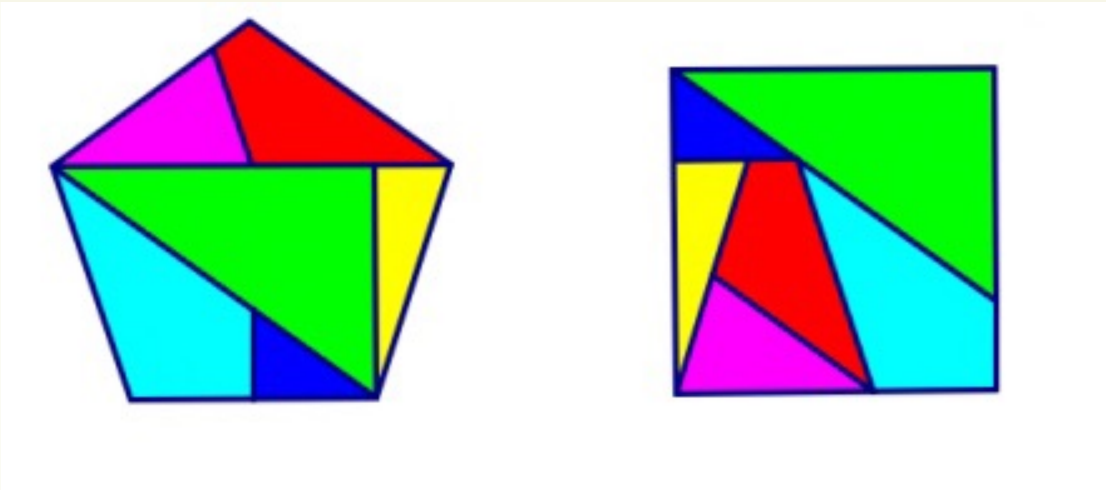
Polytope algebra

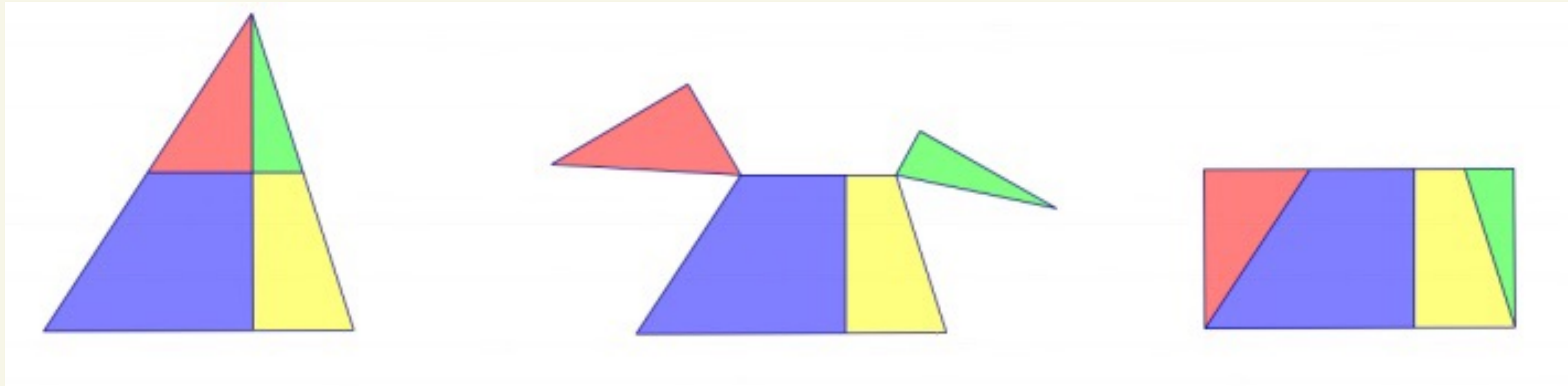
Peter McMullen

- Its group structure and scissor congruence problem
- Its "graded" algebra structure and the g-theorem

Scissor congruence Problem

Are any two polygons in $\mathbb{R}^2$ with same area related to each other by cutting, isometries (translation, rotation) and gluing





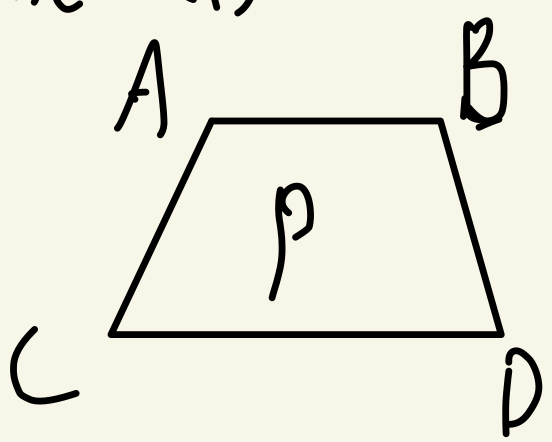
[Wallace-Bolyai-Gerwien]

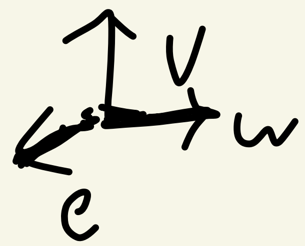
Yes! And the isometries can be restricted to just: translations + rotation by 180°

$H_v(P) = |AB| - |CD|$

$H_w(P) = 0$

$H_e(P) = -|BD|$





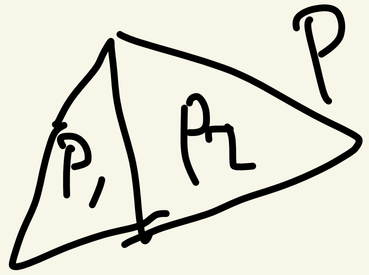
Hadwiger invariant

[Hadwiger] Polygons are scissor congruent by translations iff they have the same area and Hadwiger invariants (defined using edge length)

Higher dim | **Hilbert's 3rd problem**

already false in dim=3 [Dehn].

Defined Dehn invariant $D(P)$ s.t. $D(P) = D(P_1) + D(P_2)$

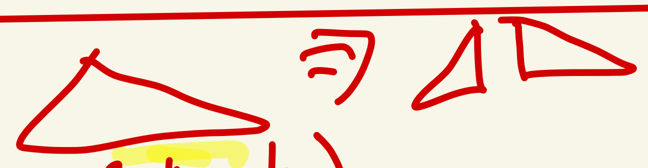


(defined using edge length + angle between faces)

showed $D(\square) = 0$, $D(\triangle) \neq 0$

[Gauss] ∞ scissor congruence $\Longleftarrow$ volume equal

[Banach-Tarski] Any two bounded subsets in $\mathbb{R}^3$ with nonempty interior are equidecomposable



(dissect)

(decompose)

Goal | Unify all this, study dim 0 to n altogether

- Generalize Hadwiger, Dehn invariants to $\mathbb{R}^n$ with translation

or any space X (with a metric) with isometry group G

$$\Pi^n \;:=\; \mathbb{R}\{P \mid \text{polytopes in } \mathbb{R}^n\} \Big/ \begin{array}{l} [P]=[P+v]\ \forall v \in \mathbb{R}^n \\ [P]+[Q]=[P\cup Q]+[P\cap Q] \end{array}$$

$0 \le \dim \le n$

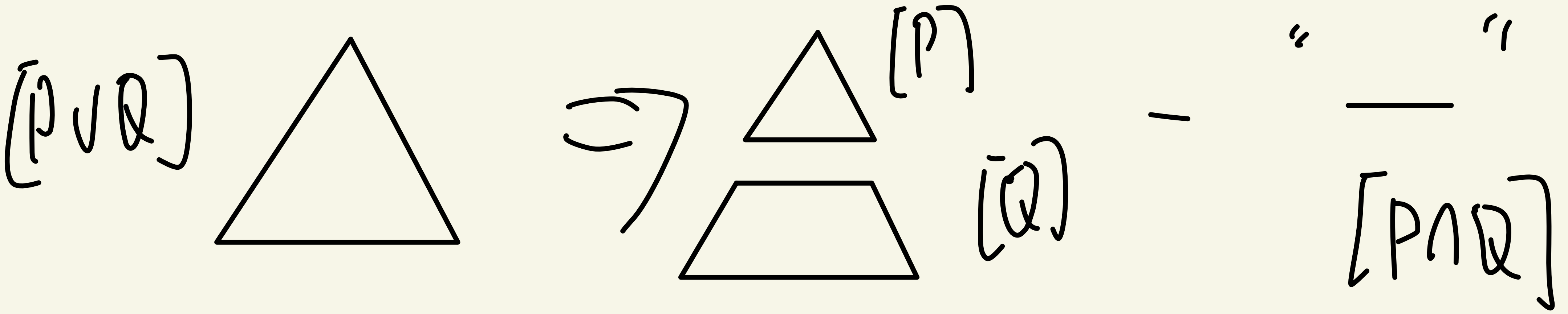
[Mcmullen, Goodwillie], If $[P]=[Q]$ in Π^n (resp. $\Pi^n(G)$), then they are scissor congruent by translations (resp. isometries in G)

Converse not true

If $X_1 = P_1 \cup Q_1$ with $P_1 = P_2 + v$ then $[X_1] = [P_1] + [Q_1] - [P_1 \cap Q_1]$

$X_2 = P_2 \cup Q_2$ $\quad Q_1 = Q_2 + w$ $\quad = [P_2] + [Q_2] - [P_1 \cap Q_1]$

$= [X_2] - [P_1 \cap Q_1] + [P_2 \cap Q_2]$

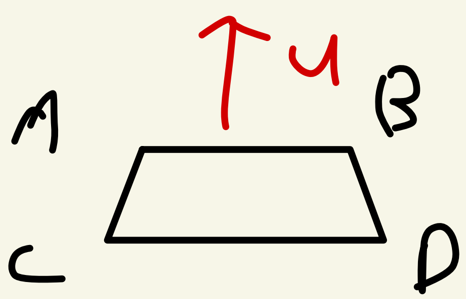


Keeping track of this is very important for the "graded" algebra structure on Π^n

So what are the invariants in Π^n that distinguish scissor congruent classes? i.e. $f_U : \Pi^n \to \mathbb{R}$ s.t. if $f_U([P]) = f_U([Q])$ $\forall$ parameter U, then $[P]=[Q]$

Answer: Frame Functionals. Defined in the same way as Hadwiger invariants, $f_U(P) = \mathrm{Vol}_{U^\perp}(P_U)$ for every set of orthogonal vectors $U := (u_1, \ldots, u_k)$

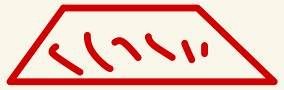
Hadwiger is special case in $n=2$! $k=1$



$$\mathrm{Had}_u(P) = |AB| - |CD|$$

"Graded" Algebra structure on Π^n

- $[P]\cdot[Q] := [P+Q]$ where $P+Q := \{x \in \mathbb{R}^n \mid x = y+z,\ y \in P,\ z \in Q\}$

$[$  $]\cdot[$ — $] = [$  $]$, $[P]\cdot[\,\cdot\,] = [P+v] = [P]$

unit

- There's a decomposition $\Pi^n = G_0 \oplus \cdots \oplus G_n$, it is indeed graded by dim, but there's a more useful characterization

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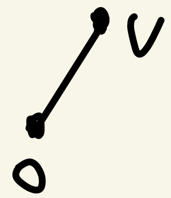
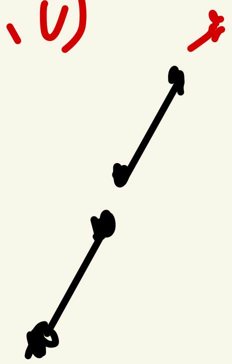
Claim: $\forall P,\ ([P]-1)^k = 0 \quad \forall k \gg 0$

proof by "e.g."

let $L := \mathrm{conv}(0, v)$, $L =$ 

$([L]-1)^2 = [L]\cdot[L]$  $-2[L]$  $+ 1\cdot = 0$

"$[2L]$"

$[2L]$

In fact. $\log([p+q]) = \log([p]\cdot[q]) = \log[p] + \log[q]$

$\mathcal{E}_r := \langle (\log[p])^r \rangle$ for $r \geq 1$

$\mathcal{E}_0 := \mathbb{Z} = \langle [\cdot] \rangle$

Dilations $\delta_\lambda : \Pi^n \to \Pi^n$

act on generators like this!

$$\delta_\lambda([p]) = [\lambda p] \neq \lambda [p]$$

$x \in \Pi^n$, $\lambda > 0$, $\lambda \neq 1$

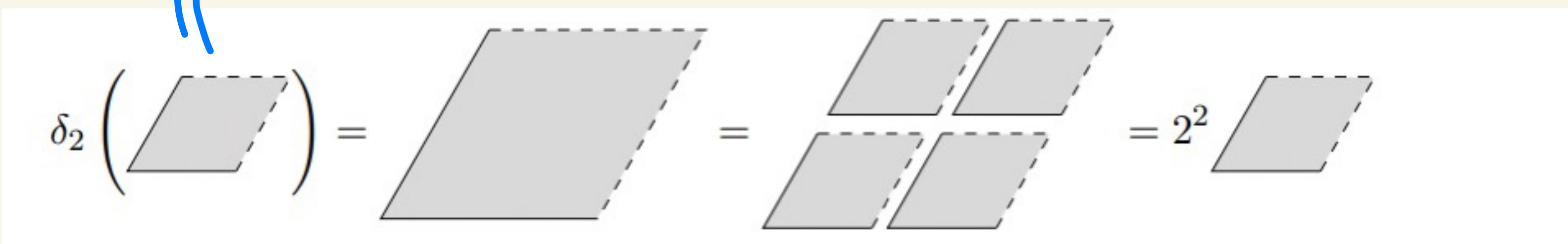
$x \in \mathcal{E}_r$ iff $\delta_\lambda x = \lambda^r x$ iff $\dim x = r$

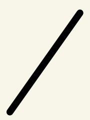
$\mathcal{E}_1 := \langle \log[p] \rangle$

e.g. $n=2$, $\lambda = 2$

$\log[L_1]\cdot\log[L_2]$

$\|$



$L_1 =$  $\log(L_1) = ([L_1] - 1) =$ 

$L_2 =$ $\log(L_2) = ([L_2] - 1) =$ 

[McMullen, Thm 1] ① Π^n has direct sum decomposition

$$\Pi^n = \Xi_0 \oplus \cdots \oplus \Xi_n .$$

② $\Xi_r \cdot \Xi_s = \Xi_{r+s}$ with $\Xi_r = 0 \ \forall r > n$

③ $\Xi_0 \cong \mathbb{Z}$, Ξ_i is vector space over $\mathbb{R}$ $\forall i > 0$

④ product in $\bigoplus_{r \geq 1} \Xi_r$ is $\mathbb{R}$-bilinear

⑤ $\delta\lambda := ([P] \mapsto [\lambda P])$ is ring hom to itself

and if $x \in \Xi_r$, $\lambda \geq 0$ then $\delta\lambda x = \lambda^n x$

$\boxed{\text{Def}}$ For polytope $P \subseteq \mathbb{R}^n$, $\Pi^n(P) = \mathbb{R}\left\{ Q \subseteq \mathbb{R}^n \;\middle|\; \begin{array}{l} P = Q + Q' \\ \text{for some } Q' \end{array} \right\} / \sim$ same relation

[McMullen] $\Pi^n(P)$ is a subalgebra of Π^n

and $\Pi^n(P) = \Xi_0(P) \oplus \cdots \oplus \Xi_n(P)$ " weight spaces

$h := (\dim(\Xi_0(P)), \dim(\Xi_1(P)), \ldots, \dim(\Xi_n(P)))$ is the h-vector of P

$\forall$ simple P, $\log(P)^{d-2r}\ \Xi_r(P) = \Xi_{d-r}(P)$ $\forall\ 0 \le r \le \frac{1}{2}d$

$\Pi^n(P)$ has Lefschetz decomposition multiplication by $\Xi_1(P)$

gives "$\Leftarrow$" of $\boxed{g\text{-theorem}}$ on h-vectors (Stanley, Billera-Lee, McMullen)
$\Leftarrow$ $\Rightarrow$

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① $h_0 = 1$, $h_{i+1} - h_i \le (h_i - h_{i-1})^{\langle i \rangle}$ ② $h_i = h_{d-i}$ $\quad 0 \le i \le \lfloor \frac{d}{2} \rfloor$ ③ $h_{i+1} \ge h_i$ $\quad 0 \le i \le \lfloor \frac{d}{2} \rfloor - 1$ $\iff$	$h = h(P)$ for a simplicial d-dim polytope P,

Goal | Unify all this, study dim 0 to n altogether

- Generalize Hadwiger. Dehn invariants to $\mathbb{R}^n$ with translation or any space X with isometry group G

$\mathcal{E}(X, G) := \left\{ \begin{array}{l} \text{free abelian group} \\ \text{generated by polytope in } X \end{array} \right\} \Big/ \begin{array}{l} [P] = [g \cdot P] \ \forall g \in G \\ [P] + [Q] = [P \cup Q] + [P \cap Q] \end{array}$

$0 \leq \dim \leq n$

In particular, McMullen's polytope group is $\mathcal{E}(\mathbb{R}^n, \mathbb{R}^n)$

[McMullen, Goodwillie], If $[P] = [Q]$ in $\mathcal{E}(\mathbb{R}^n, G)$ then P, Q are scissor congruent by isometries in G

$X_1 = P_1 \cup Q_1$ with $P_1 = g P_2$ then $[X_1] = [P_1] + [Q_1] - [P_1 \cap Q_1]$

if $X_2 = P_2 \cup Q_2$ $\quad Q_1 = \tilde{g} Q_2 \quad = [P_2] + [Q_2] - [P_1 \cap Q_1]$

5min ① Definition

— generators - relations - example - ring structure - scaling

① application (lemma 1) - Translation - invariant (simple) polytope invariant
Jessen - Thorup - Sah
polytope invariants as ring hom to $\mathbb{Z}$ or $\mathbb{Q}$

5min — iso to other algebra

③ Thms / 3min Thm3 - (volume)

Scissors vs 2D vs 3D

Log / 0.5

4min Thm Brion - iso to piecewise polynomial