STANLEY-REISNER, SIMPLICIAL COMPLEX, FAN EXERCISES

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For the first half of Spring 2021 Math 8680 Topics in Combinatorics:
Combinatorial rings and the Kahler package

1. Recall Stanley's *triangle shortcut* mentioned in lecture is a means of computing the h-vector $\mathbf{h} = (h_0, h_1, \ldots, h_d)$ from the f-vector $\mathbf{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ of a $(d-1)$-dimensional simplicial complex Δ, shown here for $d = 4$:

$$\begin{array}{cccccccccc}
 & & & & & 1 & & & & \\
 & & & & 1 & & f_0 & & & \\
 & & & 1 & & . & & f_1 & & \\
 & & 1 & & . & & . & & f_2 & \\
 & 1 & & . & & . & & . & & f_3 \\
(1 & , & h_1 & , & h_2 & , & h_3 & , & h_4) &
\end{array}$$

The dots are filled in from top row to bottom, using this local rule: $\begin{array}{ccc} x & & y \\ & y-x & \end{array}$

(a) Prove that this does indeeed compute the h-vector defined in class.

(b) Prove that the rows of the triangle are simultanously computing the h-vectors of each of the i-dimensional *skeleta* $\Delta^{(i)}$, that is, the subcomplex generated by all of the i-dimensional faces.

2. Let Δ be a simplicial complex on vertex set $\{1, 2, \ldots, n\}$, with $n \geq 1$, and all vertices present as faces, that is, $f_0(\Delta) = n$. Let $K[\Delta]$ its Stanley-Reisner ring with coefficients in some field K.

(a) Prove that the sum of all vertex variables $\theta := x_1 + x_2 + \cdots + x_n$ is always a non-zero-divisor in $K[\Delta]$.

(b) Prove that if Δ is disconnected, then the quotient ring $K[\Delta]/(\theta)$ contains a nonzero element of degree one annihilated by all elements of positive degree. Conclude that every element of positive degree is a zero-divisor.

3. Prove that if a simplicial complex Δ is (pure and) shellable, then the same is true of $\mathrm{star}_\Delta(F)$ and $\mathrm{link}_\Delta(F)$ for every face F in Δ.

4. Let $\Delta := \{\varnothing, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34\}$, the 1-dimensional simplicial complex which is the complete graph on vertex set $\{1, 2, 3, 4\}$.

(a) Prove that for a field K, the ring $K[\Delta]$ contains a linear system of parameters (that is, θ_1, θ_2 in $K[\Delta]_1$ with $K[\Delta]$ finitely generated as a $K[\theta_1, \theta_2]$-module) if and only if $\#K \geq 3$, that is, $K \neq \mathbb{F}_2$.

(b) Show that the ring $\mathbb{Z}[\Delta]$ contains **no** pair of degree one elements θ_1, θ_2 in $\mathbb{Z}[\Delta]_1$ for which $\mathbb{Z}[\Delta]$ is finitely generated over $\mathbb{Z}[\theta_1, \theta_2]$.

5. Recall that a cyclic d-polytope with n vertices $C_d(n)$ for $d \geq 2$ is the convex hull of the n vertices $\{\mathbf{x}(t_i)\}_{i=1}^n$ where $\mathbf{x}(t) := [t, t^2, \ldots, t^d]$ in $\mathbb{R}^d$ and $t_1 < t_2 < \cdots < t_n$ are any n strictly increasing real numbers. Use ideas from the argument in lecture proving $\lfloor \frac{d}{2} \rfloor$-neighborlineess of $C_d(n)$ to prove *Gale's evenness criterion*, characterizing which d-element subsets $F = \{i_1 < i_2 < \cdots < i_d\} \subset \{1, 2, \ldots, n\}$ index the vertices of a boundary facet $\{\mathbf{x}(t_i)\}_{i \in F}$ of $C_d(n)$:

> This happens if and only if any two elements in the complementary set $\{1, 2, \ldots, n\} \setminus F$ have an even number of elements of the ordered sequence $i_1 < i_2 < \cdots < i_k$ lying between them in the usual order on the integers.

6. Given two simplicial complexes Δ_1, Δ_2 on vertex sets V_1, V_2, their *simplicial join* $\Delta_1 * \Delta_2$ is the simplicial complex having the disjoint union $V_1 \sqcup V_2$ as vertex set, and with faces $F_1 \sqcup F_2$ for all (F_1, F_2) in the Cartesian product $\Delta_1 \times \Delta_2$.

(a) Defining the f-polyomial and h-polynomials of Δ by

$$f(\Delta, t) := \sum_{i=0}^{d} f_{i-1} t^i,$$
$$h(\Delta, t) := \sum_{i=0}^{d} h_i t^i$$

prove that

$$f(\Delta_1 * \Delta_2, t) = f(\Delta_1, t) f(\Delta_2, t),$$
$$h(\Delta_1 * \Delta_2, t) = h(\Delta_1, t) h(\Delta_2, t).$$

(b) Use this to compute the h-vector of the *cone* $\{\varnothing, \{v\}\} * \Delta$ and the *suspension* or *bipyramid* $\{\emptyset, \{v\}, \{v'\}\} * \Delta$ over a simplicial complex Δ, in terms of the h-vector of Δ.

(c) Show how to use the constructions in (b) to find for each d a list of simplicial d-polytopes having affinely independent h-vectors that lets you prove the following:

> The Dehn-Sommerville equations $h_i = h_{d-i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$ together with $h_0 = 1$ imply any other (affine-)linear equations satisfied by the f-vectors or h-vectors of all simplicial d-polytopes.

7. Let $n_1, n_2, \ldots, n_{d+1}$ be vectors in $\mathbb{R}^d$ which are minimally dependent, in the sense that any proper subset of them are independent.

(a) Show that, up to an overall scaling, there is only one linear dependence among them, which after re-indexing can be assumed to take the form

$$\sum_{i \in G} a_i n_i = \sum_{j \in G^c} b_j n_j$$

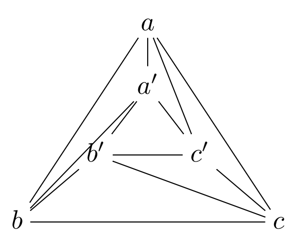
for disjoint subsets G, G^c with $G \sqcup G^c = \{1, 2, \ldots, d+1\}$, and all $a_i, b_j > 0$.

(b) Now assume further that $n_1, n_2, \ldots, n_{d+1}$ are *acyclically oriented* meaning that there is some linear function f in $(\mathbb{R}^d)^*$ having $f(n_i) > 0$ for $i = 1, 2, \ldots, d+1$. Show that this forces G, G^c to both be nonempty, proper subsets, that is, $\varnothing \neq G, G^c \subsetneq \{1, 2, \ldots, d+1\}$.

(c) Under the assumptions in (b), show that the cone $\mathbb{R}_+ n_1 + \cdots \mathbb{R}_+ n_{d+1}$ which they span has only two possible triangulations into d-dimensional simplicial cones $\sigma_F := \sum_{i \in F} \mathbb{R}_+ n_i$:

- either F runs through the sets $\{1, 2, \ldots, d+1\} \setminus \{i\}$ for $i \in G$, or
- F runs through the sets $\{1, 2, \ldots, d+1\} \setminus \{j\}$ for $j \in G^c$.

8. Consider a 3-dimensional complete simplicial fan Σ containing the following configuration of simplicial cones within an orthant spanned by rays a, b, c below:



Assume abc and $a'b'c'$ are similar triangles, placed so that one is a scaled and translated version of the other. Ignoring what might be happening inside other parts of the fan, show that Σ is not polytopal, that is, it cannot be the *face fan* of any 3-dimensional simplicial polytope, and hence also not the *normal fan* of any 3-dimensional simple polytope.

9. Let Δ be a 1-dimensional simplicial complex, that is, a simple graph, meaning one with no parallel edges and no self-loops.

(a) Prove that Δ is (pure and) shellable if and only if it is a connected graph.

(b) Prove that Δ is (pure and) partitionable if and only if it has no isolated vertices and has at most one connected component which is acyclic (that is, isomorphic to a tree).

10. Let P be a d-dimensional convex polytope in $\mathbb{R}^d$, which is k-neighborly for some $k \geq \lfloor \frac{d}{2} + 1 \rfloor$. We want to show that P this forces P to be a d-simplex, that is, it has $d+1$ vertices.

(a) Suppose P has at least $d+2$ vertices, say $v_1, v_2, \ldots, v_{d+2}$. Prove that there exist coefficients $c_1, c_2, \ldots, c_{d+2}$ in $\mathbb{R}$ with $\sum_{i=1}^{d+2} c_i v_i = \mathbf{0}$ and $\sum_{i=1}^{d+2} c_i = 0$.

(b) Show that neither of the two sets $\{i : c_i >\}$ or $\{i : c_i < 0\}$ is empty, and at least one of them has cardinality bounded above by k, so that it indexes a boundary face of P.

(c) Explain how the observations in part (b) lead to a contradiction.
(Hint: what does it mean for as set of vertices to lie on a boundary face of P?)

And here are some more exercise for the second half of Spring 2021 Math 8680 Topics in Combinatorics, in addition to those in pages 53-60 of `http://www-users.math.umn.edu/~reiner/Talks/Vienna05/Lectures.pdf`.

1. Given a finite set E, say that a function $f : 2^E \to \mathbb{R}$ is *submodular* if

$$f(I) + f(J) \geq f(I \cap J) + f(I \cup J)$$

for all subsets $I, J \subseteq E$. Say that f is *strictly submodular* if it is submodular and additionally the above inequality is strict whenever I, J are incomparable or not nested under inclusion, so$I \not\subseteq J$ and $J \not\subseteq I$.

(a) Prove that the function $f(I) := \#I \cdot (\#E - \#I)$ is strictly submodular.

(b) Prove that if M is a matroid on E, then its rank function $r_M(I)$ is submodular as a function $2^E \to \mathbb{R}$, but not in general strictly submodular.

(c) Prove that for every submodular function $f : 2^E \to \mathbb{R}$ there exists some family of strictly submodular functions $\{f_\epsilon\}$ defined for $\epsilon > 0$, such that for all subsets $I \subseteq E$ one has $f_\epsilon(I)$ a continuous function of ϵ, and $f(I) = \lim_{\epsilon \to 0^+} f_\epsilon(I)$.

2. Let $(a_0, a_1, \ldots, a_r)$ be a sequence of real numbers which are strictly positive $(a_i > 0)$ and log-concave $(a_i^2 \geq a_{i-1}a_{i+1})$, interpreting $a_i = 0$ for $i < 0$ or $i > r$.

(a) Prove that one also has the inequalities $a_i a_j \geq a_{i-k}a_{j+k}$ for $k \geq 0$.

(b) Given $(a_i)_{i=0,1,\ldots,r}$ strictly positive and log-concave, define $(c_i)_{i=0,1,2,\ldots,r+1}$ by

$$\sum_{i=0}^{r+1} c_i t^i = (1+t) \sum_{i=0}^{r} a_i t^i$$

and prove that (c_i) is also strictly positive and log-concave.

(c) Generalize part (b): given $(a_i)_{i=0,1,\ldots,r}$ and $(b_j)_{j=0,1,\ldots,s}$ be two strictly positive log-concave sequences, show that $(c_i)_{i=0,1,2,\ldots,r+s}$ defined by

$$\sum_{i=0}^{r+s} c_i t^i = \left(\sum_{i=0}^{r} a_i t^i \right) \left(\sum_{j=0}^{s} b_j t^j \right)$$

is also strictly positive and log-concave.

3. Fix a matroid M on ground set E, having lattice of flats L, with bottom, top elements $\varnothing, E$. Let's compare three closely related ways to present the Chow ring $A(M)$, starting with three polynomial rings

$$\begin{aligned} S &:= \mathbb{Z}[\{x_F : F \in L \text{ with } F \neq \varnothing\}], \\ \hat{S} &:= \mathbb{Z}[\{x_F : F \in L \text{ with } F \neq \varnothing, E\}], \\ \tilde{S} &:= \mathbb{Z}[\{x_F : F \in L \text{ with } F \neq E\} \cup \{y_i : i \in E\}]. \end{aligned}$$

so that one has obvious inclusions $\hat{S} \hookrightarrow S, \tilde{S}$ sending $x_F \mapsto x_F$ for $F \neq \varnothing, E$.

- (Feichtner-Yuzvinsky)
$$\begin{aligned} A(M) &:= S/(I+J), \text{ where} \\ I &= (\{x_F x_G : F \not\subseteq G \text{ and } G \not\subseteq F\}), \\ J &= \left(\left\{ \sum_{F: i \in F} x_F : i \in E \right\}\right). \end{aligned}$$
- (Adiprasito-Huh-Katz)
$$\begin{aligned} A(M) &:= \hat{S}/(\hat{I}+\hat{J}), \text{ where} \\ \hat{I} &= (\{x_F x_G : F \not\subseteq G \text{ and } G \not\subseteq F\}), \\ \hat{J} &= \left(\left\{ \sum_{F: i \in F} x_F - \sum_{G: j \in G} x_G : i \neq j \in E \right\}\right). \end{aligned}$$
- (Braden-Huh-Matherne-Proudfoot-Wang)
$$\begin{aligned} A(M) &\cong CH(M)/(\{y_i\}_{i \in E}), \text{ with} \\ CH(M) &:= \tilde{S}/(\tilde{I}+\tilde{J}), \text{ where} \\ \tilde{I} &= (\{x_F x_G : F \not\subseteq G \text{ and } G \not\subseteq F\}) + (\{y_i x_F : i \notin F\}), \\ J &= \left(\left\{ y_i - \sum_{F: i \notin F} x_F : i \in E \right\}\right). \end{aligned}$$

Fix an element i_0 in E,

(a) Define a ring map $S \longrightarrow \hat{S}$ on the variables sending $x_F \longmapsto x_F$ if $F \neq E$, and

$$x_E \longmapsto - \sum_{\substack{F: i_0 \in F \\ \varnothing \neq F \neq E}} x_F.$$

Prove this map and $\hat{S} \hookrightarrow S$ descend to mutually inverse ring isomorphisms

$$S/(I+J) \cong \hat{S}/(\hat{I}+\hat{J}).$$

(b) Define a ring map $\tilde{S} \longrightarrow \hat{S}$ on the variables sending $x_F \mapsto x_F$ for $F \neq \varnothing$, and

$$x_\varnothing \longmapsto - \sum_{\substack{F: i_0 \notin F \\ \varnothing \neq F \neq E}} x_F.$$

Prove this map and $\hat{S} \hookrightarrow \tilde{S}$ descend to mutually inverse ring isomorphisms

$$CH(M)/(\{y_i : i \in E\}) \cong \hat{S}/(\hat{I}+\hat{J}).$$