Math 8680 March 29, 2021

The evaluation map for complete fans

Since $R_\Sigma / A_+ R_\Sigma \xleftarrow[\varphi]{\sim} \mathbb{R}[\Delta_\Sigma] / (\underline{\Theta}_\Sigma)$,

$\quad\quad\; \|$ Fleming–Karu 2010

$H(\Sigma)$: the cohomology of the fan

we know that for Σ simplicial fan in $\mathbb{R}^d$,

- $H(\Sigma)$ is $\mathbb{R}$-spanned by $\{g_\sigma\}_{\sigma\in\Sigma}$
  $\quad\quad\updownarrow$
  $\{\underline{x}^G\}_{G\in\Delta_\Sigma}$

  and is gen'd as an $\mathbb{R}$-algebra by $\{g_{\rho_i}\}_{i=1,\dots,n}$
  $\quad\quad\updownarrow$
  $\{x_i\}_{i=1,\dots,n}$

- If Δ_Σ is <u>shellable</u>, then $\Delta_\Sigma = \bigsqcup_{i=1}^{s} [G_i, F_i]$

  $\Rightarrow H(\Sigma)$ has $\mathbb{R}$-basis $\{g_{\sigma_i}\}_{i=1,\dots,s}$
  $\quad\quad\updownarrow$
  $\{\underline{x}^{G_i}\}_{i=1,\dots,s}$

- In particular, if Σ is <u>complete in $\mathbb{R}^d$</u> simplicial (and shellable)

  so $|\Delta_\Sigma| \cong \mathbb{S}^{d-1}$ (e.g. $\Sigma = \mathcal{F}(P)$, P a simplicial polytope) then $H(\Sigma) = H^0 \oplus H^1 \oplus \cdots \oplus H^{d-1} \oplus H^d$ with $H^0 = \mathbb{R}$ and $H^d \cong \mathbb{R}^1$ via ev

because $\dim_{\mathbb{R}}(H^d) = h_d(\Delta_\Sigma) = (-1)^{d-1}\tilde{\chi}(\Delta_\Sigma) = 1$

Fleming-Karu call the isomorphism

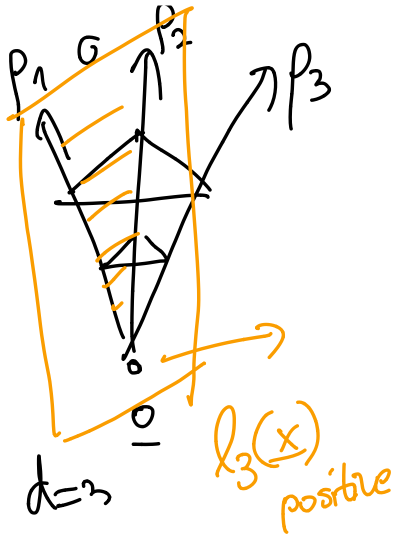
$$H^d(\Sigma) \xrightarrow{\sim} \mathbb{R}$$
$$f = (f_\sigma) \longmapsto \langle f \rangle \quad (= \mathrm{ev}(f) \text{ before})$$

We'll use $\langle f \rangle$ so much that we need a better formula for it on $H(\Sigma)$.

<u>DEF'N</u>: Given Σ a complete simplicial fan in $\mathbb{R}^d$ recall that for each maximal cone σ we have

$$g_\sigma\big|_\sigma = g_{\rho_1} \cdots g_{\rho_s} \quad \text{if } \sigma \text{ has rays } \rho_1, \ldots, \rho_d$$
$$= c \cdot \ell_1^\sigma(\underline{x}) \cdots \ell_s^\sigma(\underline{x}), \quad c \in \mathbb{R}_{>0}$$

where $\ell_i^\sigma(\underline{x})$ is the linear form defining a wall of σ, but positive on σ



Rescale g_σ so that

$$\pm \ell_1^\sigma \wedge \dots \wedge \ell_d^\sigma \in \Lambda^d(\mathbb{R}^d)^*$$

are all the same up to ± 1

i.e.

$$\det \begin{array}{cccc} \ell_1^\sigma & \ell_2^\sigma & \cdots & \ell_d^\sigma \\ \end{array} \left[\begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} \cdots \begin{pmatrix} * \\ \vdots \\ * \end{pmatrix} \right] = \pm 1 \quad \forall \sigma$$

<u>DEF'N:</u> $R_\Sigma \xrightarrow{\pi_\Sigma} \mathrm{Frac}(A) \cong \mathbb{R}(y_1, \dots, y_d)$ $\quad A = \mathbb{R}[y_1, \dots, y_d]$ = poly functions on $\mathbb{R}^d$

$$f = \underset{\substack{\text{max cones} \\ \sigma \in \Sigma}}{(f_\sigma)} \longmapsto \pi_\Sigma(f) = \sum_{\substack{\text{max cones} \\ \sigma}} \frac{f_\sigma}{g_\sigma}$$

<u>THM</u> The map π_Σ actually maps $R_\Sigma \xrightarrow{\pi_\Sigma} A$

and for Δ_Σ shellable, it is the unique map which is (up to $\mathbb{R}$-scaling)

- A-linear
- decreasing degree by d
- nonzero.

It therefore induces an evaluation isomorphism

$$\begin{array}{ccc} H^d(\Sigma) & \xrightarrow{\sim} & \mathbb{R} \\ \| & & \| \\ (R_\Sigma / A_+ R_\Sigma)_d & & A/A_+ \end{array}$$

$$f \longmapsto \langle f \rangle$$

that sends each $g_\sigma \longmapsto 1$. for max cones $\sigma \in \Sigma$

proof: Assume π_Σ maps $R_\Sigma \to A$

for the moment.

Then the A-linearity is easy
and lowering degree by d is easy.

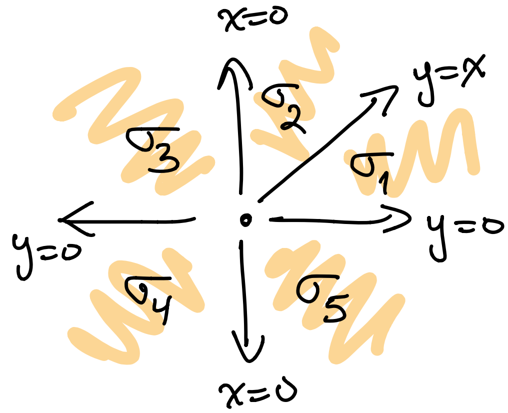
But this determines π_Σ uniquely since
R_Σ is a free A-module with A-basis $\{g_{\sigma_i}\}$

$\Delta_\Sigma = \bigcup_{i=1}^{s} (G_i; F_i)$ $\quad \updownarrow \; \underline{x}^{G_i}$

so π_Σ on R_Σ is determined by choice of
the $\pi_\Sigma(g_{\sigma_i})$, and only g_{σ_s} has degree d,
so $\pi_\Sigma(g_{\sigma_i}) = 0$ because $\deg(g_{\sigma_s}) < d$.

EXAMPLE

$$\Sigma = $$
$$\cap$$
$$\mathbb{R}^2$$
coords x, y



$\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1$

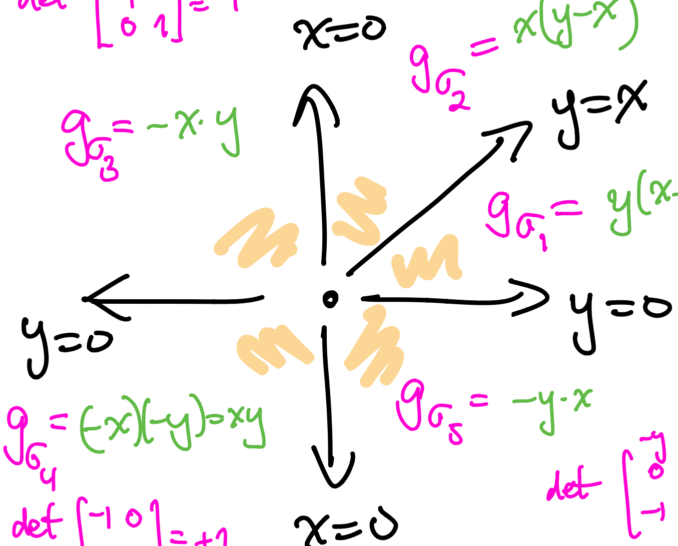
$g_{\sigma_3} = -x \cdot y$

$g_{\sigma_2} = x(y-x)$

$\det \begin{array}{c} \begin{array}{cc} x & y-x \end{array} \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \end{array} = +1$

$g_{\sigma_1} = y(x-y)$

$\det \begin{array}{c} \begin{array}{cc} y & x-y \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \end{array} = -1$



$(\mathbb{R}^2)^*$ – basis:
$\{x, y\}$

$g_{\sigma_4} = (-x)(-y) = xy$

$\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = +1$

$g_{\sigma_5} = -y \cdot x$

$\det \begin{array}{c} \begin{array}{cc} -y & x \end{array} \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array} = +1$

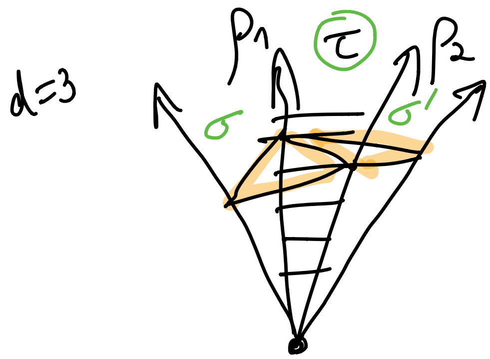
$$\Rightarrow \quad \pi_\Sigma\Big(\underset{(f_{\sigma_i})_{i=1,2,3,4,5}}{\overset{f}{\shortparallel}}\Big) := \frac{f_{\sigma_1}}{y(x-y)} + \frac{f_{\sigma_2}}{x(y-x)} + \frac{f_{\sigma_3}}{-xy} + \frac{f_{\sigma_4}}{xy} + \frac{f_{\sigma_5}}{-xy}$$

lies in $A = \mathbb{R}[x,y]$ if $f \in R_\Sigma$

Note that if $A = \mathbb{R}[y_1, \dots, y_d]$ then $\pi_\Sigma(f) \in \mathrm{Frac}(A)$

has denominator dividing $\prod_{\substack{(d-1)\text{-cones} \\ \tau \in \Sigma}} l_\tau(\underline{x})$

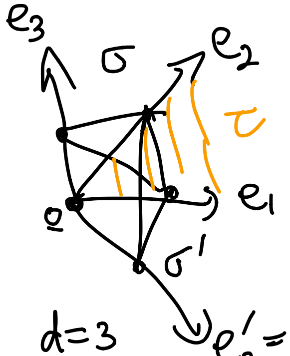
where $l_\tau(\underline{x})$ is the linear form defining $\mathrm{Lin}(\tau)$



Hence it's enough to check a magical cancellation:
for each such $(d-1)$-cone τ bounding σ, σ' d-cones
the sum $\dfrac{f_\sigma}{g_\sigma} + \dfrac{f_{\sigma'}}{g_{\sigma'}}$

has no $l_\tau(\underline{x})$ in its denominator.

KEY IDEA:
Change coordinates so that $\sigma = \underbrace{\mathbb{R}_{\geq 0} \cdot e_1 + \dots + \mathbb{R}_{\geq 0} \cdot e_{d-1}}_{\tau} + \mathbb{R}_{\geq 0} \cdot e_d$



$e_3' = \begin{bmatrix} a_1 \\ a_2 \\ -1 \end{bmatrix}$

$\sigma' = \mathbb{R}_{\geq 0} \cdot e_1 + \dots + \mathbb{R}_{\geq 0} \cdot e_{d-1} + \mathbb{R}_{\geq 0} \cdot e_d'$

where $e_d' = \begin{bmatrix} a_1 \\ \vdots \\ a_{d-1} \\ -1 \end{bmatrix}$

for some $a_1, \dots, a_{d-1} \in \mathbb{R}$

Now check $\{l_1^{\sigma},\dots,l_d^{\sigma}\} = \{y_1,\dots,y_d\}$ so $g_\sigma = y_1\cdots y_d$

and $\{l_1^{\sigma'},\dots,l_d^{\sigma'}\} = \{y_1+a_1y_d,\dots,y_{d-1}+a_{d-1}y_d, -y_d\}$

so $g_{\sigma'} = -y_d \prod_{i=1}^{d-1}(y_i+a_iy_d)$

This means that

$$\frac{f_\sigma}{g_\sigma}+\frac{f_{\sigma'}}{g_{\sigma'}} = \frac{f_\sigma}{y_1\cdots y_d} + \frac{f_{\sigma'}}{-y_d\prod_{i=1}^{d-1}(y_i+a_iy_d)}$$

$$= \frac{-1}{y_1\cdots y_{d-1}\,y_d\prod_{i=1}^{d-1}(y_i+a_iy_d)}\Bigg[\underbrace{f_\sigma\prod_{i=1}^{d-1}(y_i+a_iy_d) - f_{\sigma'}\cdot y_1\cdots y_{d-1}}_{f_\sigma\cdot y_1\cdots y_{d-1} + \left(\substack{\text{terms}\\\text{divisible}\\\text{by } y_d}\right) - f_{\sigma'}\cdot y_1\cdots y_{d-1}}\Bigg]$$

$$= \underbrace{\underbrace{\left(f_\sigma - f_{\sigma'}\right)}_{\text{divisible by } y_d} y_1\cdots y_{d-1} + \left(\substack{\text{terms divisible}\\\text{by } y_d}\right)}_{\text{divisible by } y_d}$$

this factor of y_d disappears from the denominator $\Leftarrow$ divisible by y_d

∎

Math 8680 March 31, 2021

Ooh, what's in that (Kähler) package?

For $\Delta = \partial P$, P a simplicial d-polytope
having $\Sigma = \underset{\text{face fan}}{\mathcal{F}}(P)$,

we've seen $H(\Sigma) := R_\Sigma / A_+ R_\Sigma = \mathbb{R}[\Delta]/(\Theta_\Sigma)$

satisfies

(0) (evaluation map) $H(\Sigma) = \underset{\mathbb{R}}{\overset{\shortparallel}{H^0}} \oplus H^1 \oplus \cdots \oplus H^{d-1} \oplus \underset{\mathbb{R}}{\overset{\wr\wr}{H^d}}$, $H^d \xrightarrow{ev} \mathbb{R}$, $f \mapsto \langle f \rangle$

and

(1) (Poincaré duality) P.D. $H^k \times H^{d-k} \xrightarrow{\langle \cdot,\cdot \rangle} \mathbb{R}$, $(x, y) \longmapsto \langle x \cdot y \rangle$

is a perfect/non-degen. pairing $\forall k$

and we will show...

(2) THM: (Hard Lefschetz) HL (McMullen '93, '96) (Fleming-Karu 2010) (Stanley 1980)

$\exists$ Lefschetz elements $\ell \in H^1(\Sigma)$ or $(R_\Sigma)_1$

that is, those for which

$$H^k \xrightarrow[\sim]{\cdot \ell^{d-2k}} H^{d-k}$$

is an $\mathbb{R}$-linear isomorphism.

Specifically we will show it suffices to take $\ell \in (\mathbb{R}_\Sigma)$, that are ample or strictly convex on $\Sigma \subset \mathbb{R}^d$

PL on Σ, but on no coarser fan than Σ

concave up, or the graph of $y \geq \ell(x)$ in $\mathbb{R}^d \times \mathbb{R}$ is a convex set

or equivalently if $\ell = \sum_{j=1}^{n} c_j \, \underbrace{g_{\rho_j}}_{\text{test function for rays } \rho_j}$

then $\underbrace{P^\Delta}_{\text{polar dual polytope to } P} = \bigcap_{j=1}^{n} \{ v_j \cdot x \leq c_j \}$

HL lets us turn the bilinear form $H^k \times H^{d-k} \to \mathbb{R}$ into a symmetric bilinear form on H^k (quadratic)

via $Q_\ell(x) = \langle x \cdot \ell^{d-2k} \cdot x \rangle \quad \forall x \in H^k$
$= \langle x^2 \cdot \ell^{d-2k} \rangle \in \mathbb{R}$

with corresponding bilinear form on H^k

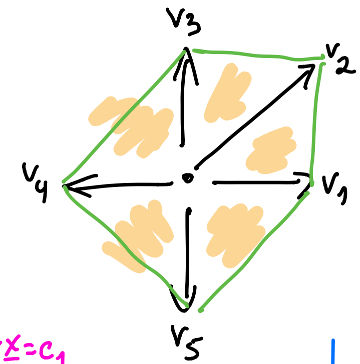
$B_\ell(x,y) := \langle x \cdot \ell^{d-k} \cdot y \rangle = \langle x y \ell^{d-2k} \rangle \in \mathbb{R}$
$\forall x, y \in H^k$ $\quad = \frac{1}{2}\left[Q_\ell(x+y) - \left(Q_\ell(x) + Q_\ell(y) \right) \right]$

and $Q_\ell(x) = B_\ell(x,x)$

So $B_\ell(x,y) = x^T M y$ for some symmetric $M \in \mathbb{R}^{h_k \times h_k}$ ...

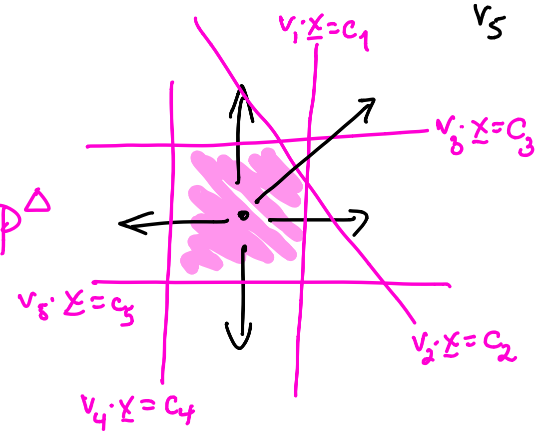
EXAMPLE:

P simplicial polytope

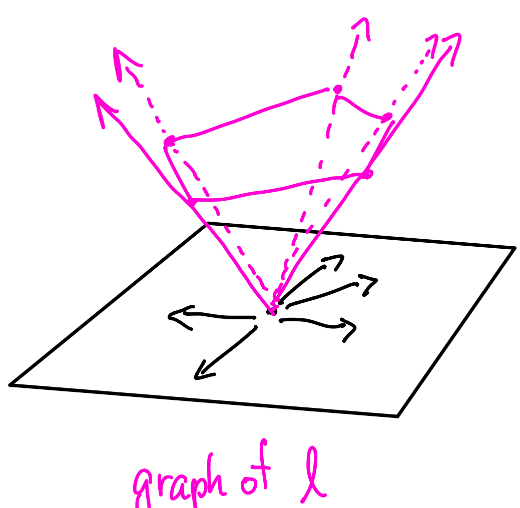


$\Sigma = \mathcal{F}(P)$ its face fan

P^Δ

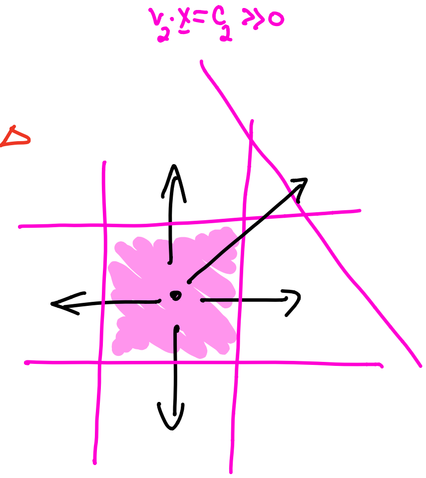


GOOD ("ample") $\ell = \sum_{j=1}^{n} c_j g_{\rho_j}$ gives simple polytope P^Δ with $\Sigma = N(P^\Delta)$ its normal fan

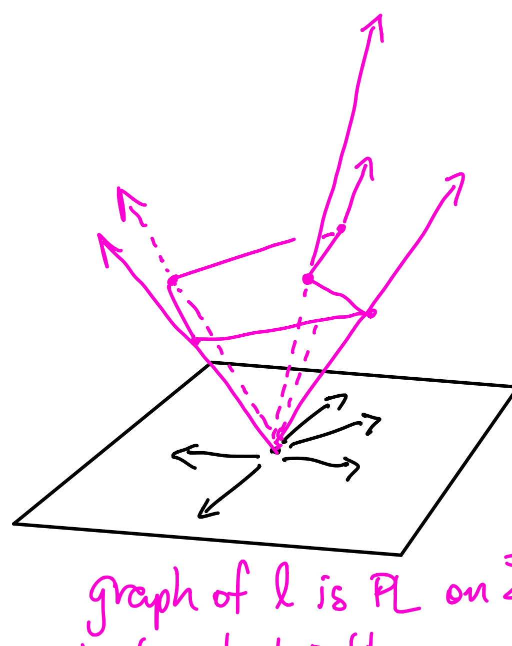


graph of ℓ strictly convex on Σ

not P^Δ



BAD $\ell = \sum_{j=1}^{n} c_j g_{\rho_j}$



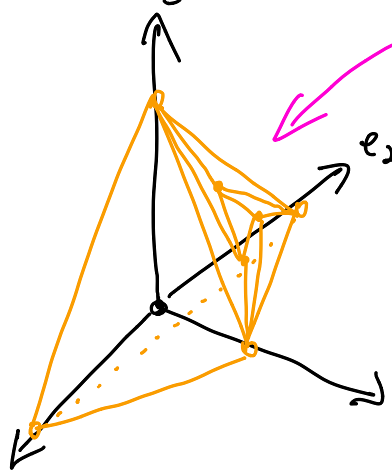
graph of ℓ is PL on Σ, but not strictly convex

NON-EXAMPLE:

This complete simplicial fan Σ in $\mathbb{R}^3$ <u>looks like</u> it could be $\Sigma = \mathcal{F}(P)$ for a simplicial polytope P, but it's <u>not</u>: there is no <u>ample</u> $\ell \in (R_\Sigma)_1$

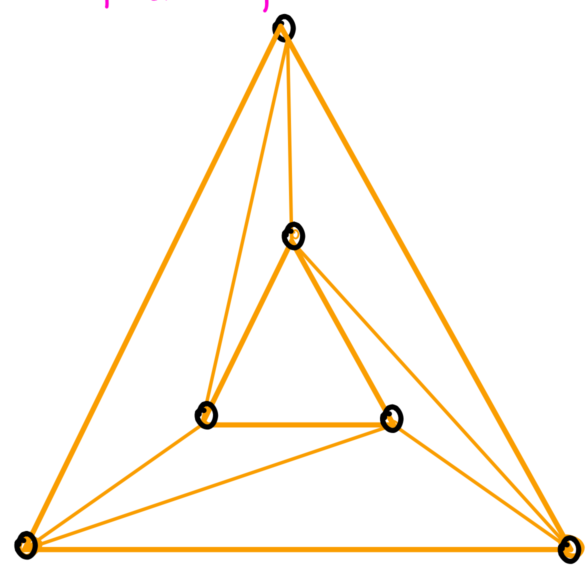
no such P with $\Sigma = \mathcal{F}(P)$

no P^Δ with $\Sigma = \mathcal{N}(P^\Delta)$



a nonpolytopal complete simplicial fan Σ

1st orthant picture:



(EXERCISE 8 on HW!)

We can ask for the <u>signature</u> (n_+, n_-, n_0) for

$Q_\ell(-)$ or $B_\ell(-,-)$

i.e. after a doing a change-of-basis on H^k
orthonormal

making $Q_\ell(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_{h_k} x_{h_k}^2$

where $\lambda_1, \lambda_2, \ldots$ are eigenvalues of M

and
$$n_+ = \#\{i : \lambda_i > 0\}$$
$$n_- = \#\{i : \lambda_i < 0\}$$
$$n_0 = \#\{i : \lambda_i = 0\}$$

pos. def. $\Leftrightarrow n_+ = h_k$

neg. def. $\Leftrightarrow n_- = h_k$

EXERCISE:
<u>NOTE</u>: HL, PD $\Rightarrow n_0 = 0$

(3) <u>THM</u> (Hodge-Riemann-Minkowski) HRM inequalities

When $\ell \in (R_\Sigma)_1$ is ample/strictly convex as above,

then the <u>primitive cohomology</u>
$$PH^k := \ker\left(H^k \xrightarrow{\ell^{d-2k+1}} H^{d-k+1}\right) \subset H^k$$

has $Q_\ell \big|_{PH^k}$ pos. def. if k even, so all $\lambda_i > 0$
neg. def. if k odd, so all $\lambda_i < 0$.

We can get an equivalent rephrasing of HRM by iterating this <u>orthogonal decomposition</u>
$$\forall k \leq \tfrac{d}{2},\ H^k = \ell \cdot H^{k-1} \oplus PH^k$$

Why are $\ell \cdot H^{k-1}$ and PH^k orthogonal?

Any $x \in \ell \cdot H^{k-1}$, $y \in PH^k$ have

$$\begin{aligned} B(x,y) &= \langle x \cdot \ell^{d-2k} y \rangle \\ &= \langle \ell \cdot x' \, \ell^{d-2k} y \rangle \\ &= \langle x' \cdot \ell^{d-2k+1} y \rangle \\ &= \langle x' \cdot 0 \rangle \\ &= 0 \end{aligned}$$

Recall: $V = X \oplus Y$ is <u>orthogonal</u> w.r.t. $Q(-)$ if or $B(-,-)$

$\forall x \in X, y \in Y$

$B(x,y) = 0$

or $Q(x+y) = Q(x) + Q(y)$

Iterating, we get the <u>Lefschetz</u> <u>decomposition</u> of H^k

$$H^k = \ell^k \cdot PH^0 \oplus \ell^{k-1} PH^1 \oplus \dots \oplus \ell PH^{k-1} \oplus PH^k$$

and HRM says Q_ℓ is

pos. def. | neg. def. | ... | $(-1)^{k-1}$ def. | $(-1)^k$ def.

... on these summands

Equivalently, Q_ℓ on H^k has signature (n_+, n_-, n_0) with $n_0 = 0$ and $n_+ - n_- = \sum_{i=0}^{k} (-1)^{k-i} \underbrace{\dim_{\mathbb{R}} PH^i}_{\dim_{\mathbb{R}} H^i - \dim_{\mathbb{R}} H^{i-1}}$

$= h_i - h_{i-1} =: g_i$

Math 8680 Apr. 2, 2021

Spring break next week – no class Apr 5–9!

Course planning:
- 20 min. student talks on Apr 28, 30, May 3
- no HW required from 2nd part of course, just 4 problems from 1st part, by end of Spring Break

"Lefschetz linear algebra" – see Adiprasito-Huh-Katz §7.1, Robles §3–4

The HL isomorphism $H^k \xrightarrow[\sim]{\cdot \ell^{d-2k}} H^{d-k}$ $\forall k \leq d/2$

$H^k = H^k(\Sigma)$ via a Lefschetz element $\ell \in H^1$

immediately implies...

PROP 1: $\forall k \leq d/2$

$$H^k = \ell \cdot H^{k-1} \oplus \underbrace{PH^k}_{\underset{\text{DEFIN}}{:=} \ker\left(H^k \xrightarrow{\cdot \ell^{d-2k+1}} H^{d-k+1}\right)}$$

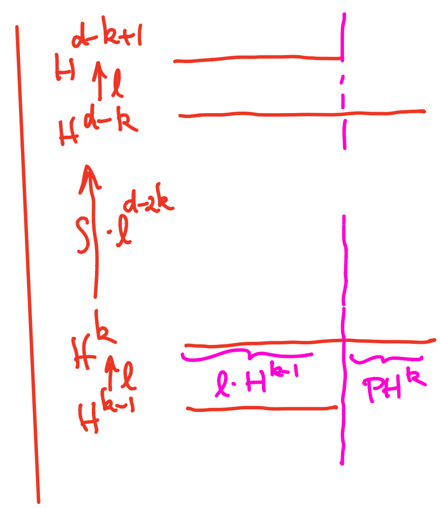
primitive cohomology

and checked these are $\perp$ w.r.t. $Q_\ell(-)$ on H^k

$\ell \cdot H^{k-1} \xrightarrow[\sim]{\cdot \ell^{d-2k}} \operatorname{im}\left(H^k \xrightarrow{\ell^{d-2k+1}} H^{d-k+1}\right)$

proof:

So dim on left = sum of dims on right, and only need to check...



$\ell \cdot H^{k-1} \cap PH^k = \{0\}$

If $x \in \ell \cdot H^{k-1}$ so $x = \ell \cdot x'$ for $x' \in H^{k-1}$

and $x \in PH^k$ then

$$
\begin{aligned}
0 &= \ell^{d-k+1} \cdot x \\
&= \ell^{d-k+2} \cdot x' \\
&\Rightarrow x' = 0 \quad \text{since HL on } H^{k-1} \\
&\Rightarrow x = 0 \quad \blacksquare
\end{aligned}
$$

PROP 2: $\forall k \leq d/2$

the isomorphism $H^{k-1} \xrightarrow[\sim]{\cdot \ell} \ell \cdot H^{k-1}$

preserves Q_ℓ, i.e. it's an isometry for the quad. forms Q_ℓ

proof: Want for $x' \in H^{k-1}$ that

$$
\begin{array}{ccc}
Q_\ell(x') & = & Q_\ell(\ell \cdot x') \\
\shortparallel & & \| \\
\langle (x')^2 \cdot \ell^{d-2(k-1)} \rangle & = & \langle (\ell \cdot x')^2 \cdot \ell^{d-2k} \rangle
\end{array}
$$

$\blacksquare$

Iterating PROP 1 gave us the Lefschetz decomposition:

$H^k = \ell^k \cdot PH^0 \oplus \ell^{k-1} PH^1 \oplus \dots \oplus \ell \cdot PH^{k-1} \oplus PH^k$

and HRM s $-1)^k Q(-)$ is positive definite on PH^k

The general picture for HL & HRM

HL gives Lefschetz orthogonal decomposition

$$
\begin{array}{ccccc}
H^d & = & \ell^d PH^0 & & \\
\uparrow & & \uparrow & & \\
H^{d-1} & = & \ell^{d-1} PH^0 & \oplus & \ell^{d-2} PH^1 \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\hline
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
H^2 & = & \ell^2 PH^0 & \oplus & \ell PH^1 \quad \oplus \quad PH^2 \\
\uparrow & & \uparrow & & \uparrow \\
H^1 & = & \ell PH^0 & \oplus & PH^1 \\
\ell \uparrow & & \uparrow & & \\
H^0 & = & PH^0 & &
\end{array}
\qquad d/2
$$

HRM: Q_ℓ is ___. pos def | neg def | pos def

HL and HRM for d=6:

h-vector $(h_0, h_1, h_2, h_3, h_4, h_5, h_6)$, $h_0 = 1$
interpreted by $H(\Sigma)$

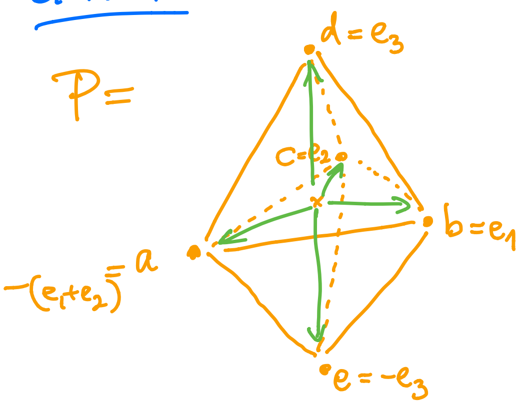
g-vector (g_0, g_1, g_2, g_3) $= (1,\ h_1-h_0,\ h_2-h_1,\ h_3-h_2)$
interpreted by $H(\Sigma)/(\ell)$
or by $PH(\Sigma)$ $= PH^0 \oplus PH^1 \oplus PH^2 \oplus PH^3$

$$
\begin{array}{l|llll}
\text{dim} & & & & \\
h_6 & H^6 = \ell^6 \cdot PH^0 \overset{ev}{\xrightarrow{\sim}} \mathbb{R} & & & \\
 & \uparrow & & & \\
h_5 & H^5 = \ell^5 \cdot PH^0 & \oplus\ \ell^4 \cdot PH^1 & & \\
 & \uparrow & \uparrow & & \\
h_4 & H^4 = \ell^4 \cdot PH^0 & \oplus\ \ell^3 \cdot PH^1 & \oplus\ \ell^2 \cdot PH^2 & \\
 & \uparrow & \uparrow & \uparrow & \\
h_3 & H^3 = \ell^3 \cdot PH^0 & \oplus\ \ell^2 \cdot PH^1 & \oplus\ \ell \cdot PH^2 & \oplus\ PH^3 \\
 & \uparrow & \uparrow & \uparrow & \text{dim } g_3 \\
h_2 & H^2 = \ell^2 \cdot PH^0 & \oplus\ \ell \cdot PH^1 & \oplus\ PH^2 & \\
 & \uparrow & \uparrow & \text{dim } g_2 & \\
h_1 & H^1 = \ell \cdot PH^0 & \oplus\ PH^1 & & \\
 & \uparrow & \text{dim } g_1 & & \\
h_0 & H^0 = PH^0 \ \text{dim } g_0 = 1, \ \cong \mathbb{R} & & & \\
 & & & & \\
Q_\ell \text{ is} & \text{pos. def.} & \text{neg. def.} & \text{pos. def.} & \text{neg. def.}
\end{array}
$$

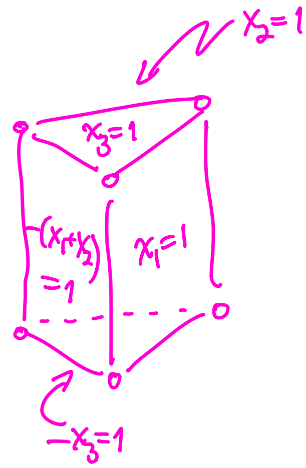
vertical arrows ↑ are all multiplication by ℓ

EXAMPLE:

$P =$



$P^\Delta =$



$\Sigma = \mathcal{F}(P)$

$= \mathcal{N}(P^\Delta)$

with $\ell = 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot d + 1\cdot e$

(i.e. pick $v_i \cdot \underline{x} \le \underset{c_i}{\overset{\shortparallel}{1}}$ $\forall$ vertices v_i of P to define P^Δ)

Then

$$H(\Sigma) = R_\Sigma / A_+ R_\Sigma \cong \mathbb{R}[\Delta]/(\underline{\Theta}_\Sigma) \qquad \begin{cases} \Theta_1 = a-b \\ \Theta_2 = a-c \\ \Theta_3 = d-e \end{cases}$$

$$= \mathbb{R}[a,b,c,d,e]/(abc, de, a-b, a-c, d-e) \qquad \begin{matrix} b=a \\ c=a \\ e=d \end{matrix}$$

$$\cong \mathbb{R}[a,d]/(a^3, d^2)$$

$$= \mathbb{R}\text{-span of } \{ \underbrace{1,}_{H^0} \ \underbrace{a, d,}_{H^1} \ \underbrace{a^2, ad,}_{H^2} \ \underbrace{a^2 d}_{H^3 \cong \mathbb{R}} \}$$

with $\ell = a+b+c+d+e$

$\equiv 3a + 2d$

$a^2 d \equiv bcd \overset{ev}{\longmapsto} \underset{=+1}{\langle bcd \rangle}$

since $bcd = g_\sigma$

for $\sigma = \mathbb{R}_{\ge 0} e_1 + \mathbb{R}_{\ge 0} e_2 + \mathbb{R}_{\ge 0} e_3$

Check HL for $H^0 \xrightarrow{\cdot \ell^3} H^3$: $\langle 1\cdot \ell^3\rangle = \langle (3a+2d)^3\rangle$

$$= \left\langle \binom{3}{1}\cdot 3^2\cdot 2^1 \cdot a^2 d\right\rangle$$

$$= +54 \neq 0 \quad \text{(invertible ✓)}$$

which also checks HRM for Q_ℓ on $H^0 \, (= PH^0)$,

since $Q_\ell(1) = \langle 1\cdot 1\cdot \ell^3\rangle = +54 > 0$ (pos. def. ✓)

Check HL for $H^1 \xrightarrow{\cdot \ell^1} H^2$:

$$\begin{array}{c} \cdot\ell^1 \text{ acts as} \\ \| \\ \cdot(3a+2d) \end{array} \quad \begin{array}{c|cc} & a & d \\ \hline a^2 & 3 & 0 \\ ad & 2 & 3 \end{array} \quad \text{(invertible ✓)}$$

and check HRM

for Q_ℓ on $PH^1 = \ker(H^1 \xrightarrow{\cdot \ell^2} H^3)$ $\ell^2 = (3a+2d)^2 = 9a^2 + 12ad$

$$= \mathbb{R}\text{-span of } \{3a-4d\}$$

with $Q_\ell(3a-4d)$

$$= \langle (3a-4d)^2 \cdot \ell\rangle$$

$$= \langle (9a^2 - 24ad + \cancel{16d^2}^{\,0})\cdot(3a+2d)\rangle$$

$$= \langle 18a^2d - 72a^2d\rangle$$

$$= -54 \quad \text{(neg. def. ✓)}$$

H^3	ℓ^3	
H^2	ℓ^2	$\ell(3a-4d)$
H^1	$\ell = 3a+2d$	$\underbrace{3a-4d}_{PH^1}$
H^0	1	
Q_ℓ is	pos. def.	neg. def.

McMullen / Fleming–Karu strategy for HL + HRM of $H(\Sigma)$

$\Sigma = \mathcal{N}(P)$ P simple
$= \mathcal{F}(P^\Delta)$ P^Δ simplicial

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① Prove HL & HRM for d-simplices by easy direct calculation

② "Dunk" the simple polytope P slowly according to a generic functional h with $h(v_i) \neq h(v_j)$ $\forall v_i \neq v_j$ in P vertices

and prove HL/HRM via induction for all the simple $P_t := P \cap \{p : h(p) \leq t\}$

with ~~Lefschetz~~ elements ℓ_t

③ Call $P_{t_i} \rightsquigarrow P_{t_{i+1}}$ a flip

and prove an orthogonal decomposition

identified later

$$H(\Sigma_-) = H(\Sigma_+) \oplus K$$

using various $Q_{\ell_t}(-)$ on the different spaces

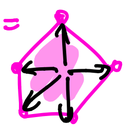
④ Use a local-to-global argument for showing HRM for (d-1)-dim'l fans $\Rightarrow$ HL for d-dim'l fans.

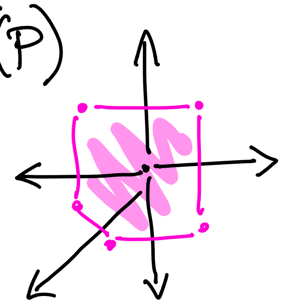
⑤ Use a continuity argument for signature of $Q_{\ell_t}(-)$ to show HL for d-dim'l fans $\Rightarrow$ HRM for d-dim'l fans

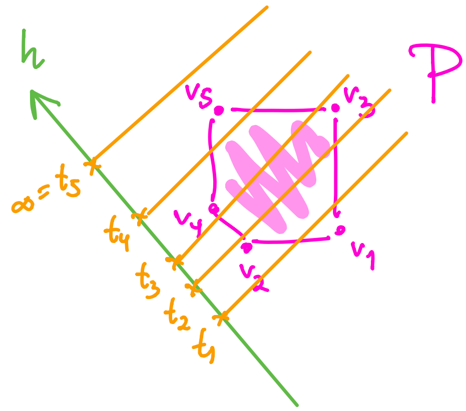
Polytope-dunking examples

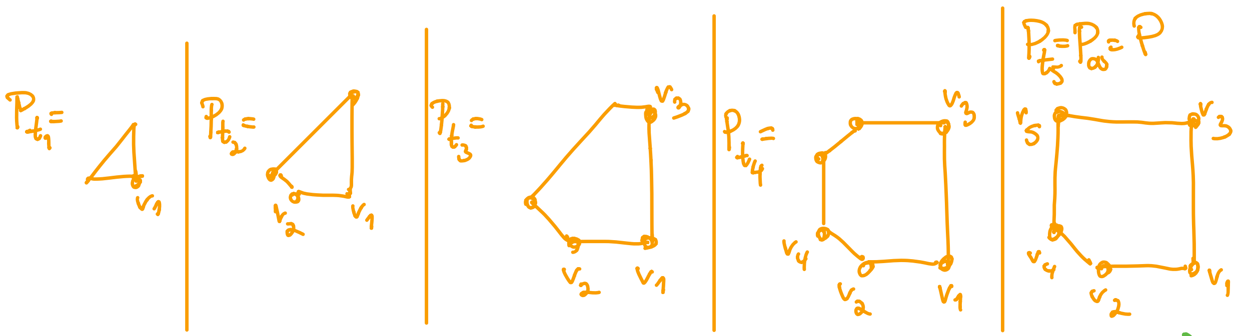
① $\Sigma = \mathcal{N}(P) = \mathcal{F}(P^{\Delta})$

$P^{\Delta} =$







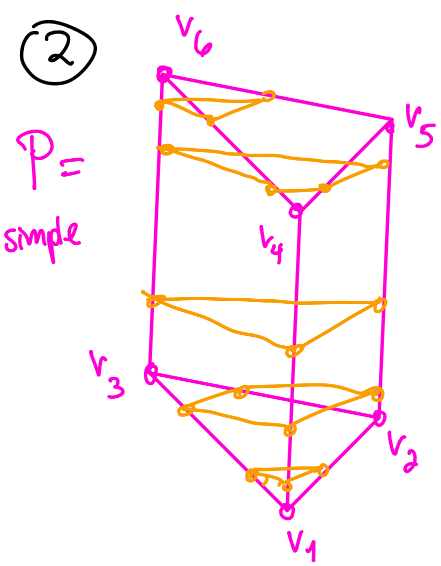


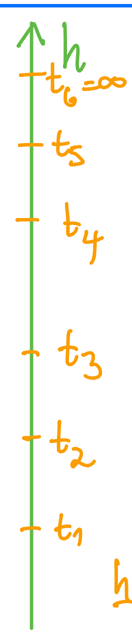
$\underline{h} = (1,1,1)$ $(1,2,1)$ $(1,3,1)$ $(1,4,1)$ $(1,3,1)$

---

②

$P =$ simple





$(1,2,2,1)$

$(1,3,3,1)$

$(1,3,3,1)$

$(1,2,2,1)$

$(1,2,2,1)$

$\underline{h} = (1,1,1,1)$