theta_1, \ldots, \Theta_k]$-module

In particular, when $(\underline{\Theta}) = (\Theta_1, \ldots, \Theta_k)$ are an R-regular sequence,

$\dim(R/(\underline{\Theta})) = \dim R - k$

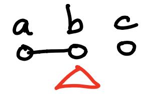
Math 8680 March 12, 2021

Recall for R a comm. graded K-algebra ...

$\dim R := \max \{ d : \exists\, \theta_1, \ldots, \theta_d$ homog. in R algeb. indep. $\}$

Krull dim

$= $ order of pole at $t=1$ in $\mathrm{Hilb}(R,t)$

e.g. $R = K[$  $] = K[a,b,c]/(ac,bc)$

has $\dim R = 2$, e.g. $(\theta_1, \theta_2) = (a,b)$ are alg. indep. inside R

$\dim K[\Delta] = \dim \Delta + 1$

$$\mathrm{Hilb}(R,t) = \frac{h_0 + h_1 t + h_2 t^2}{(1-t)^2} = \frac{1+t-t^2}{(1-t)^2}$$

$f_{-1}\ f_0\ f_1$

$\underline{f} = (1,3,1)$

$1 = f_{-1}$

$3 = f_0$

$1 = f_1$

$1 \quad 1 \quad 2$

$(1, 1, -1) = \underline{h}$

$h_0 \quad h_1 \quad h_2$

$(\theta_1, \ldots, \theta_d)$ for $d = \dim R$ homog. are called an <u>h.s.o.p.</u>

if $R/(\underline{\theta}) = R/(\theta_1, \ldots, \theta_d)$ is fin. dim'l K

or equivalently, Krull dim 0.

... and $\theta_1, \ldots, \theta_k$ are a <u>regular sequence</u> on an R-module M

if θ_1 is an NZD on M

θ_2 — " — on $M/\theta_1 M$

$\vdots$

θ_k — " — on $M/(\theta_1 M + \theta_2 M + \cdots + \theta_{k-1} M)$

PROP: For $M \neq 0$, $(\underline{\theta}) = (\theta_1, \ldots, \theta_k)$ is an M-regular sequence

$$\Longleftrightarrow \operatorname{Hilb}(M,t) = \underbrace{\prod_{i=1}^{k} \frac{1}{1-t^{\deg \theta_i}}}_{\operatorname{Hilb}(K[\theta_1,\ldots,\theta_k],t)} \operatorname{Hilb}(M/\theta_1 M + \cdots + \theta_k M), t)$$

$\Longleftrightarrow$ M is a free $K[\theta_1, \ldots, \theta_k]$-module and $\theta_1, \ldots, \theta_k$ are alg. indep. inside R

In particular, if θ is a NZD on R then $\dim(R/(\theta)) = \dim R - 1$

if $\theta_1, \ldots, \theta_k$ is an R-regular sequence then $\dim(R/(\theta_1, \ldots, \theta_k)) = \dim R - k$

proof: The first $\Leftrightarrow$ comes from iterating this:

LEMMA: θ homog. in R has

$$\operatorname{Hilb}(M,t) \leq \underbrace{\frac{1}{1-t^{\deg\theta}}}_{(1+t^{\deg\theta}+t^{2\deg\theta}+\cdots)} \operatorname{Hilb}(M/\theta M, t) \quad \text{in } \mathbb{Z}[[t]]$$

coefficient wise in $\mathbb{Z}[[t]]$

with equality $\Leftrightarrow$ θ is a NZD on M.

proof: Let's prove ...

$$\boxed{\mathrm{Hilb}(M,t) \overset{(*)}{=} \frac{1}{1-t^{\deg\theta}}\left(\mathrm{Hilb}(M/\theta M,t) - t^{\deg\theta}\,\mathrm{Hilb}(\underbrace{\mathrm{Ann}_M(\theta)}_{0 \iff \theta \text{ is an NZD}},t)\right)}$$

by constructing the ker-coker exact sequence

$$0 \to \ker(f) \to A \xrightarrow{f} B \to B/f(A) \to 0$$

for

$$0 \to \mathrm{Ann}_M(\theta)(-\delta) \to \underbrace{M(-\delta)}_{} \xrightarrow{\cdot\theta} M \to M/\theta M \to 0$$

$$m \longmapsto \theta m \qquad \text{if } \deg\theta = \delta$$

same module M,
but everyone's degree shifted up by δ

EXERCISE:

$$0 \to M_0 \to M_1 \to \cdots \to M_k \to 0 \quad \text{exact, graded maps of graded vector spaces}$$

$$\Longrightarrow \sum_{i=0}^{k} (-1)^i\, \mathrm{Hilb}(M_i,t) = 0$$

since $0 \to V_0 \to V_1 \to \cdots \to V_k \to 0$ exact as vector spaces

$$\Longrightarrow \sum_{i=0}^{k} (-1)^i \dim_k(V_i) = 0$$

This gives

$$t^\delta \operatorname{Hilb}(\operatorname{Ann}_M(\theta)) - t^\delta \operatorname{Hilb}(M) + \operatorname{Hilb}(M) - \operatorname{Hilb}(M/\theta M) = 0$$

$$\operatorname{Hilb}(M)(1-t^\delta) = \operatorname{Hilb}(M/\theta M) - t^\delta \operatorname{Hilb}(\operatorname{Ann}_M(\theta))$$

equivalent to $(*)$.

Let's leave the proof of the 2nd $\Longleftarrow$ for office hour. $\blacksquare$

DEF'N: Say the graded (comm.) K-algebra R is Cohen-Macaulay if $\exists\ \underline{\theta} = (\theta_1, \ldots, \theta_d)$ in R an h.s.o.p. which is also an R-regular sequence.

(In fact, one can define

$$\operatorname{depth}(R) := \max\{k : \exists \text{ an } R\text{-regular sequence of length } k\}$$

and then R is Cohen-Macaulay $\iff \operatorname{depth}(R) = \underset{\uparrow \text{ Krull dim}}{\dim(R)}$

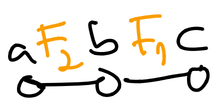
($\leq$ always holds)

(Nontrivial, surprising...)

THM: When R is Cohen-Macaulay, every h.s.o.p. is an R-regular sequence. (!)

Part of "the magic of C-M-ness"

EXAMPLES:

① $K\Big[$  $\Big] = K[a,b,c]/(ac)$

Δ

is Cohen-Macaulay since Δ is shellable

so any l.s.o.p. $\underline{\Theta} = (\theta_1, \theta_2)$

e.g. $\theta_1 = a+b+c$, $\theta_2 = b$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

has $K[\Delta]$ a free $K[\theta_1,\theta_2]$-module with basis $\{1, a\}$ from the shelling

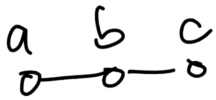
$$\mathrm{Hilb}(K[\Delta],t) = \frac{1+t}{(1-t)^2} = \mathrm{Hilb}(K[\theta_1,\theta_2])(1+t)$$

But this means <u>any</u> hsop (θ_1,θ_2) should be a $K[\Delta]$-regular sequence,

e.g. $\theta_1 = a+b+c \longleftarrow \deg 1$

$\theta_2 = ab+bc \longleftarrow \deg 2$

$$K[\Delta]/(\theta_1,\theta_2) = K[a,b,c]/(ac,\ a+b+c,\ ab+bc)$$



$b = -(a+c)$

$$\cong K[a,c]/\left(ac,\ -\underbrace{(a+c)^2}_{-(a^2+c^2+\overset{0}{\cancel{2ac}})}\right)$$

$$\cong K[a,c]/(ac,\ a^2+c^2)$$

$$= K\text{-span of } \{\underset{0}{1},\ |\ \underset{1}{a, c},\ |\ \underset{2}{a^2}\}$$

$c^3 \equiv a^2\cdot c$
$\equiv 0$

$$\mathrm{Hilb}(K[\Delta]/(\Theta), t) = 1+2t+t^2$$

$$\mathrm{Hilb}(K[\Delta], t) = \frac{1+t}{(1-t)^2} = \frac{1+2t+t^2}{(1-t^{1})(1-t^{2})}$$

$\deg \Theta_i$

Math 8680 March 15, 2021

R graded comm. K-algebra

Noether's Normalization Lemma

$\Rightarrow$ any partial h.s.o.p. $(\Theta_1, \ldots, \Theta_k)$ in R
(i.e. $\dim R/(\Theta_1, \ldots, \Theta_\ell) = \dim R - \ell$)
extends to a (full) h.s.o.p. i.e.
$\dim(R/(\Theta_1, \ldots, \Theta_d)) = 0$
and $\dim R = d$.

$(\Theta_1, \ldots, \Theta_k)$ is an R-regular seq. if ....
and $\mathrm{depth}(R) =$ length of longest R-regular sequence.

R is C-M if $\exists$ an hsop which is also an R-regular sequence,

i.e. $\mathrm{depth}(R) = \dim(R)$

($\leq$ always)

In fact, every R-regular sequence can be extended to one of length $\mathrm{depth}(R)$.

more general "magic of C-M-ness & depth"

EXAMPLES:

① $K[a - b - c]$ is C-M and

and $\theta_1 = a+b+c$, $\theta_2 = b$

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

is an hsop & regular seq.

but also $\theta_1 = a+b+c$, $\theta_2 = ab+bc$ is an hsop, and therefore a regular sequence

② $K[\overbrace{a - b \quad c}^{\Delta}]$ is not C-M since for any (θ_1, θ_2) an l.s.o.p. one will not have

$\underline{f} = (1,3,1)$

$\underline{h} = (1,1,-1)$

$$\mathrm{Hilb}(K[\Delta]/(\underline{\Theta}),t) = (1-t)^2 \cdot \underbrace{\mathrm{Hilb}(K[\Delta],t)}_{= \frac{1+t-t^2}{(1-t)^2}}$$

$$= 1+t-t^2$$

e.g. $\Theta_1 = a+b+c$
$\Theta_2 = b$

$$\begin{array}{ccc} a & b & c \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is an l.s.o.p. for $K[\Delta]$,
but not a regular seq.

(Hw2 asks you to check
$\sum_{i=1}^{n} x_i$ is always a NZD on $K[\Delta]$ if $\Delta \neq \{\emptyset\}$
but if Δ is disconnected
then $\operatorname{depth} K[\Delta] = 1$)

Reisner's Theorem

Q: What makes $K[\Delta]$ C-M ?

It relates to (reduced) simplicial homology with coefficients in K for links of faces F in Δ

$\tilde{H}_i(-;K) = K^{\beta_i}$ where β_i "=" # of i-dim'l holes in space

e.g. $\tilde{H}_0(-;K) = K^{\#(\text{conn. comps})-1}$

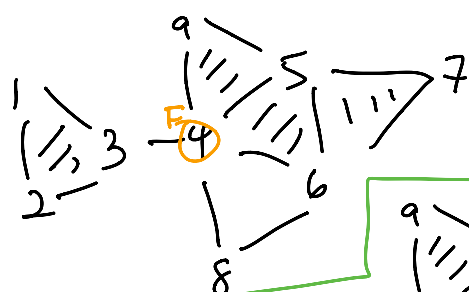
DEF'N: For F a face of Δ,

$$\text{star}_\Delta(F) := \{G \in \Delta : G \cup F \in \Delta\}$$

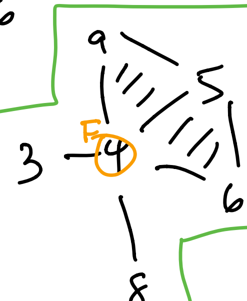
$$\cup$$

$$\text{link}_\Delta(F) := \{G \in \Delta : G \sqcup F \in \Delta\}$$

i.e. $\begin{cases} G \cup F \in \Delta \\ G \cap F = \emptyset \end{cases}$

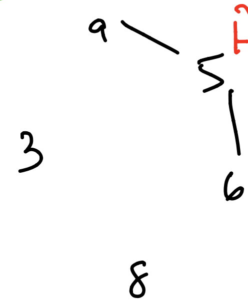
e.g. $\Delta =$ 

$$\tilde{H}_i(\Delta;K) = \begin{cases} 0 & \text{if } i > 2 \\ 0 & \text{if } i = 2 \\ K^1 & \text{if } i = 1 \\ K^0 = 0 & \text{if } i = 0 \\ 0 & \text{if } i = -1 \end{cases}$$

has $\text{star}_\Delta(\{4\}) =$ 

$\tilde{H}_i(\text{star}_\Delta(F);K) = 0 \ \forall i$ since they're cones, so contractible.

$\cup$

$\text{link}_\Delta(\{4\}) =$ 

$$\tilde{H}_i((\text{link}_\Delta\{4\})) = \begin{cases} 0 & \text{if } i > 0 \\ K^2 & \text{if } i = 0 \\ 0 & \text{if } i = -1 \end{cases}$$

THM (Reisner 1976) $K[\Delta]$ is C-M

$\iff \tilde{H}_i(\Delta; K) = 0$ for $i < \dim \Delta$

$\tilde{H}_i(\text{link}_\Delta(\emptyset); F) =$

and $\forall$ faces $F \in \Delta$

$\tilde{H}_i(\text{link}_\Delta(F); K) = 0$ for $i < \dim(\text{link}_\Delta(F))$

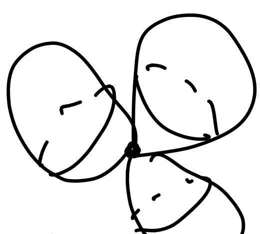
EXAMPLES:

(1) Δ (pure) shellable $\Rightarrow \tilde{H}_i(\Delta; K) = 0 \ \forall i < \dim \Delta$

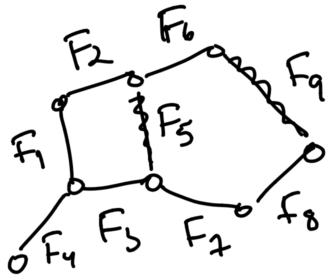
via an easy Mayer-Vietoris argument

(In fact, one can show shellability implies

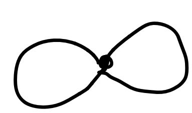
$$\|\Delta\| \underset{\text{homotopy equiv.}}{\approx} \underbrace{\mathbb{S}^d \vee \mathbb{S}^d \vee \dots \vee \mathbb{S}^d}_{\substack{\text{a wedge of spheres,}\\ \text{as many as } \#\{i : G_i = F_i\} \\ \text{in the shelling}}} \quad \text{if } d = \dim \Delta$$
)



$\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2$



$\approx \mathbb{S}^1 \vee \mathbb{S}^1$



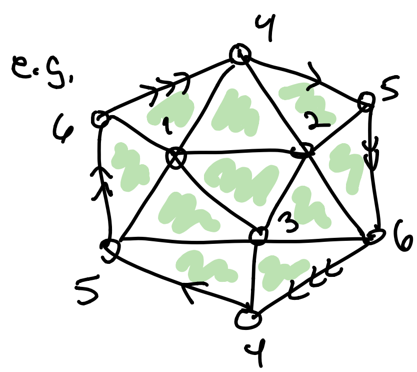
$\underset{F_1}{\circ} \quad \underset{F_2 = G_2}{\circ} \quad \underset{F_3 = G_3}{\circ} \quad \underset{F_4 = G_4}{\circ} \quad \approx \quad \mathbb{S}^0 \vee \mathbb{S}^0 \vee \mathbb{S}^0$

<u>AND</u> (HW 3) Δ shellable
$\Rightarrow$ $\mathrm{star}_\Delta(F)$, $\mathrm{link}_\Delta(F)$ shellable $\forall$ faces F

(2) 1-dim'l Δ have $K[\Delta]$ C-M
$\Leftrightarrow$ Δ shellable
$\Leftrightarrow$ Δ connected
regardless of K

(3) When $\dim \Delta \geq 2$, it can depend upon K:
e.g. if $\|\Delta\| \underset{\text{homeomorphic to}}{\cong} \mathbb{RP}^2$ = real projective plane

e.g.



then $\tilde{H}_i(\Delta;K) = \begin{cases} K & \text{for } i=1,2 \text{ when } \mathrm{char}(K)=2 \\ 0 & \text{else} \end{cases}$

so not C-M if $\mathrm{char}(K)=2$

but $\tilde{H}_i(\Delta;K)=0$ $\forall i$ when $\mathrm{char}(K)\neq 2$
and can check it's C-M if $\mathrm{char}(K)\neq 2$.

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④ If $\|\Delta\| \underset{\text{homeomorphic to}}{\cong} \mathbb{S}^{d-1}$, shellable or not,

then $K[\Delta]$ is C-M (independent of K)

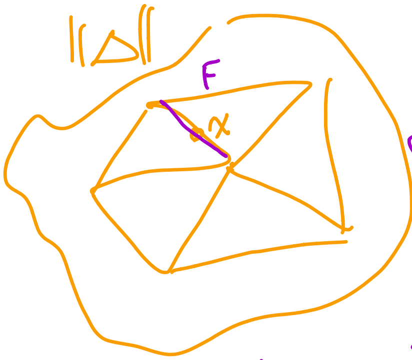
because $\tilde{H}_i(\Delta;K) = \begin{cases} 0 & \text{if } i < d-1 \\ K & \text{if } i = d-1 \end{cases}$

and one can show

$$\tilde{H}_i(\mathrm{link}_\Delta F;K) = \tilde{H}_i(\mathbb{S}^{\dim(\mathrm{link}_\Delta F)};K)$$

⑤ THM: (Munkres 1984) Using similar ideas,

$K[\Delta]$ is C-M $\iff$



$$\begin{cases} \tilde{H}_i(\|\Delta\|;K) = 0 \text{ for } i < \dim(\Delta) \\ \underbrace{\tilde{H}_i(\|\Delta\|, \|\Delta\|-x;K)}_{\text{called the local homology at the point } x} = 0 \text{ for } i < \dim(\Delta) \text{ and } \forall x \in \|\Delta\| \end{cases}$$

depends only on $\|\Delta\|$ not on triangulation.

Gorenstein-ness & Poincaré duality

DEF'N: Recall the (reduced) Euler characteristic

$$\tilde{\chi}(\Delta) := \sum_{i \geq -1} (-1)^i f_i = -f_{-1} + f_0 - f_1 + f_2 - f_3 + \dots + (-1)^d f_d$$

Euler-Poincaré Thm.

$$= \sum_{i \geq -1} (-1)^i \underbrace{\tilde{\beta}_i(\Delta; K)}_{\dim_K \tilde{H}_i(\Delta; K)}$$

e.g. $\tilde{\chi}(\mathbb{S}^d) = (-1)^d$

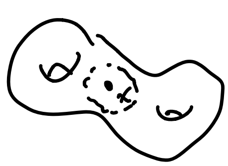
since $\tilde{H}_i(\mathbb{S}^d) = \begin{cases} K & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$

Call Δ an Euler complex if $\begin{cases} \Delta \text{ is pure} \\ \text{AND} \\ \tilde{\chi}(\mathrm{link}_\Delta(F)) = \tilde{\chi}(\mathbb{S}^{\dim(\mathrm{link}_\Delta(F))}) \\ \quad = (-1)^{\dim(\mathrm{link}_\Delta(F))} \end{cases}$

$\forall$ faces $F \in \Delta$

e.g. $\|\Delta\| = \mathbb{S}^d$ is an Euler complex

also $\|\Delta\|$ any (compact) odd-dimensional manifold is an Euler complex



Call Δ a Gorenstein* complex (over K) (or a K-homology sphere) if $\begin{cases} K[\Delta] \text{ is C-M} \\ \text{and } \Delta \text{ is an Euler complex} \end{cases}$

$\iff \tilde{H}_i(\mathrm{link}_\Delta(F); K) = \tilde{H}_i(\mathbb{S}^{\dim(\mathrm{link}_\Delta(F))}; K)$

$\forall$ faces $F \in \Delta$

e.g. $\|\Delta\| = \mathbb{S}^d$ a sphere is Gorenstein* over K for all fields K

Euler complexes have a reciprocity relation for $K[\Delta]$'s finely graded Hilbert series...

PROP: (a) For any Δ,

$$\mathrm{Hilb}(K[\Delta], \underset{t_1,\dots,t_n}{\underline{t}})\Big|_{\substack{t_i = 1/t_i \\ \forall i=1,\dots,n}} = \sum_{G\in\Delta} \prod_{j\in G} \frac{t_j}{1-t_j} (-1)^{\#G+1} \tilde{\chi}(\mathrm{link}_\Delta G)$$

(b) Δ is an Euler complex $\iff$

$$\left\{ \begin{array}{l} \Delta \text{ is pure} \\ \quad \text{AND} \\ \mathrm{Hilb}(K[\Delta], \underline{t})\Big|_{t_i = 1/t_i} = (-1)^{\dim\Delta+1} \mathrm{Hilb}(K[\Delta], \underline{t}) \end{array} \right.$$

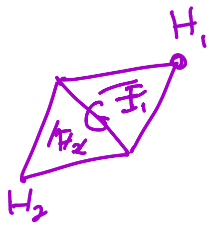
proof: (a)

$$\mathrm{Hilb}(K[\Delta], \underline{t})\Big|_{t_i = 1/t_i} = \sum_{F\in\Delta} \prod_{j\in F} \frac{t_j^{-1}}{1-t_j^{-1}} \cdot \frac{t_j}{t_j}$$

$$= \sum_{F\in\Delta} \prod_{j\in F} \frac{1}{\underbrace{t_j - 1}_{-\left(1+\frac{t_j}{1-t_j}\right)}}$$

$$= \sum_{F\in\Delta} (-1)^{\#F} \prod_{j\in F} \left(1 + \frac{t_j}{1-t_j}\right)$$

$$= \sum_{F\in\Delta} (-1)^{\#F} \sum_{G : G\subseteq F} \prod_{j\in G} \frac{t_j}{1-t_j}$$



$$= \sum_{G\in\Delta} \prod_{j\in G} \frac{t_j}{1-t_j} \sum_{\substack{F\in\Delta: \\ G\subseteq F}} (-1)^{\#F}$$

$H = F \setminus G$

$$= \sum_{G\in\Delta} \prod_{j\in G} \frac{t_j}{1-t_j} (-1)^{\#G+1} \underbrace{\sum_{H\in \mathrm{link}_\Delta(G)} (-1)^{\#H-1}}_{\tilde{\chi}(\mathrm{link}_\Delta(G))}$$

(b) Δ is an Euler complex $\iff$ Δ pure and reciprocity

since

$$\mathrm{Hilb}(K[\Delta], \underline{t})\Big|_{t_j = 1/t_j} \overset{\text{reciprocity}}{=} (-1)^{\dim\Delta+1} \underbrace{\mathrm{Hilb}(K[\Delta], \underline{t})}_{\displaystyle \sum_{G\in\Delta} \prod_{j\in G} \frac{t_j}{1-t_j}}$$

$$\mathrm{Hilb}(K[\Delta], \underline{t})\Big|_{t_j = 1/t_j} = \sum_{G\in\Delta} \prod_{j\in G} \frac{t_j}{1-t_j} (-1)^{\#G+1} \tilde{\chi}(\mathrm{link}_\Delta F)$$

$\Updownarrow$ since $\prod_{j\in G} \frac{t_j}{1-t_j}$ are lin. indep in $\mathbb{Z}[[t]]$ or $\mathbb{Q}[[t]]$

$$\tilde{\chi}(\mathrm{link}_\Delta F) = (-1)^{\dim\Delta - \#G} \quad \forall\, G\in\Delta$$

$$\overset{\Delta \text{ is pure}}{=} (-1)^{\dim(\mathrm{link}_\Delta G)}$$

, the Euler complex condition ∎

PROP: If Δ is an Euler complex, then

$$h_k(\Delta) = h_{d-k}(\Delta) \quad \text{where } d = \dim(\Delta)-1$$

$$\forall k$$

(Dehn-Sommerville eqns)

proof: Δ an Euler complex $\Longrightarrow$

$$\mathrm{Hilb}(K[\Delta], \underline{t})\Big|_{t_j = 1/t_j} = (-1)^{\dim \Delta + 1}\, \mathrm{Hilb}(K[\Delta], \underline{t})$$

$$\Big\Downarrow \; t_j = t \;\; \forall j$$

$$\mathrm{Hilb}(K[\Delta], t)\Big|_{t = 1/t} = (-1)^d\, \mathrm{Hilb}(K[\Delta], t)$$

$$\mathrm{Hilb}(K[\Delta], t)\Big|_{t = 1/t} = \frac{t^d}{t^d} \cdot \frac{h_0 + h_1 t^{-1} + \dots + h_d t^{-d}}{(1-t^{-1})^d} = \cancel{(-1)^d}\, \frac{h_0 t^d + h_1 t^{d-1} + \dots + h_d}{\cancel{(1-t)^d}}$$

$$(-1)^d\, \mathrm{Hilb}(K[\Delta], t) = \cancel{(-1)^d}\, \frac{h_0 + h_1 t + \dots + h_d t^d}{\cancel{(1-t)^d}}$$

compare coeffs on t^k on both sides $\blacksquare$