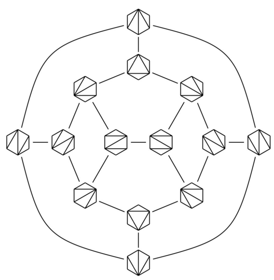
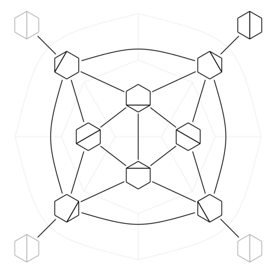
Generalized Cluster Complex



Math-8680, April 2021

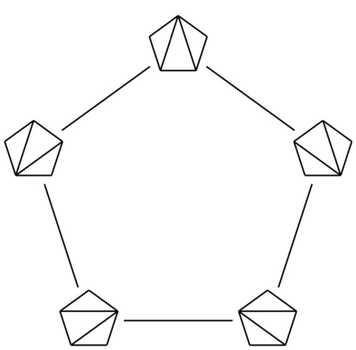
Libby Farrell



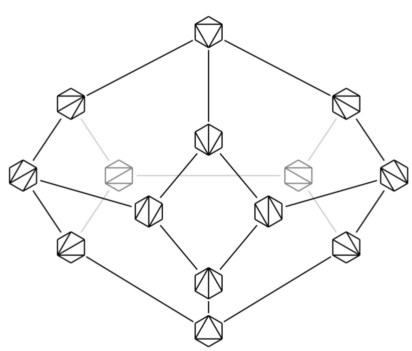
The Associahedron

A polytope with many constructions …

- Triangulations of a polygon
- Exchange graph of a type A_{n-1} cluster algebra



n = 3

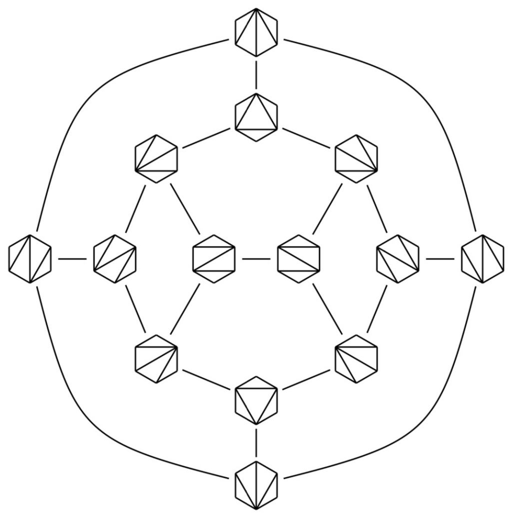


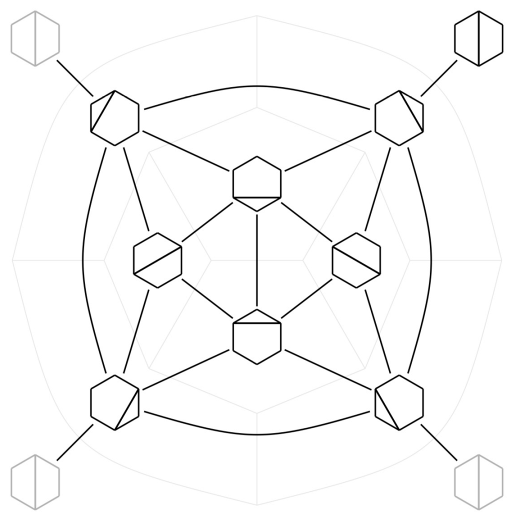
n = 4

…

The Cluster Complex

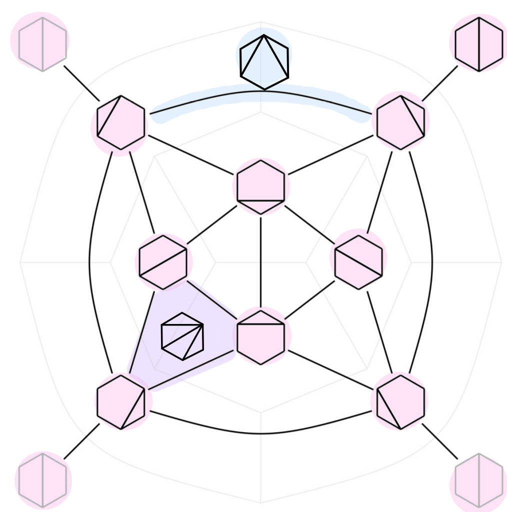
The associahedron has a *dual simplicial complex* called the cluster complex.





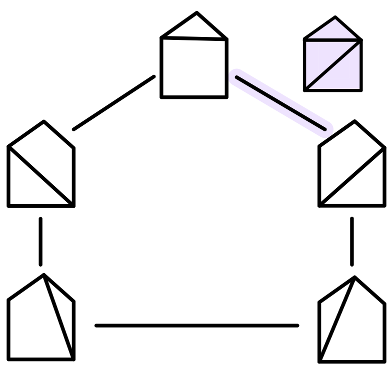
The Cluster Complex

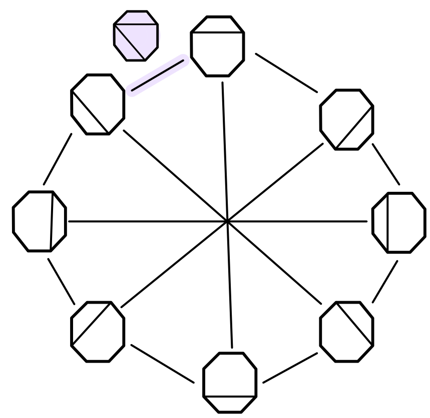
- Vertices: diagonals
- Simplices: partial triangulations
- Facets: triangulations



Type A Cluster Complex

The cluster complex is constructed via triangulations of polygons.
The (type A) generalized cluster complex is constructed via $(m+2)$-angulations of polygons.



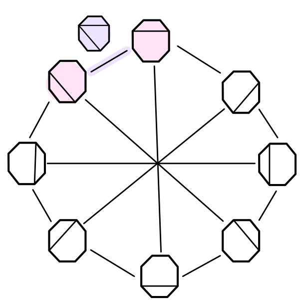


Type A Cluster Complex

- Vertices: diagonals
- Simplices: partial $(m+2)$-angulations
- Facets: $(m+2)$-angulations of an $(mn+2)$-gon

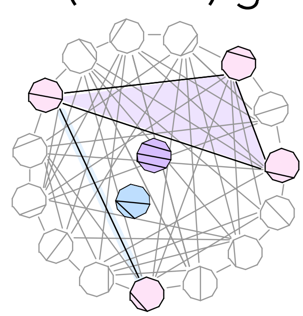
$m = 2$

$n = 3$



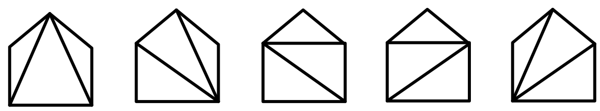
$m = 2$

$n = 4$



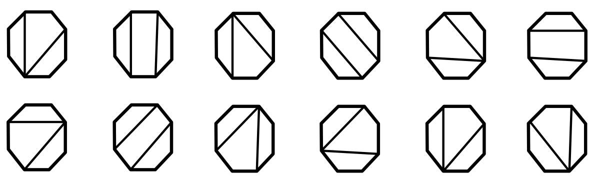
Fuss-Catalan Combinatorics

Catalan numbers count the number of triangulations of an $(n+2)$-gon.



$$\frac{1}{n}\binom{2n}{n-1}$$

Fuss-Catalan numbers count $(m+2)$-angulations of an $(mn+2)$-gon.



$$\frac{1}{n}\binom{mn+n}{n-1}$$

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f-Vectors

Component i-1 of the f-vector of the type A cluster complex is

$$\frac{1}{n}\binom{mn+i+1}{i}\binom{n}{i+1} \qquad i = 0, 1, \ldots n\text{-}1$$

In other words, this is the number of partial $(m+2)$-angulations of an $(mn+2)$-gon that have i diagonals.
(Przytycki & Sikora, 2000)

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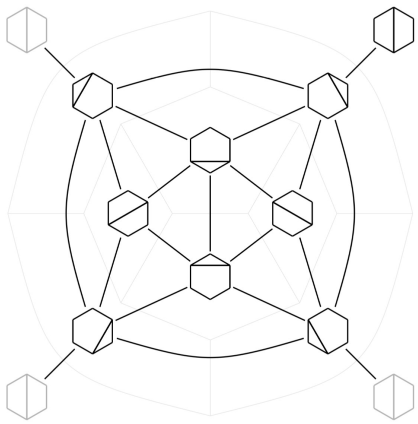
f-Vector Example



m = 1, n = 4

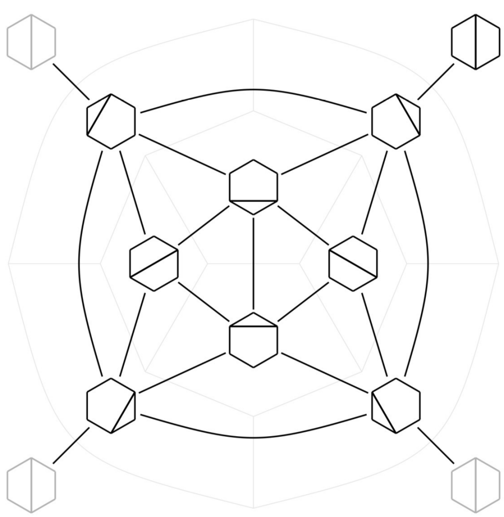
f = (1, 9, 21, 14)

h-Vectors

Component i of the h-vector of the type A cluster complex is

$$\frac{1}{i+1}\binom{n-1}{i}\binom{mn}{i} \qquad i = 0, 1, \ldots n\text{-}1$$

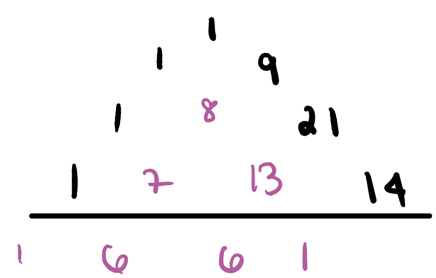
(Tzanaki, 2005)



m = 1, n = 4

f = (1, 9, 21, 14)

h = (1, 6, 6, 1)



Narayana Numbers

In the m = 1 case , component i of the h-vector of the type A cluster complex is the Narayana number N(n, i).

$$\frac{1}{i+1}\binom{n-1}{i}\binom{n}{i} \qquad i = 0, 1, \ldots n\text{-}1$$

Narayana numbers give refinement of the Catalan numbers. The sum of N(n, i) from i = 0 to n-1 gives the nth Catalan number.

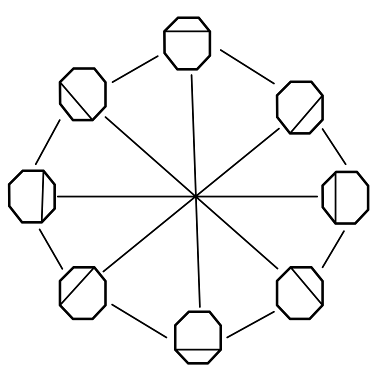
Narayana Numbers

Thus the h-vector of the type A cluster complex provides a type A generalization of Narayana numbers.

$$\frac{1}{i+1}\binom{n-1}{i}\binom{mn}{i} \qquad i = 0, 1, \ldots n\text{-}1$$

m = 2

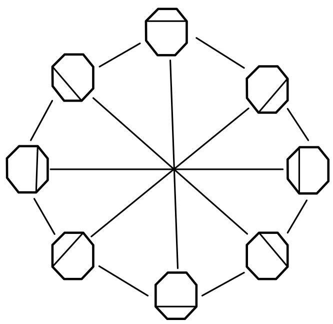
n = 3



h = (1, 6, 5)

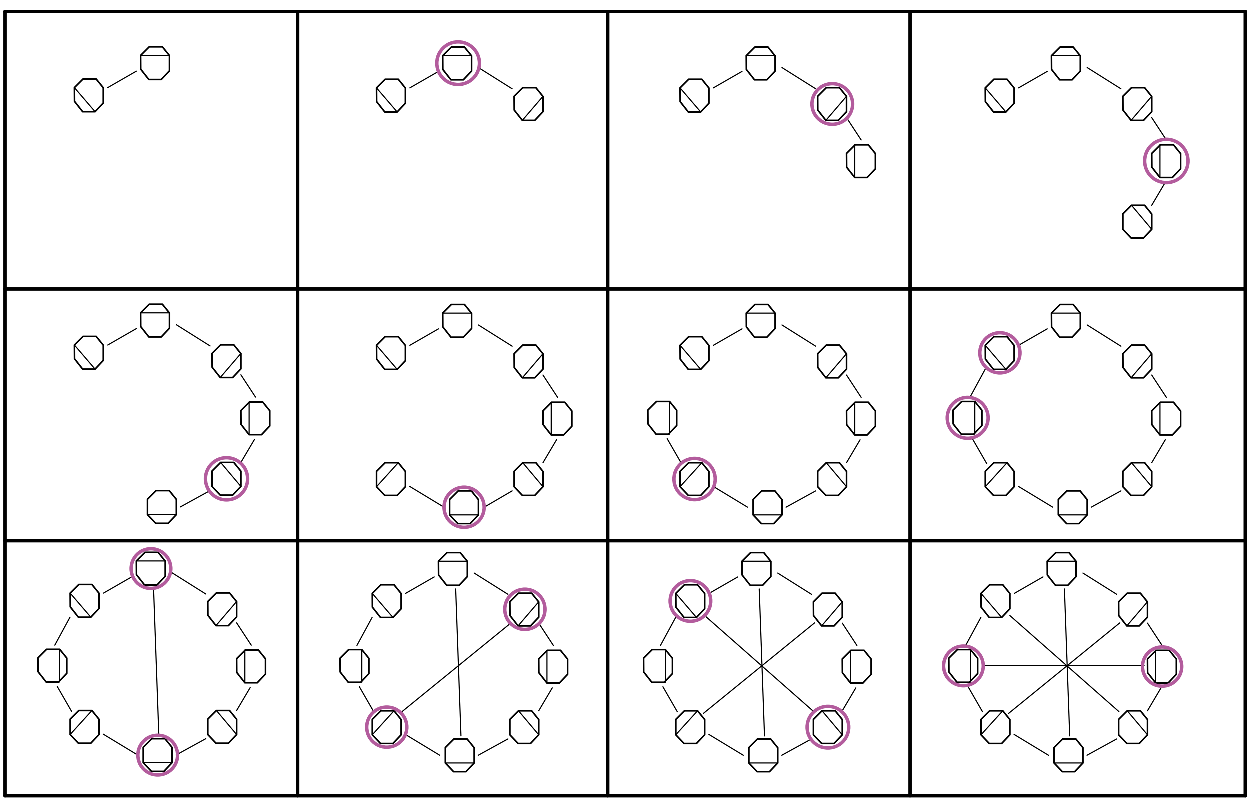
Shellability

Although the type A cluster complex is not polytopal, it is shellable. (Tzanaki, 2005)



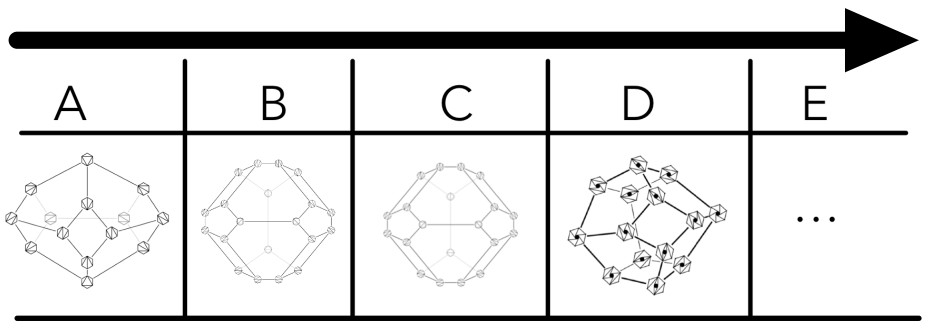
$m = 2, n = 3$

Shelling of this complex:



Generalized Associahedron

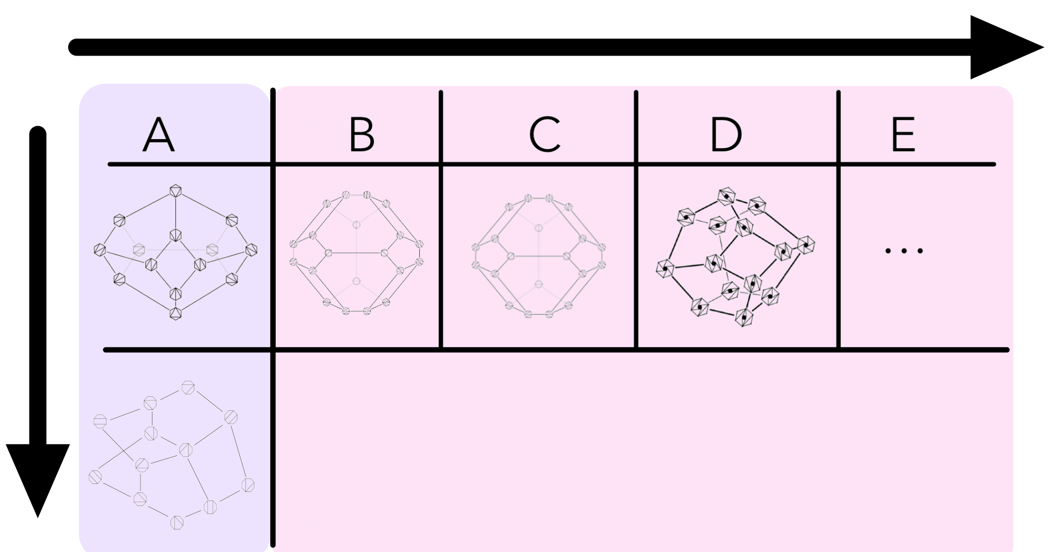
A polytope that generalizes the associahedron by *Dynkin type*.



Constructed as the exchange graph of a Dynkin type cluster algebra.

Generalized Cluster Complex

The generalized cluster complex generalizes the associahedron within and *across* types.



Properties of the Generalized Cluster Complex

There exist expressions for the *h*-vector of the generalized cluster complex of any type and any *m*. (Fomin & Reading, 2006)

The generalized cluster complex of any type and any *m* is shellable. (Athanasiadis & Tzanaki, 2007)

References

- Athanasiadis & Tzanaki, 2007. *Shellability and Higher Cohen-Macaulay Connectivity of Generalized Cluster Complexes*.
- Fomin & Reading, 2006. *Generalized Cluster Complexes and Coxeter Combinatorics*.
- Fomin & Reading, 2008. *Root Systems and Generalized Associahedra*.
- Tzanaki, 2005. *Polygon Dissections and Some Generalizations of Cluster Complexes*.
