A simple way to tell a simple polytope from it's graph

— Gil Kalai —

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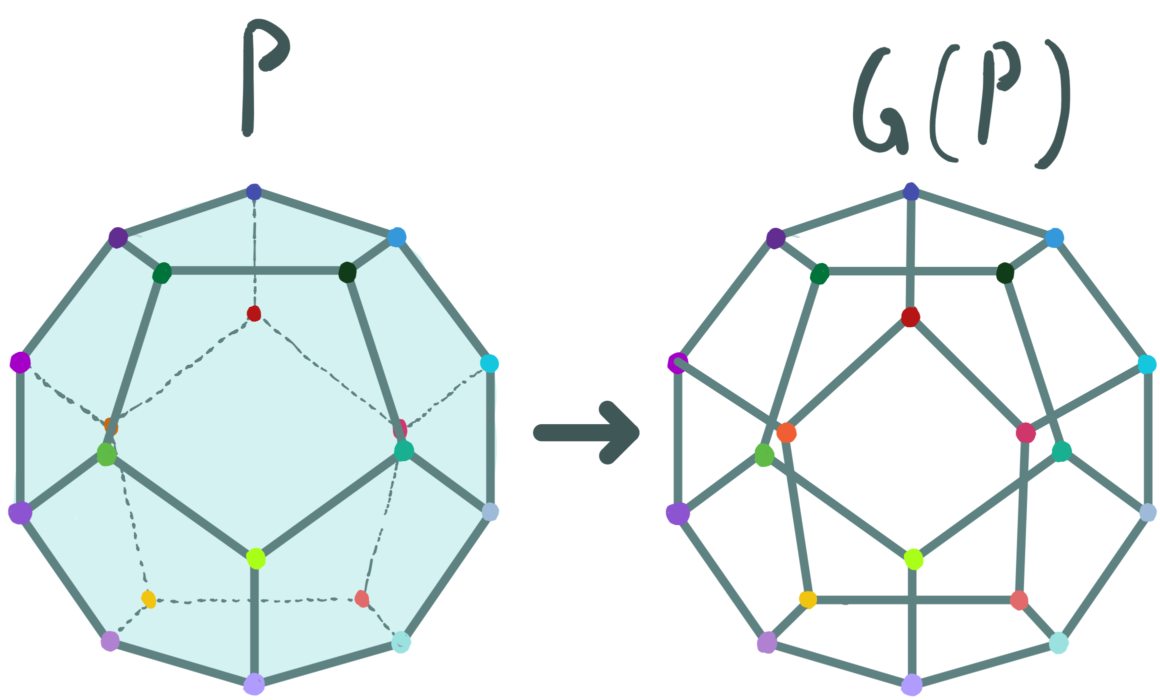
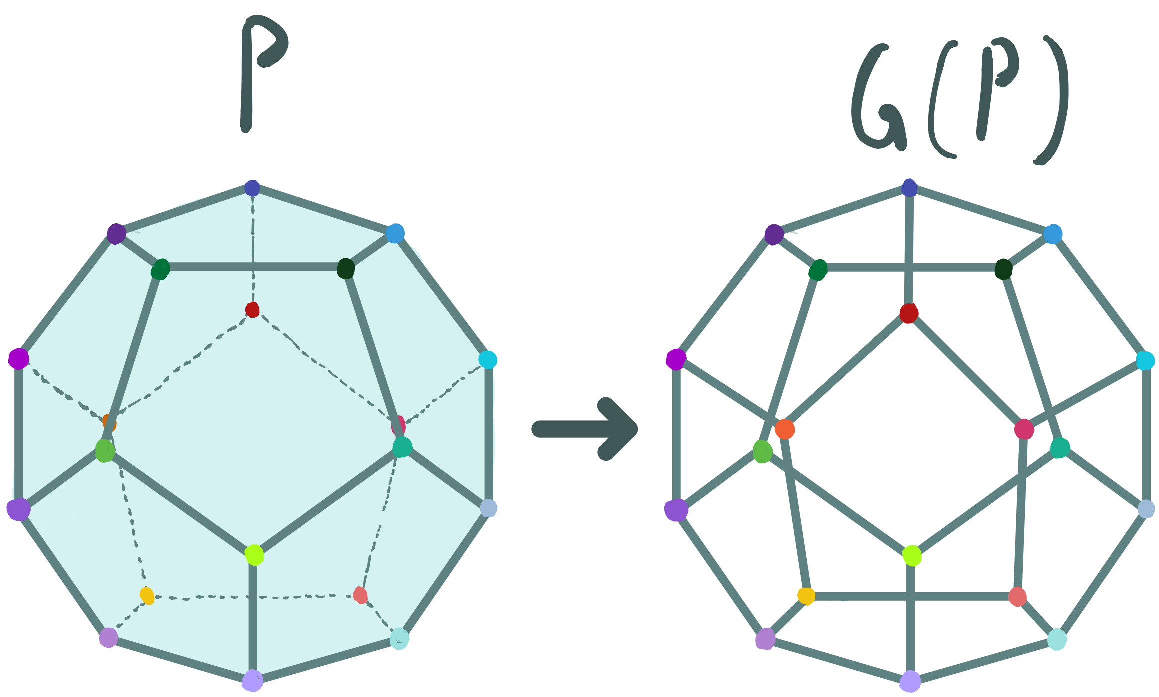


<u>Outline</u>

- Introduction/Defs.
- Acyclic Orientations
- Finding Faces

Graph of a polytope

Def: The graph a polytope P, is the collection of its vertices and edges (ie its one skeleton) and is denoted $G(P)$.



<u>Question:</u>

Is there a class of polytopes for which $G(P)$ determines its facial poset?

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Answer:

$G(P)$ determines this for all simple polytopes.

Simple Polytopes

Def: A d-dimensional polytope is simple if every vertex is contained in d-edges.

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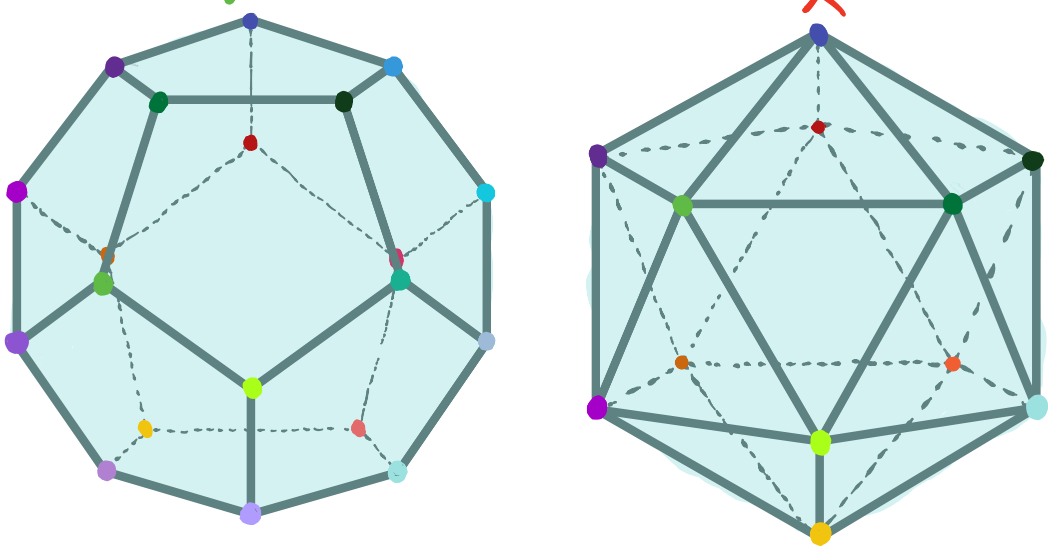

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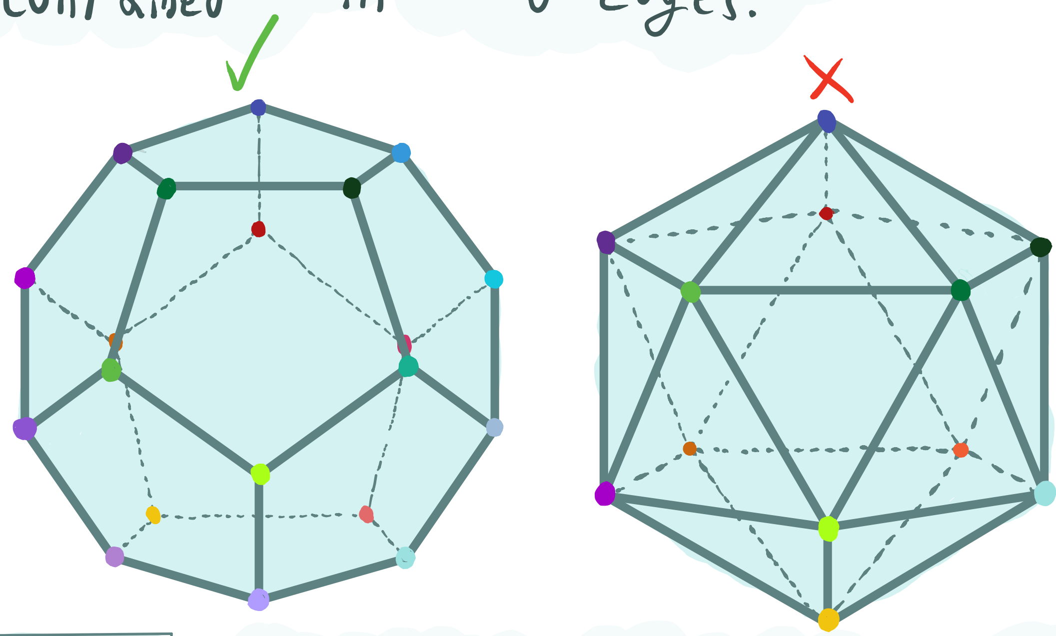
<u>Def</u>: A d-dimensional polytope is simple if every vertex is contained in d-edges.

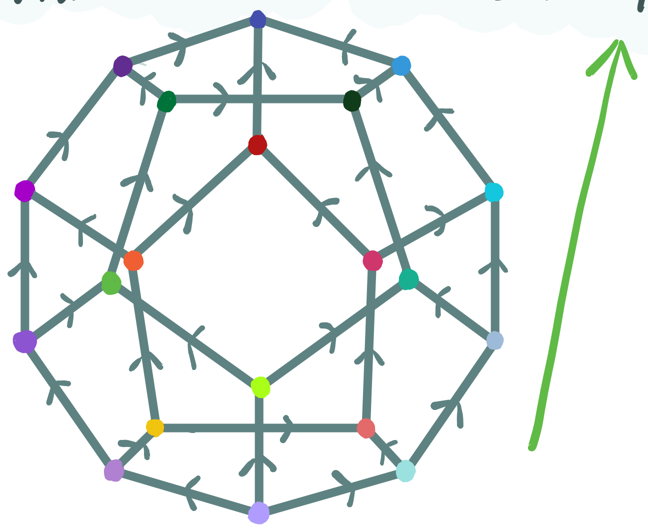


Fact 1. Every face of a simple polytope is also a simple polytope.

Fact 2. Given a vertex v and k edges adjacent to v in a simple polytope, there is a unique k-face containg all of the k edges and our vertex v.

Acyclic Orientations :

An acyclic orientation on $G(P)$ is one without any cycles. We can construct acyclic orientations via linear functionals.



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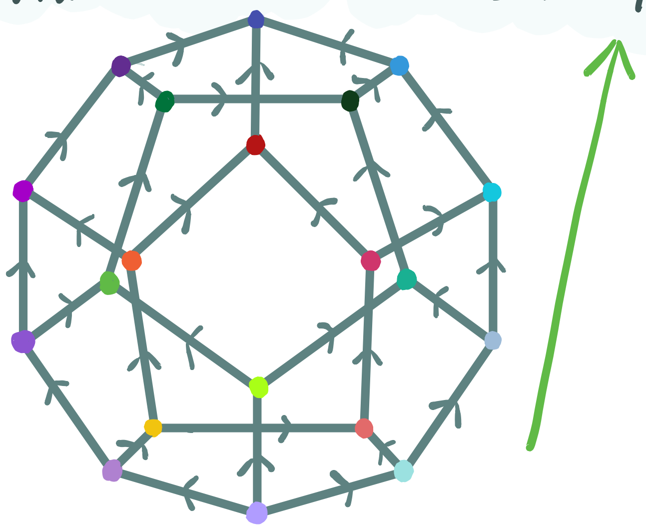
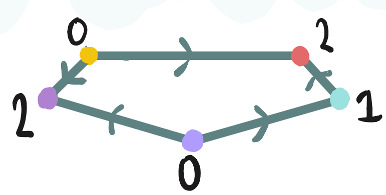


Def: Given an acyclic orientation on $G(P)$, let $\deg(v)$ be the # of adjacent edges directing into it.



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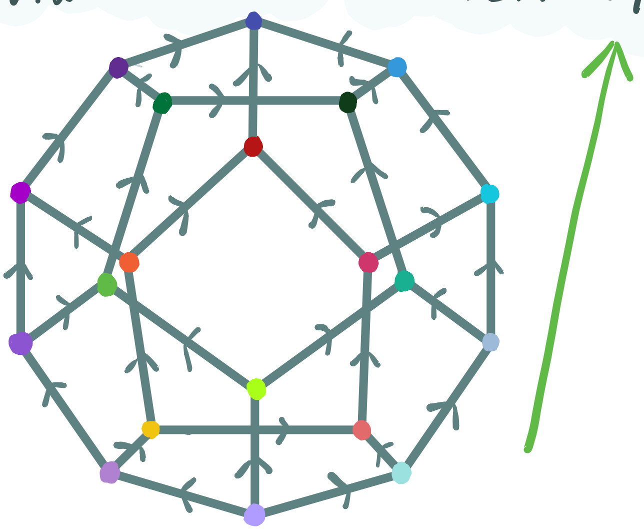

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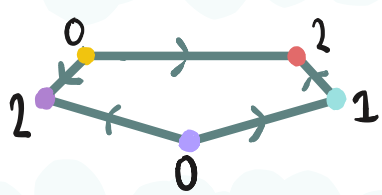
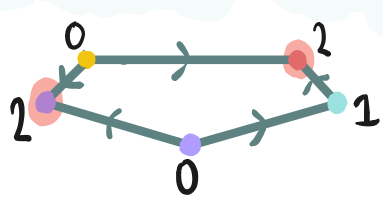
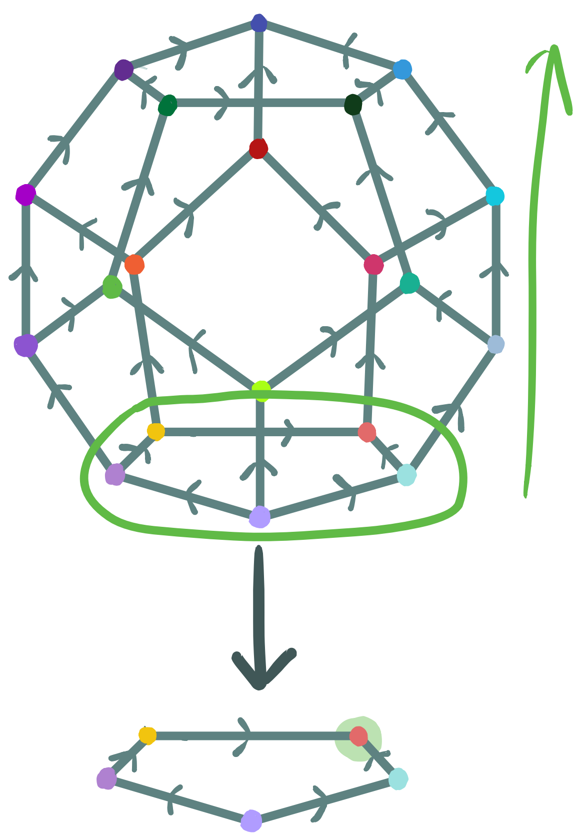


Def: If all edges of a vertex are directed into it, we call it a sink.

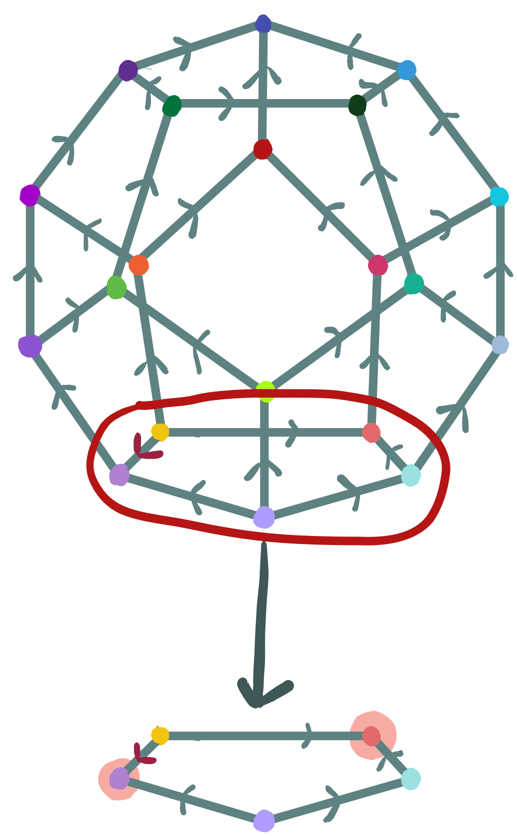


Def: An acyclic orientation on $G(P)$ is good if for each face $f \in P$, $G(f)$ has exactly one sink.

Good



Bad



Idea:

Given an acyclic orientation on $G(P)$, we will count the # of (sink, face) pairs to determine if our orientation is good.

Lower Bound:

Every face of P has at least one sink. So the # of (sink, face) pairs is $\geq$ the # of faces of P.

Idea:

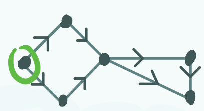
Given an acyclic orientation on $G(P)$, we will count the # of (sink, face) pairs to determine if our orientation is good.

Lower Bound:

Given P a simple polytope with any acyclic ordering on it, every face of P will have at least one sink. So the # of (sink, face) pairs is $\geq$ the # of faces of P.

"Pf":

- Every acyclic graph has at least one sink.



- Every subgraph of an acyclic graph is acyclic

- The graph $G(f)$ where f is a face of P, is a subgraph of $G(P)$.

Rk:

A good orientation is one with a unique sink for each face, so we reach our lower bound exactly when the orientation is good.

Prop:

#(sink, face) pairs $\geq$ # of faces of P, with equality exactly when our orientation is good.

Rk: A good orientation is one with a unique sink for each face, so we reach our lower bound exactly when the orientation is good.

Prop: #(sink, face) pairs $\geq$ # of faces of P, with equality exactly when our orientation is good.

How to count?

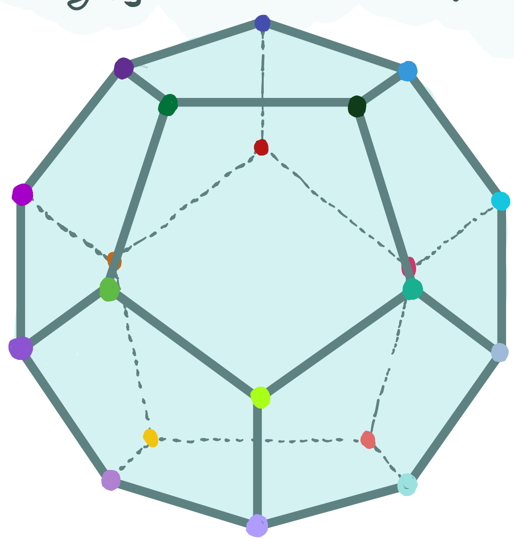
Given an acyclic orientation on $G(P)$, let h_i be the # of vertices with deg $= i$.

Then:

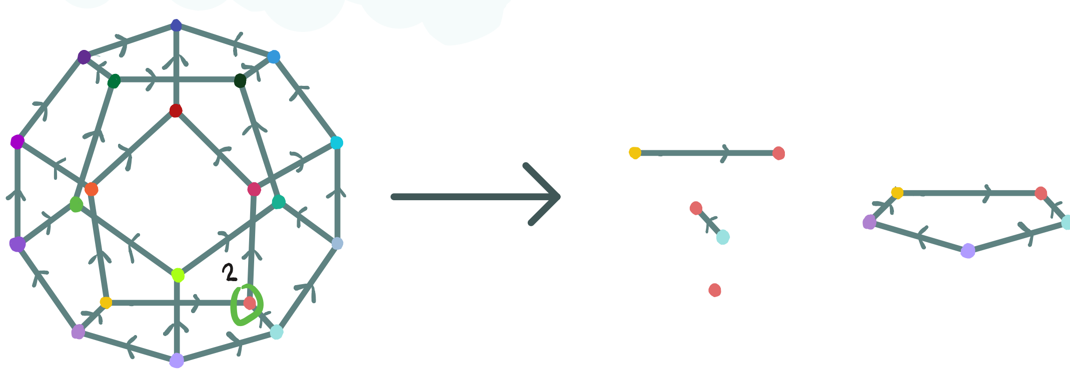
$$\#(\text{sink}, \text{face}) = h_0 + 2h_1 + \dots + 2^d h_d$$

Fact 2.

Given a vertex v and k edges adjacent to v in a simple polytope, there is a unique k-face containg all of the k edges and our vertex v.



So if a vertex v, has in-degree i, there are 2^i faces for which i is a sink.

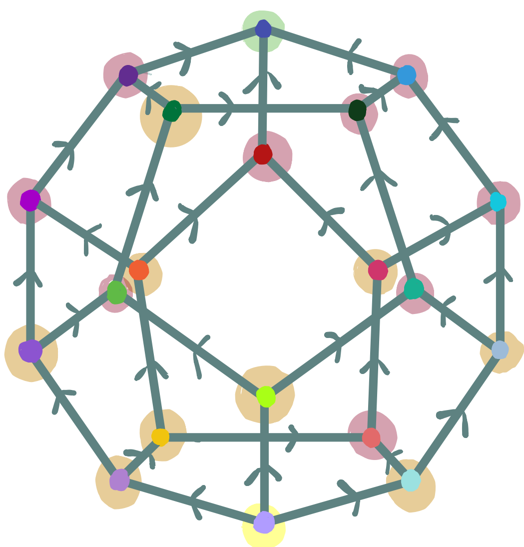


Example

$1 + \#\text{vertices} + \#\text{edges} + \#\text{facets} = \#\text{faces}$

$1 + 20 + 30 + 12 = 63$

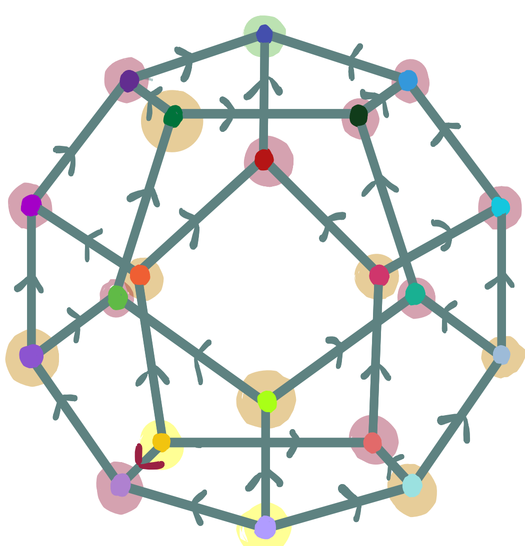
Good



- deg(0) : $1 \cdot 2^0$
- deg(1) : + $9 \cdot 2^1$
- deg(2) : + $9 \cdot 2^2$
- deg(3) : + $1 \cdot 2^3$

63

Bad



- deg(0) : $1 \cdot 2^0$
- deg(1) : + $7 \cdot 2^1$
- deg(2) : + $10 \cdot 2^2$
- deg(3) : + $1 \cdot 2^3$

64

Finite = Computable

1. Given $G(P)$ calculate:
$$\#(\text{sink}, \text{face}) = h_0 + 2h_1 + \dots + 2^d h_d$$
for all acyclic orientations of $G(P)$.

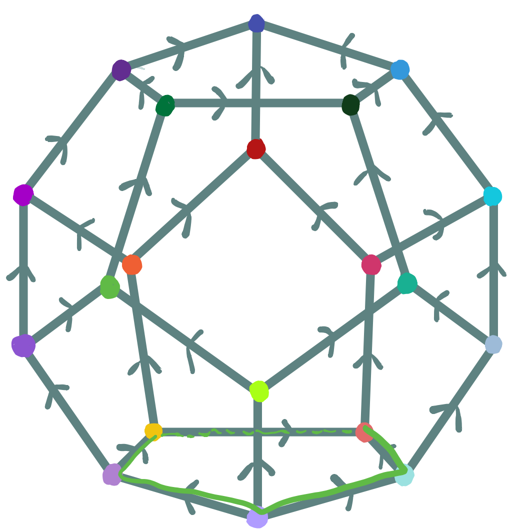
2. Since good orientations are guarenteed to exist, $\#f = \min\{\#(\text{sink}, \text{face})\}$.

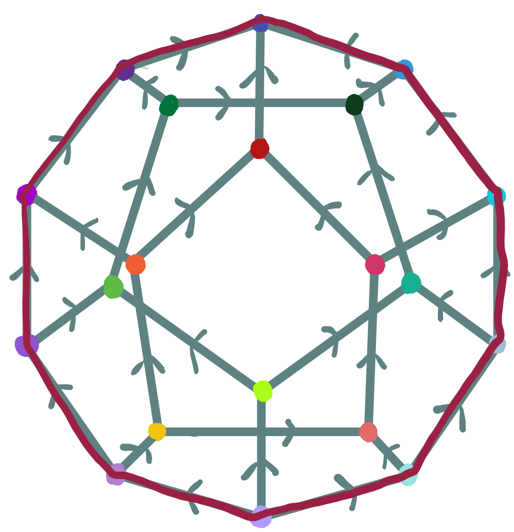
3. The good orientations are those such that $\#(\text{sink}, \text{face}) = \#f$.

So what?

Thrm:
A connected induced k-regular subgraph of G(P) is the graph of a face of P iff its vertices are initial w.r.t. some good ordering on G(P).

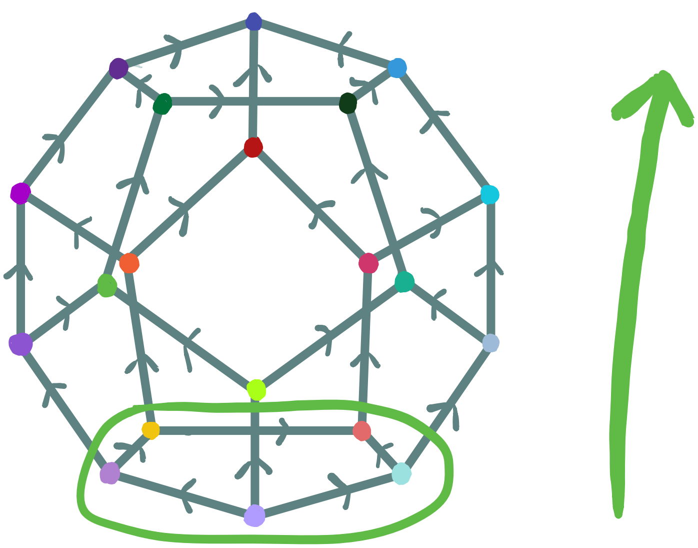
Ex:





Face ⇒ initial

"Use the linear functional defined by the supporting hyperplane of our face"



Initial $\Rightarrow$ face

"Subtle"

Key Points

- The graph of a face of a polytope is a connected, induced, and k-regular subgraph.

- $\boxed{\text{Fact 2.}}$

- $H \subseteq G$ minimal w.r.t an ordering $O \Rightarrow \forall v \in V(H)$ and $\forall w \in V(G)$ s.t. $w \leq_O v$, $w \in V(H)$.

- If H, G connected and k-regular and $H \subseteq G$ then $H = G$

Final Algorithm

Given $G(P)$:

1. Compute all good acyclic orientations

2. Compute all k-regular, induced, connected subgraphs of $G(P)$

3. Check which of those are initial w.r.t. a good ordering

4. Construct a poset by inclusion from the initial subgraphs.