McMullen / Fleming–Karu strategy for HL + HRM of $H(\Sigma)$

$\Sigma = \mathcal{N}(P)$ P simple
$= \mathcal{F}(P^\Delta)$ P^Δ simplicial

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① Prove HL & HRM for d-simplices by easy direct calculation

② "Dunk" the simple polytope P slowly according to a generic functional h with $h(v_i) \neq h(v_j)$ $\forall v_i \neq v_j$ in P vertices

and prove HL/HRM via induction for all the simple $P_t := P \cap \{p : h(p) \le t\}$

with ~~Lefschetz~~ elements ℓ_t

③ Call $P_{t_i} \rightsquigarrow P_{t_{i+1}}$ a flip

and prove an orthogonal decomposition

$$H(\Sigma_-) = H(\Sigma_+) \oplus K$$

identified later

using various $Q_{\ell_t}(-)$ on the different spaces

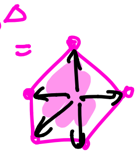
④ Use a local-to-global argument for showing HRM for $(d-1)$-dim'l fans $\Rightarrow$ HL for d-dim'l fans.

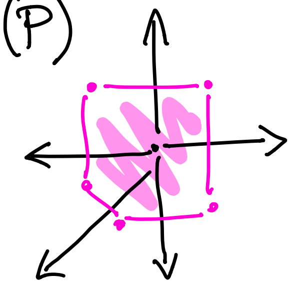
⑤ Use a continuity argument for signature of $Q_{\ell_t}(-)$ to show HL for d-dim'l fans $\Rightarrow$ HRM for d-dim'l fans

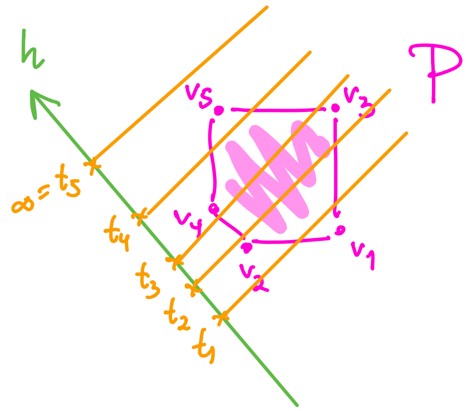
Polytope-dunking examples

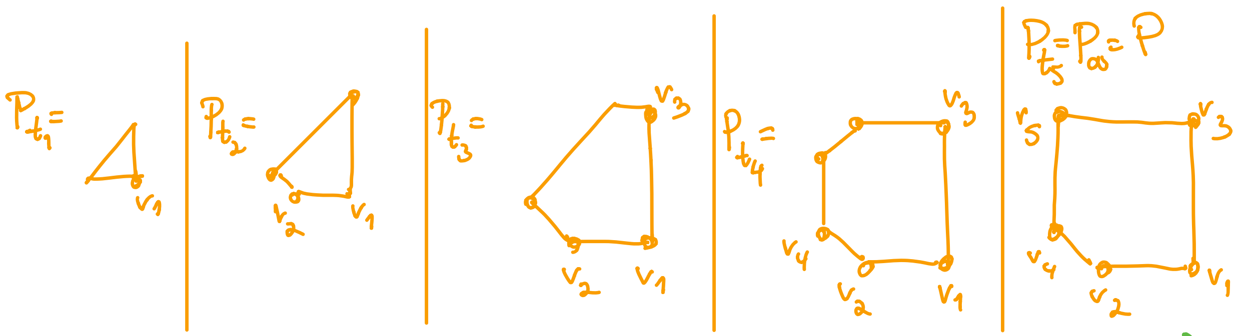
(1) $\Sigma = \mathcal{N}(P)$
$= \mathcal{F}(P^\Delta)$

$P^\Delta =$









$h = (1,1,1)$ $(1,2,1)$ $(1,3,1)$ $(1,4,1)$ $(1,3,1)$

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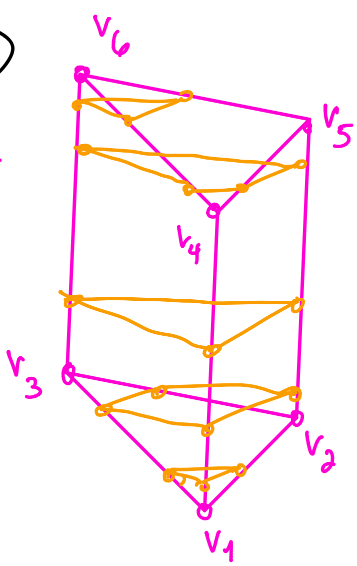
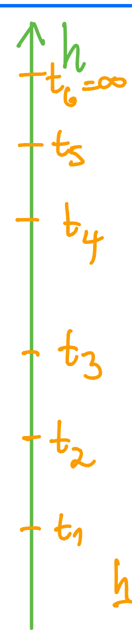
(2)

$P =$ simple





$(1,2,2,1)$

$(1,3,3,1)$

$(1,3,3,1)$

$(1,2,2,1)$

$(1,2,2,1)$

$h = (1,1,1,1)$

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Ideas from the McMullen/Fleming–Karu proof of HL, HRM

Σ = complete simplicial polytopal fan in $\mathbb{R}^d$

$= N(P)$ ← simple

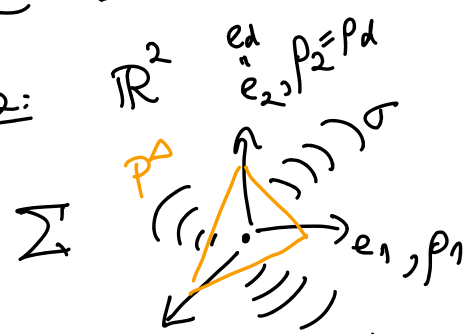
$= F(P^\Delta)$ ← simplicial

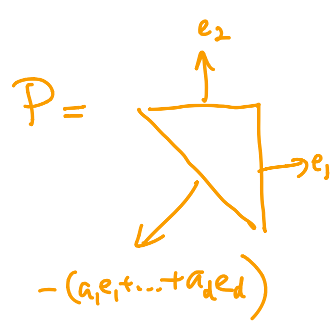
IDEA 1:

The simplex case

When P, P^Δ are simplices, it's easiest to change bases in $\mathbb{R}^d$ so Σ has a d-cone $\sigma = \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_d$

$d=2$: $\mathbb{R}^2$





$\rho_{d+1} = \mathbb{R}_{\geq 0}\left(-(a_1e_1+\ldots+a_de_d)\right)$

$a_i > 0 \ \forall i = 1, \ldots, d$

h-vector $\underline{h} = (\underset{h_0}{1}, \underset{h_1}{1}, \ldots, \underset{h_d}{1})$

Let's calculate in . . .

$$H(\Sigma) \cong \mathbb{R}[\Delta_\Sigma]/(\Theta_\Sigma)$$

$$= \mathbb{R}[x_1, x_2, \dots, x_d, x_{d+1}] \Big/ \big(x_1x_2\cdots x_d x_{d+1},\ \overbrace{x_1 - a_1 x_{d+1}}^{\Theta_1},\ \overbrace{x_2 - a_2 x_{d+1}}^{\Theta_2},\ \vdots\ ,\ \overbrace{x_d - a_d x_{d+1}}^{\Theta_d}\big)$$

$$\cong \mathbb{R}[x_{d+1}] \Big/ (x_{d+1}^{d+1})$$

$$\cong \mathbb{R}\text{-span of } \{1,\ \underbrace{x_{d+1}}_{\boxed{H^1(\Sigma) = \mathbb{R}^1}},\ x_{d+1}^2,\ \dots,\ x_{d+1}^d\}$$

$x_{d+1}^d \,/\!\!/\, x_1x_2\cdots x_d = g_\sigma$ up to ⊕ constant

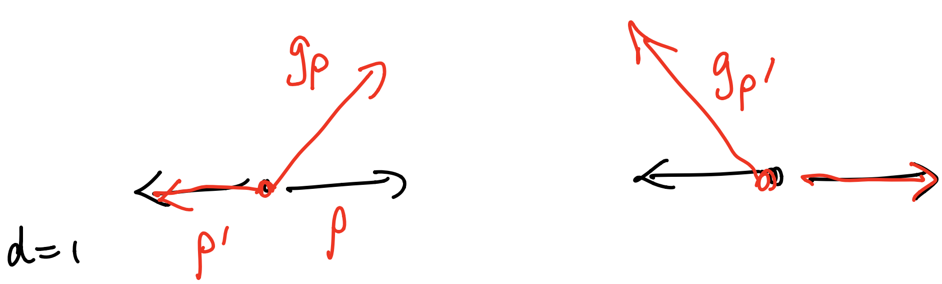
$x_{d+1}^d \xrightarrow{\text{ev}} +1$

Lefschetz element

$$l = l_\Sigma = x_1 + x_2 + \dots + x_d + x_{d+1}$$

$$\equiv c \cdot x_{d+1} \quad \text{with } c > 0$$

→ all $x \in H^1(\Sigma)$ are same up to linear maps and a scaling; those with pos. scalar are the strictly convex on Σ, including all g_ρ = tent functions of rays



In $H(\Sigma) \cong \mathbb{R}[x_{d+1}]/(x_{d+1}^{d+1})$,

$\ell = x_{d+1}$ has HL property:

$$\begin{array}{ccc} & x_{d+1}^{d-2k} & \\ & \| & \\ H^k & \xrightarrow[\sim]{\cdot \ell^{d-2k}} & H^{d-k} \\ \| & & \| \\ \mathbb{R}\cdot x_{d+1}^k & & \mathbb{R}\cdot x_{d+1}^{d-k}\end{array}$$

and has HRM property:

on $PH^0 = H^0 = \mathbb{R}\cdot 1$

$$Q_\ell(1) = \langle 1^2 \cdot \ell^d \rangle = \langle x_{d+1}^d \rangle = +1 > 0$$

IDEA 2: Local-to-global
HRM HL

We've seen for subfans $\Sigma' \subset \Sigma$ that $R_\Sigma \xrightarrow{\text{res}} R_{\Sigma'}$ surjects and an important case is $\Sigma' = \mathrm{star}_\Sigma(\tau) = \{\text{subfan gen'd by } \sigma \supseteq \tau\}$

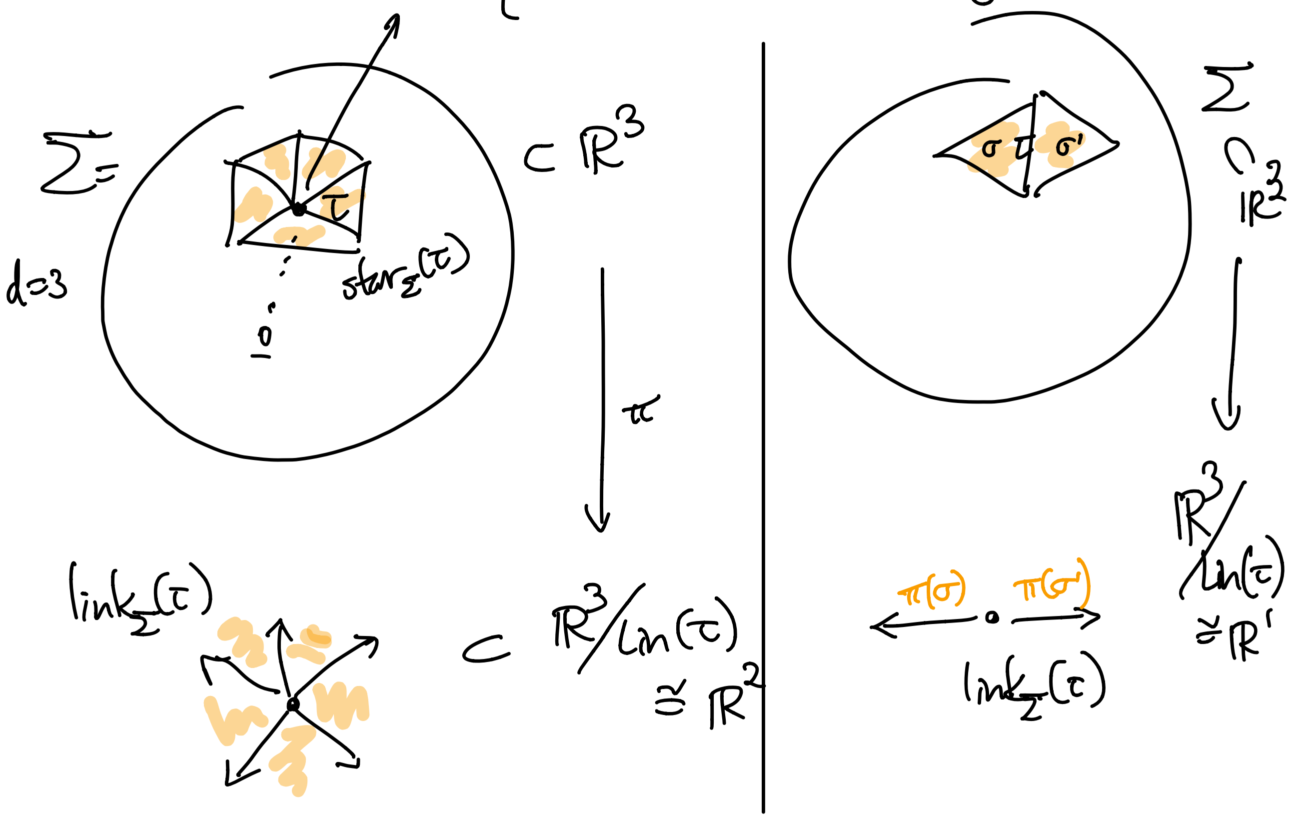
On the other hand, one can use the linear map

$$\mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^d / \underbrace{\mathrm{Lin}(\tau)}_{\mathbb{R}\text{-lin. span of } \tau}$$

to define a fan

$$\mathrm{link}_\Sigma(\tau) \subset \mathbb{R}^d/\mathrm{Lin}(\tau) \cong \mathbb{R}^{d-\dim(\tau)}$$

$$:= \{\pi(\sigma) : \tau \subseteq \sigma \in \Sigma\}$$



One gets a map $R_{\mathrm{link}_\Sigma(\tau)} \xrightarrow{\pi^*} R_{\mathrm{star}_\Sigma(\tau)}$

$$f \longmapsto f \circ \pi$$

$$\begin{matrix} \| & & \| \\ (f_\sigma) & & (f_\sigma \circ \pi) \end{matrix}$$

We claim this π^* induces an iso. on $H(-)$:

$$H(\mathrm{link}_\Sigma(\tau)) \xrightarrow[\sim]{\pi^*} H(\mathrm{star}_\Sigma(\tau))$$

because one can pick a shelling of Δ_Σ (or Σ) $= \partial(P^\Delta)$

that starts by shelling $\mathrm{star}_\Sigma(\tau)$

(pick a functional on vertices of simple P that minimize on vertices of dual face τ^* in P)

which gives also a shelling of $\mathrm{link}_\Sigma(\tau)$

and the iso. π^* sends shelling basis to shelling basis!

<u>LEMMA</u>: (local-to-global; HRM, HL) Σ as above

(Fleming–Karu Lemma 5.1)

Assume $\ell = \ell_\Sigma \in (R_\Sigma)_1$ and $H^1(\Sigma)$ has

$\ell(v) > 0 \quad \forall v \in \mathbb{R}^d - \{\underline{0}\}$

AND

$\forall$ rays $\rho \in \Sigma$ the unique $\ell_\rho \in H^1(\mathrm{link}_\Sigma(\rho))$

satisfying

$$\begin{array}{ccccc} H^1(\Sigma) & \xrightarrow{\mathrm{res}_\rho} & H^1(\mathrm{star}_\Sigma(\rho)) & \xleftarrow{\pi^*} & H^1(\mathrm{link}_\Sigma(\rho)) \\ \ell & \longmapsto & \mathrm{res}_\rho(\ell) & & \\ & & \| & & \\ & & \pi^*(\ell_\rho) & \longleftarrow & \ell_\rho \end{array}$$

ℓ_ρ satisfies HRM on $H(\mathrm{link}_\Sigma(\rho))$

Then ℓ satisfies HL on $H(\Sigma)$.

proof: Want $H^k(\Sigma) \xrightarrow{\ell^{d-2k}} H^{d-k}(\Sigma)$ an iso.,

but by Poincaré Duality, only need injectivity.
(or just Dehn-Sommerville)

So assume $f \in H^k(\Sigma)$ has $\ell^{d-2k} \cdot f = 0$ in $H^{d-k}(\Sigma)$,

and we'll show $g_\rho \cdot f = 0$ $\forall$ rays ρ.

This suffices since $\{g_\rho\}_{\text{rays } \rho}$ generate $H(\Sigma)$, so then f is $\perp$ to all of $H^{d-k}(\Sigma)$, and f is zero by P.D..

Assuming $\ell^{d-2k} \cdot f = 0$ in $H^{d-k}(\Sigma)$

$$
\begin{tikzcd}
\ell^{d-2k} \cdot f = 0 \text{ in } H^{d-k}(\Sigma) \arrow[d, "\mathrm{res}_\rho"] & \\
\mathrm{res}_\rho(\ell^{d-2k} \cdot f) = 0 \text{ in } H^{d-k}(\mathrm{star}_\Sigma(\rho)) & \mathrm{res}_\rho(f) \\
\ell_\rho^{d-2k} \cdot f_\rho = 0 \text{ in } H^{d-k}(\mathrm{link}_\Sigma(\rho)) \arrow[u, "\pi^*", "\wr"'] & f_\rho \arrow[u, mapsto]
\end{tikzcd}
$$

So $\ell_\rho^{(d-1)-2k+1} f_\rho = 0 \Rightarrow f_\rho \in PH^k(\mathrm{link}_\Sigma(\rho))$

$\xRightarrow{\text{HRM for } \ell_\rho \text{ on } H(\mathrm{link}_\Sigma(\rho))}$ $(-1)^k Q_{\ell_\rho}(f_\rho) \geq 0$

with equality $\Longleftrightarrow f_\rho = 0$

Now write $\overset{\ell_\Sigma}{\overset{\shortparallel}{\ell}} = \sum_{\text{rays } \rho \in \Sigma} c_\rho \cdot g_\rho$ with $c_\rho > 0$ because $\ell(v) > 0$ on $\mathbb{R}^d - \{0\}$

Then

$$0 \leq (-1)^k Q_{\ell_\rho}(f_\rho) = (-1)^k \langle \ell_\rho^{(d-1)-2k} f_\rho^2 \rangle_{\mathrm{link}_\Sigma(\rho)} \quad \forall \rho$$

with equality $\Longleftrightarrow f_\rho = 0$

$\Downarrow$

$$0 \leq (-1)^k \sum_{\text{rays } \rho} c_\rho \langle \ell_\rho^{(d-1)-2k} f_\rho^2 \rangle_{\mathrm{link}_\Sigma(\rho)}$$

$$= (-1)^k \sum_\rho c_\rho \langle g_\rho \ell^{(d-1)-2k} f^2 \rangle_\Sigma$$

$$= (-1)^k \langle \sum_\rho c_\rho g_\rho \cdot \ell^{(d-1)-2k} f^2 \rangle_\Sigma$$

A subtle calculational point: for any cone τ of Σ, then

$$\begin{array}{ccccc} H(\Sigma) & \xrightarrow{\text{res}} & H(\mathrm{star}_\Sigma(\tau)) & \xleftarrow{\pi^*} & H(\mathrm{link}_\Sigma(\tau)) \\ y & \longmapsto & \mathrm{res}(y) & & \\ & & \shortparallel & & \\ & & \pi^*(x) & \longleftarrow\!\shortmid & x \end{array}$$

then $\langle x \rangle_{\mathrm{link}_\Sigma(\tau)} = \langle y \cdot g_\tau \rangle_\Sigma$.

$= (-1)^k \langle \ell^{d-2k} f^2 \rangle_\Sigma$ since $\ell = \sum_\rho c_\rho g_\rho$

$= 0$ since $\ell^{d-2k} \cdot f = 0$.

Hence one must have equality in all of the inequalities $0 \leq (-1)^k Q_{\ell_\rho}(f_\rho)$, so $\forall$ rays ρ one has

$$f_\rho = 0 \text{ in } H^k(\mathrm{link}_\Sigma(\rho))$$
$$\Rightarrow \mathrm{res}_\rho(f) = 0 \text{ in } H^k(\mathrm{star}_\Sigma(\rho))$$
$$\Rightarrow g_\rho \cdot f = 0 \text{ in } H^k(\Sigma),$$ as desired. $\blacksquare$

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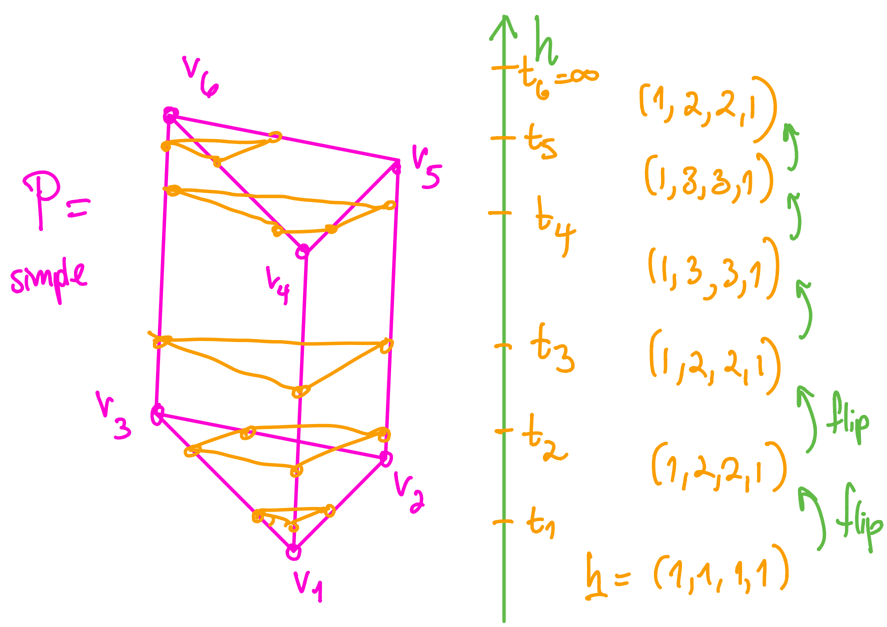
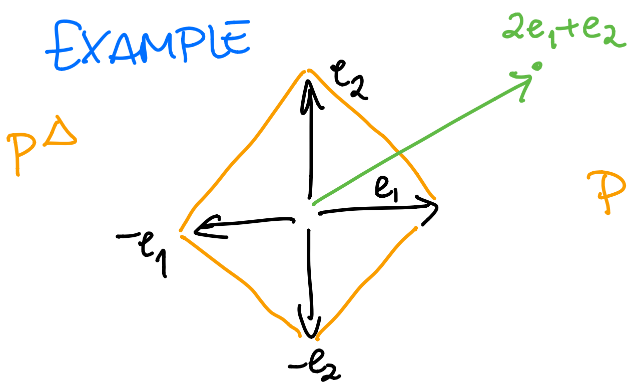
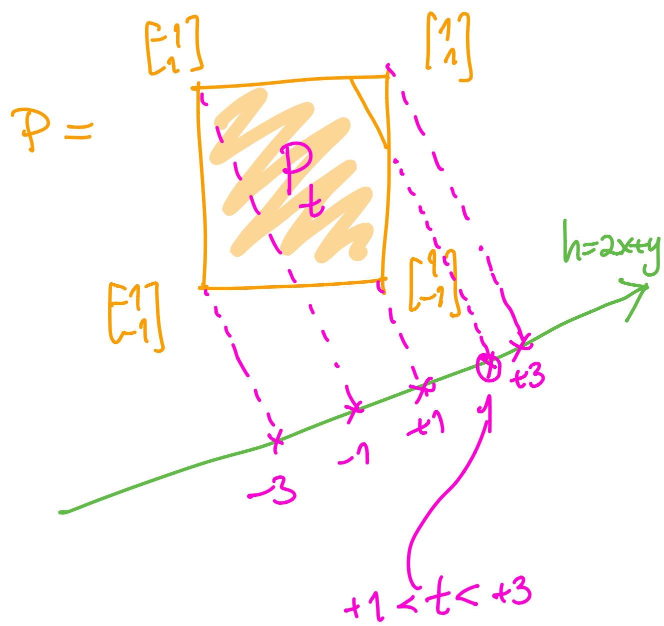
IDEA 3: Strictly between the flips, HRM for $Q_{\ell_t}(-)$ is maintained by continuity if we know HL holds.



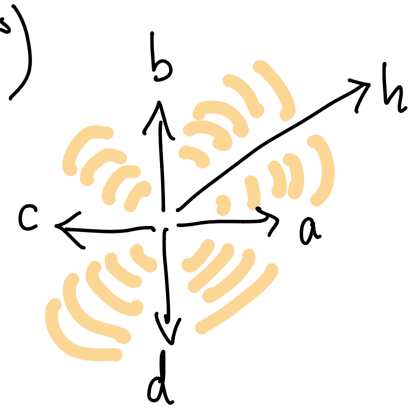
This is because ℓ_t and $Q_{\ell_t}(-)$ vary continuously in t, so the signature (n_+, n_-, n_0) can't change if $n_0 = 0$ throughout.

EXAMPLE





$$\Sigma_t = \mathcal{N}(P_t) = \mathcal{F}(P_t^\Delta)$$



$$H(\Sigma_t) \cong \mathbb{R}[a,b,c,d,h] \big/ (ab, ch, bd, ac, dh, \overset{a+2h-c}{\overset{\parallel}{\Theta_1}}, \overset{b+h-d}{\overset{\parallel}{\Theta_2}})$$

$$= \mathbb{R}\text{-span of } \{ \underbrace{1,}_{H^0} \mid \underbrace{b, \ c, \ d,}_{H^1} \mid \underbrace{cd}_{H^2} \}$$

$$cd \xrightarrow{ev = \langle \cdot \rangle} +1$$

$\ell_t = a+b+c+d+t\cdot h$ for $+1 < t < +3$ ← the only thing varying!

$Q_{\ell_t}(x) = \langle x^2 \cdot \ell_t^{d-2k} \rangle$ for $x \in H^k$

corresponds to a symmetric matrix

column x_j

row x_i

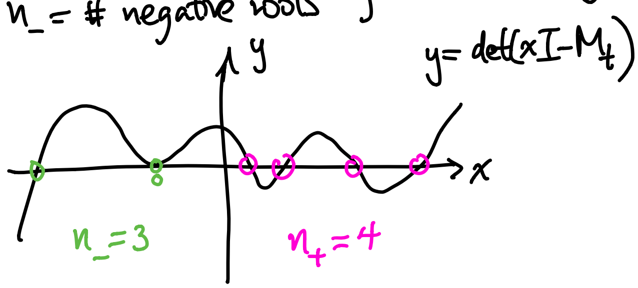
$$M_t = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_{h_k} \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_{h_k} \end{array} & \begin{bmatrix} & & \vdots & \\ \cdots & & \langle x_i x_j \ell_t^{d-2k} \rangle & \\ & & & \end{bmatrix} \end{array}$$

whose entries are continuous functions in t

$\Longrightarrow$ its eigenvalues, roots of $\det(xI - M_t)$, vary continuously in t.

Then since HL $\Rightarrow$ no roots are zero $\forall t$,

$$\left.\begin{array}{l} n_+ = \#\text{ positive roots} \\ n_- = \#\text{ negative roots} \end{array}\right\} \text{cannot change}$$

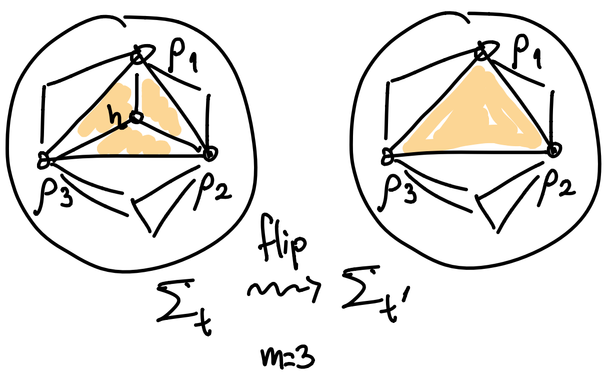


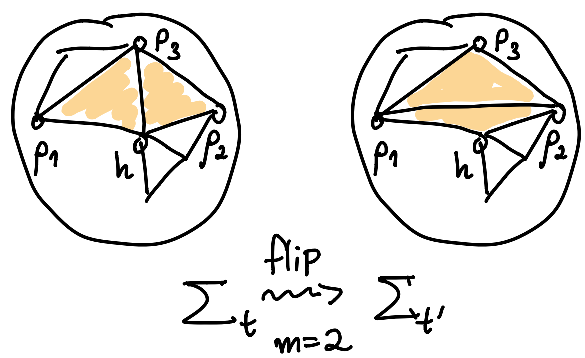
IDEA 4: When $\Sigma_t \overset{\text{flip}}{\rightsquigarrow} \Sigma_{t'}$ passes through a vertex $v \in P$, lying on facets with $\underset{\text{dual}}{\text{normal}}$ rays $\underset{\mathbb{R}_{\geq 0} \cdot n_1}{\overset{\shortparallel}{\rho_1}}, \ldots, \underset{\mathbb{R}_{\geq 0} \cdot n_d}{\overset{\shortparallel}{\rho_d}}$, the only change to Σ_t is a generic bistellar flip in the triangulation of the cone spanned by $\{h, n_1, \ldots, n_d\}$:

writing $h = \sum_{i=1}^{d} c_i n_i$ uniquely, with $c_i \neq 0 \;\forall i$ and $c_1, \ldots, c_m > 0$, $c_{m+1}, \ldots, c_d < 0$

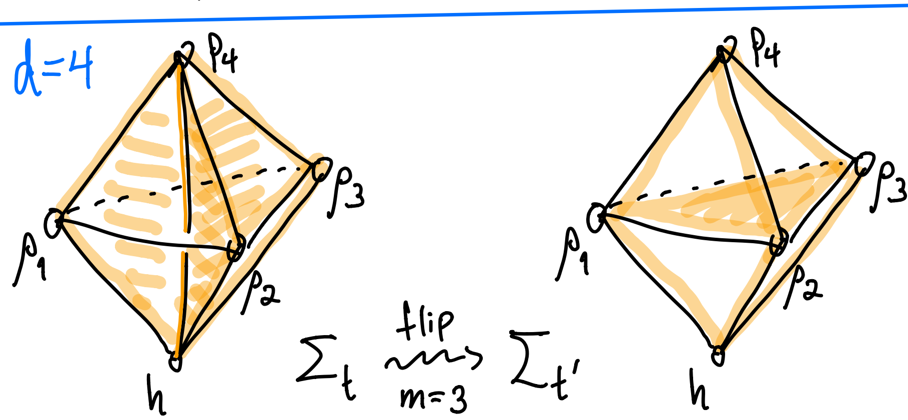
the flip changes it from
m d-cones: $\{n_1, \ldots, n_d, h\} - \{n_i\}$ for $i = 1, 2, \ldots, m$
to $d+1-m$ d-cones: $\{n_1, \ldots, n_d, h\} - \{n_j\}$ for $j = m+1, m+2, \ldots, d$
plus $\{n_1, \ldots, n_d\}$

(Schematic)
EXAMPLES with $d=3$





$d=4$

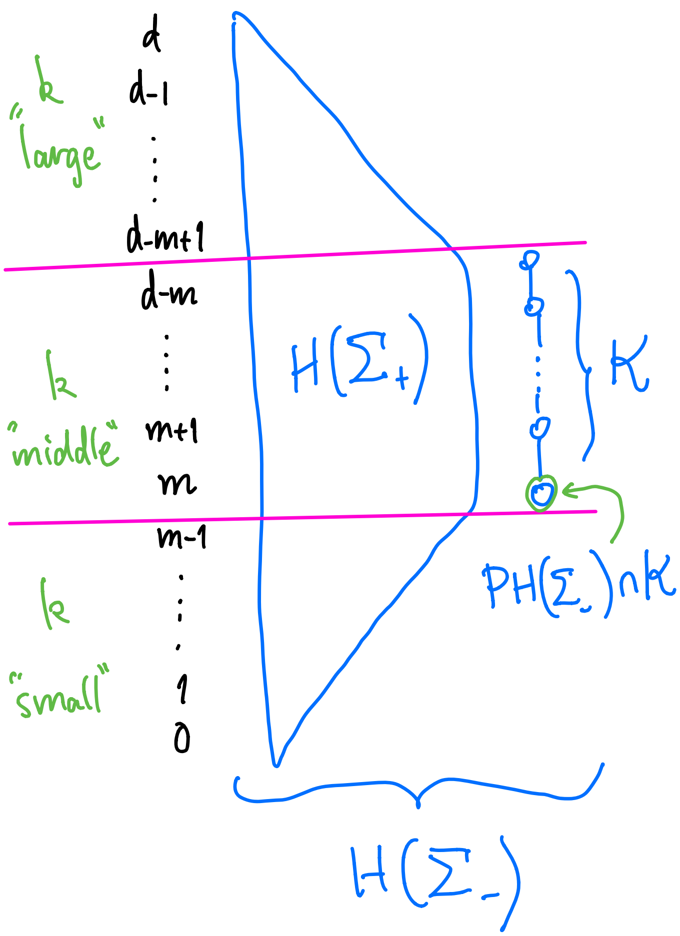


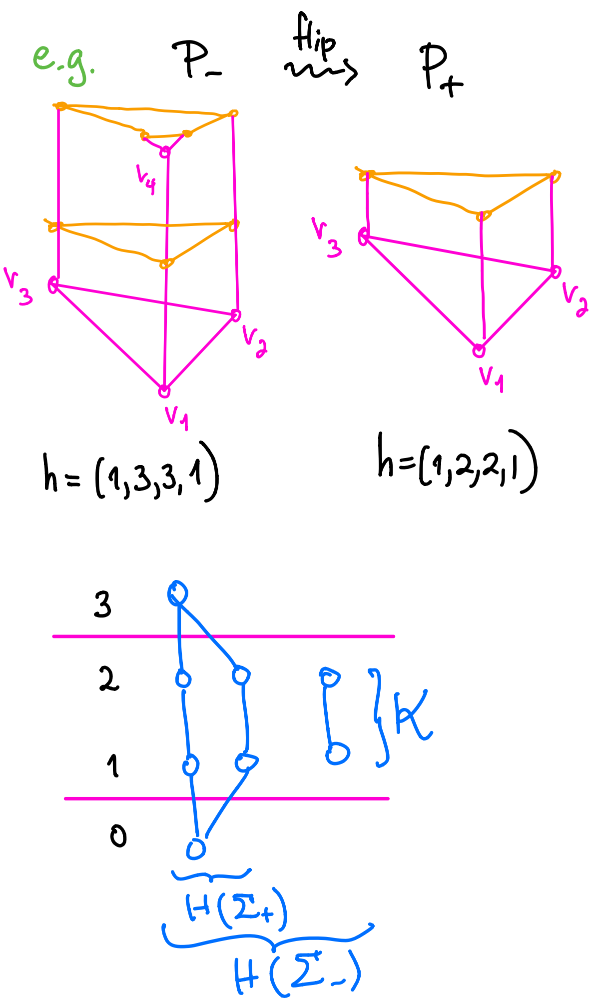
One can rename $\Sigma_t, \Sigma_{t'}$ as Σ_-, Σ_+ so that $1 \le m \le \frac{d+1}{2}$, and one has that this cone spanned by $\{h, n_1, \ldots, n_d\}$ has

Σ_- triangulating it as $\mathrm{star}_{\Sigma_-}(\tau_-) =: \Delta_-$ with $d+1-m$ d-cones
and $\mathrm{link}_{\Sigma_-}(\tau_-) \cong N\big((d-m)\text{-simplex fan}\big)$

Σ_+ triangulating it as $\mathrm{star}_{\Sigma_+}(\tau_+) =: \Delta_+$ with m d-cones
and $\mathrm{link}_{\Sigma_+}(\tau_+) \cong N\big((m-1)\text{-simplex fan}\big)$

One can check that one has this relation between $H(\Sigma_\pm)$, h-vectors:



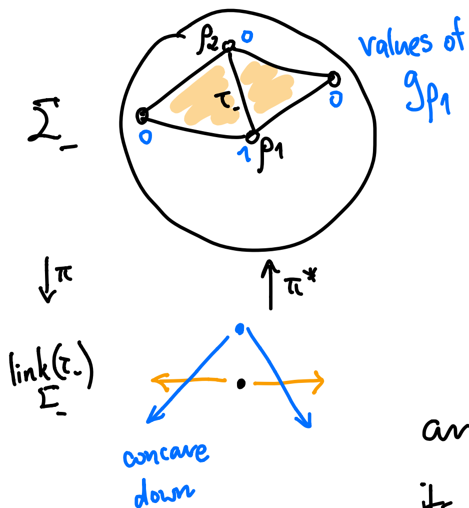


One can then produce an explicit decomposition, orthogonal with respect to the $Q_{\ell_t}(-)$

$$H(\Sigma_-) = H(\Sigma_+) \oplus \underbrace{K}$$

$$= \bigoplus_{k \text{ middle}} \overbrace{H^k(\Delta_-, \partial \Delta_-)}^{f \in R_{\Sigma_-} \text{ that vanish outside } \Delta_- = \mathrm{star}_{\Sigma_-}(\tau_-)}$$

$$\underbrace{H^k(\Delta_-, \partial \Delta_-)} = \underbrace{g_{\tau_-}}_{g_{\rho_1} g_{\rho_2} \cdots g_{\rho_m}} \cdot H^{k-m}(\Delta_-)$$



and for each of the rays $\rho_1, \rho_2, \dots, \rho_m \subset \tau_-$, its tent function g_{ρ_i} has

$$H^1(\Delta_-) = H^1(\mathrm{star}_{\Sigma_-}(\tau_-)) \xleftarrow[\sim]{\pi^*} H^1(\overbrace{\mathrm{link}_{\Sigma_-}(\tau_-)}^{(d-m)\text{-simplex fan}})$$

$$g_{\rho_i} \longleftarrow\!\shortmid \underbrace{(\pi^*)^{-1}(g_{\rho_i})}_{\text{concave down (not convex)}} = -\ell \quad \text{for Lefschetz element } \ell$$

which is roughly why $Q_{\ell_t}(-)$ ends up $(-1)^m$-definite on $PH^m(\Sigma_-) \cap K$.

∎