A Characterization of Gorenstein Rings

Math 8680 Student Talk

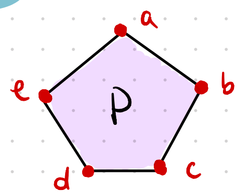
Sasha Pevzner

References

- Stanley: "Invariants of finite groups and their applications to combinatorics"

- Fløystad, Mccullough, Peeva: "Three themes of syzygies"

Free resolutions

Let $\Delta = \partial P$, $P =$ 

$$k[\Delta] = k[a,b,c,d,e] \Big/ (ac,\ ad,\ bd,\ be,\ ce)$$

homogeneous

$$= S / I_\Delta$$

Here is a free resolution of $k[\Delta]$ (as an S-module):

$$f_i \mapsto$$

$$0 \leftarrow S/I_\Delta \leftarrow S \longleftarrow S(-2)^5 \longleftarrow S(-3)^5 \longleftarrow S(-5) \leftarrow 0$$

$$\begin{aligned}
ac &\longleftarrow\!\shortmid\ e_1 & \qquad d\cdot e_1 - c\cdot e_2 &\longleftarrow\!\shortmid\ f_1 \\
ad &\longleftarrow\!\shortmid\ e_2 & \qquad e\cdot e_1 - a\cdot e_5 &\longleftarrow\!\shortmid\ f_2 \\
&\ \ \vdots & & \\
ce &\longleftarrow\!\shortmid\ e_5 & &
\end{aligned}$$

The same resolution, with maps:

$$\begin{tikzcd}[column sep=large]
0 & S/I_\Delta \arrow[l] & S \arrow[l] \arrow[d, equal] & S(-2)^5 \arrow[l, "{\begin{bmatrix} ac & ad & bd & be & ce \end{bmatrix}}"'] \arrow[d, equal] & S(-3)^5 \arrow[l, "{\begin{bmatrix} 0 & -d & -e & 0 & 0 \\ -b & c & 0 & 0 & 0 \\ a & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & -c & d \\ 0 & 0 & a & b & 0 \end{bmatrix}}"'] \arrow[d, equal] & S(-5) \arrow[l, "{\begin{bmatrix} -ce \\ -be \\ bd \\ -ad \\ -ac \end{bmatrix}}"'] \arrow[d, equal] & 0 \arrow[l] \\
 & & F_0 & F_1 & F_2 & F_3 &
\end{tikzcd}$$

$\beta_1 = 5$ $\qquad$ $\beta_{1,2} = 5, \quad \beta_{1,j} = 0 \quad j \neq 2$

Some facts

① Minimal free resolutions are unique (up to isomorphism).

② Hilbert Syzygy Theorem (1890):

Every finitely-generated $k[x_1, \ldots, x_n]$-module has a finite free resolution, of length $\leq n$.

Information from the minimal free resolution

$$0 \leftarrow k[\Delta] \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_i \leftarrow \cdots \leftarrow F_h \leftarrow 0$$

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$$

graded Betti numbers of $k[\Delta]$

projective / homological dimension of $k[\Delta]$ $= \mathrm{pd}_S(k[\Delta])$

* Note: "ungraded" Betti numbers are

$$\beta_i = \sum_j \beta_{ij}$$

Auslander - Buchsbaum formula: $\mathrm{pd}_S(k[\Delta]) = n - \mathrm{depth}(k[\Delta])$

For our example: pd () $= 5 - (\dim(\Delta) + 1) = 3$

Hilbert series from free resolutions

Recall An exact sequence $(0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_\ell \leftarrow 0)$ of graded S-modules has $\sum_{i=0}^{\ell} (-1)^i \operatorname{Hilb}(M_i, t) = 0$.

Computing $\operatorname{Hilb}(k[\Delta], t)$ $\quad 0 \leftarrow k[\Delta] \leftarrow S \leftarrow S(-2)^5 \leftarrow S(-3)^5 \leftarrow S(-5) \leftarrow 0$

$$\operatorname{Hilb}(k[\Delta], t) = \operatorname{Hilb}(S,t) - \operatorname{Hilb}(S(-2)^5, t) + \operatorname{Hilb}(S(-3)^5, t) - \operatorname{Hilb}(S(-5), t)$$

$$= \frac{1 - 5t^2 + 5t^3 - t^5}{(1-t)^5}$$

In general $\quad \operatorname{Hilb}(M, t) = \left[\sum_{i \geq 0} (-1)^i \sum_j t^{\beta_{ij}} \right] \Big/ (1-t)^n$

The functor $(-)^* = \mathrm{Hom}_S(-, S)$

For an S-module M, define $M^* = \mathrm{Hom}_S(M, S)$

$$\begin{tikzcd} \varphi : M \arrow[r] \arrow[d, rightsquigarrow] & N \\ \varphi^* : N^* \arrow[r] & M^* \end{tikzcd}$$

$\varphi : M \longrightarrow N$ an S-module map

$\varphi^* : N^* \longrightarrow M^*$, $f \longmapsto f \circ \varphi$

For M, N free,
if A, A^* represent φ, φ^*, then $A^* = A^T$.

Applying $(-)^*$ to the minimal free resolution

$$0 \longrightarrow \underset{\overset{\shortparallel}{0}}{k[\Delta]^*} \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_h^* \longrightarrow 0 \qquad (\star)$$

Theorem If $k[\Delta]$ is Cohen-Macaulay, then $(\star)$ is exact, except possibly at F_h^*.

To make it exact, we append the 😱🧟‍♀️ **canonical module** $\Omega(k[\Delta])$.

The canonical module

$$0 \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_h^* \longrightarrow \Omega(k[\Delta]) \longrightarrow 0 \qquad (*)$$

Some "easy" observations:

- $(*)$ is a minimal free resolution of $\Omega(k[\Delta])$ (as an S-mod).
- $\Omega(k[\Delta])$ is also a $k[\Delta]$-module, with $\beta_h(k[\Delta])$ minimal generators.

Upshot: $\beta_h(k[\Delta])$ depends only on $k[\Delta]$!

Type

type $k[\Delta] := \beta_h(k[\Delta])$

$= \#$ of minimal generators of $\Omega(k[\Delta])$

as a $k[\Delta]$ (and S) - module.

Type is an important invariant of Cohen-Macaulay rings.

What happens when type $k[\Delta] = 1$?

$k[\Delta]$ and $\Omega(k[\Delta])$ are isomorphic

(as $k[\Delta]$-modules)

Gorenstein rings

Say Cohen-Macaulay $k[\Delta]$ is Gorenstein if type $k[\Delta] = 1$.

2 minimal free resolutions of $k[\Delta]$:

$$0 \leftarrow k[\Delta] \leftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_h \leftarrow 0$$

$$0 \leftarrow \underset{\substack{\cong \\ \Omega(k[\Delta])}}{k[\Delta]} \longleftarrow F_h^* \longleftarrow \cdots \longleftarrow F_1^* \longleftarrow F_0^* \longleftarrow 0$$

Consequences

$$F_i \cong F_{h-i}^* \qquad 0 \le i \le h$$

$$\beta_i(k[\Delta]) = \beta_{h-i}(k[\Delta])$$

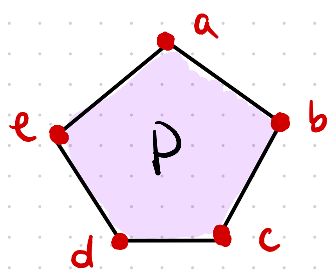
"The minimal free resolution is self-dual."

Another definition of type

Let $A = k[\Delta] / (\underline{\theta})$

an h.s.o.p.

$\text{type } k[\Delta] = \dim_k \text{soc } A$, where $\text{soc } A = \left\{ \begin{array}{l} \text{elements of } A \\ \text{annihilated by } k[\Delta]_+ \end{array} \right\}$



$k[\Delta] = k[a,b,c,d,e] / (ac, ad, bd, be, ce)$

$\underline{\theta} = (a+c+e,\ b+d+e)$

Basis for A: $\{1 \mid c, d, e \mid ae\}$

$\text{soc } A = \text{span}_k \{ae\}$

$a \cdot ae = e(c+e)(c+e) = 0$

$b \cdot ae = 0$

$c \cdot ae = 0$

$d \cdot ae = 0$

$e \cdot ae = a(a+c)(a+c) = 0$

Summary

We now have several ways to think about Gorenstein rings:

① Poincaré duality

$A = \mathrm{span}_k \{ 1, \mid c, d, e, \mid ae \}$

$$\begin{array}{c|ccc} & c & d & e \\ \hline c & & & \\ d & & \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \\ e & & & \end{array}$$

$\det = 1$

② Symmetry in the minimal free resolution

$$0 \leftarrow k[\Delta] \leftarrow S^1 \leftarrow S^5 \leftarrow S^5 \leftarrow S^1 \leftarrow 0$$

③ Socle of $k[\Delta]/(\underline{\theta})$ has k-dimension 1

$\mathrm{Soc}\, A = \mathrm{span}_k \{ae\}$