# The Beauty of Groups and Symmetry

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# Outline

# Introduction

The goal in this talk is to share some of the beauty to be found in symmetrical objects, especially crystals, and to show how the use of mathematics can help us to understand them, use them, and appreciate them better.

In the following pictures, which are the work of M.C. Escher, you will be impressed by the wonder of the design, but may find it hard to say what important differences there are between the pictures.


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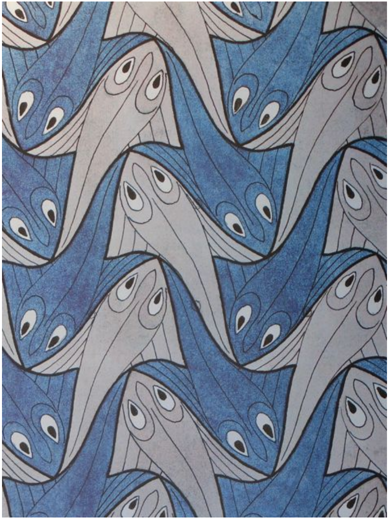
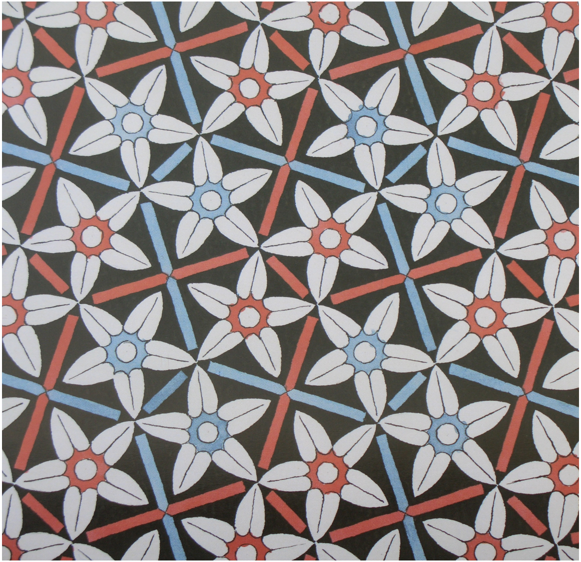
BAARN (Holland)



# A pattern which suggests 5-fold symmetry



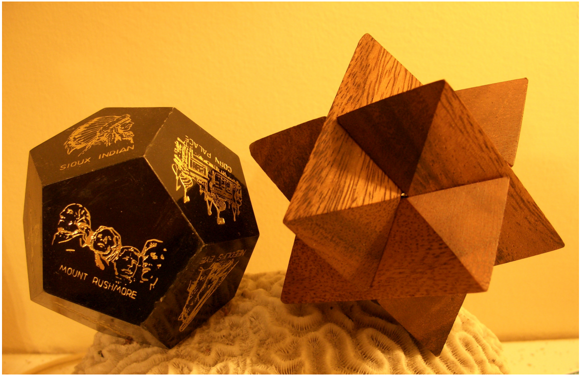
# A question to which I do not know the answer

Why is it that we should find symmetry so attractive?

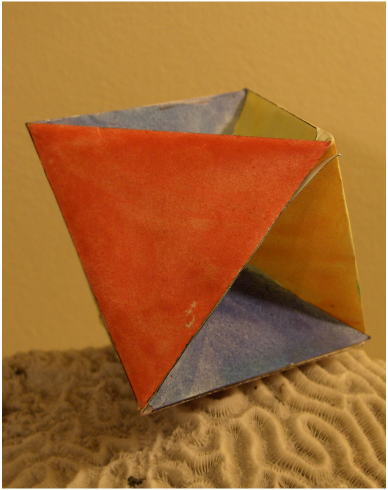
It is a famous quotation that 'God gave us the natural numbers, the rest is man's invention.' We might also say that God gave us symmetry.

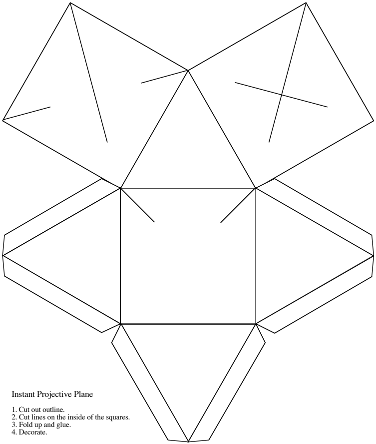
Here are some more pictures.

# Four Platonic solids and a puzzle

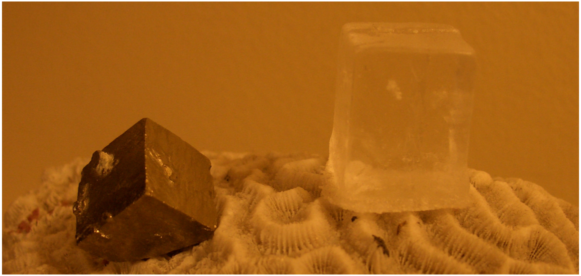


# Paper model of a projective plane and its net





# Two cubical crystals



# The role of mathematics



Mathematics can help us to understand better the structure of these patterns and objects, and how they are made. Without mathematics we think the patterns are beautiful, and complicated, but find it hard to say why one should differ from another.

In the next section of this talk we introduce the following mathematical concepts:

- isometry of Euclidean space,
- group
- crystal structure in Euclidean space

# What is an isometry?

Write $\mathbb{E}^n$ for *n*-dimensional (Euclidean) space. We live in $\mathbb{E}^3$.

An isometry (or rigid motion) of $\mathbb{E}^n$ is a transformation $\mathbb{E}^n \to \mathbb{E}^n$ which preserves distance.

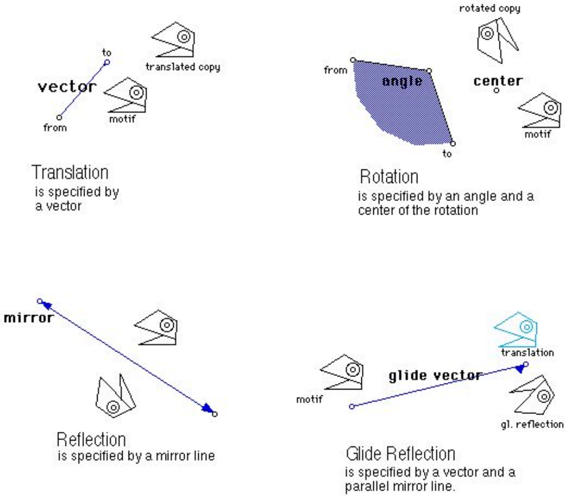
Each subset of $\mathbb{E}^n$ (an object of some kind) has a group of isometries. This is the set of all isometries of $\mathbb{E}^n$ which send the subset exactly to itself.

## Examples

A cube has 48 elements in its group of isometries, a tetrahedron has 24, and an icosahedron has 120. A regular polygon with $n$ sides has $2n$ isometries.

## Theorem

*The possible isometries of 2-dimensional space are translations, rotations and glide reflections.*



# In general, what is a group?

The definition is a little abstract, but we will need it later. A group is a set whose elements can be multiplied in some formal way, satisfying properties listed below. If $a$ and $b$ are elements of the group, there is defined an element $ab$ which is the product of $a$ and $b$. The following must hold.

- Multiplication is associative, that is, $(ab)c = a(bc)$ always.
- There is an identity element, that is, an element written 1 with the property that $1a = a1 = a$ always.
- Every element is invertible, that is, for every element $a$ there is an element $a^{-1}$ so that $aa^{-1} = a^{-1}a = 1$.

### Example

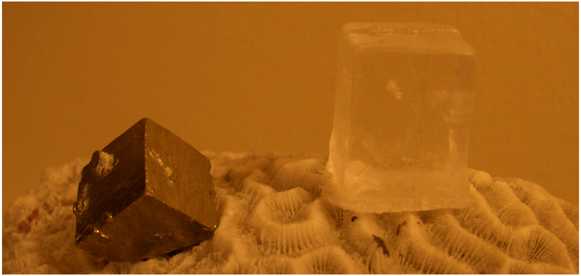
A group of isometries is a group.

# What is a crystal (informal idea)?

The property that distinguishes a crystal is that there are planes along which it will fall apart cleanly.

If we take a sodium chloride crystal (a cube), place a knife on it parallel to one of the faces, and strike the back of the knife with a hammer, the crystal will divide cleanly along the plane of the knife. There are many places in each of 3 independent directions we could have put the knife to achieve this.

How can we model these properties mathematically?



# What is not a crystal?

A cube of glass (which is amorphous) if treated similarly will shatter into little pieces. Glass does not have the structure of a crystal, even though it may be in a shape which looks like a crystal.

# What else is not a crystal?

There are types of structure called Fullerenes which are sometimes viewed as crystals, but which are not crystals in our sense. It is possible for atoms to arrange themselves with the symmetry of an icosahedron around a point. These are not crystals because they favour a distinguished point.

In two dimensions we have the Penrose tiles, which tile the plane in many ways, but always without periodicity. The wikipedia entry `http://en.wikipedia.org/wiki/Penrose_tiling` has an account of this.

# Formal mathematical definition of a crystal





A crystal structure is a subset of $\mathbb{E}^n$ whose group of isometries

- contains $n$ independent translations, and
- has the property that every isometry moves every point it does not fix at least as far as some prescribed minimum distance.

The group of isometries of a crystal structure is called a space group, or crystallographic group.

# Wallpaper patterns

In 2-dimensional space we call crystal structures wallpaper patterns. The definition says that a crystal in dimension 2 must repeat itself in two independent directions (one direction along the roll of paper, the other direction coming as strips of paper are laid next to one another) and that the pattern may not be continuous in any direction.

One of our goals is to explain what is meant by the following:

**Theorem**

*There are precisely 17 different wallpaper patterns.*

# Some questions

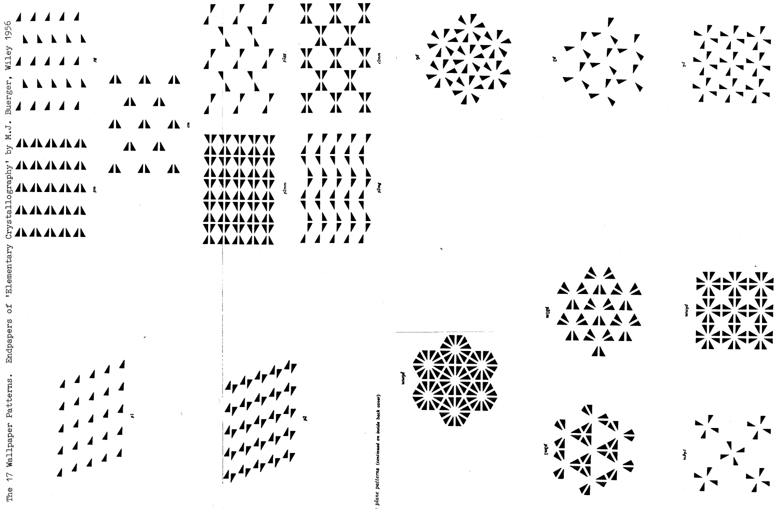
The Escher pictures shown at the beginning are wallpaper patterns.

What does this mean to say that there are 17 wallpaper patterns? Clearly there are more than 17 wallpaper patterns!

How can we recognize them?

How can we prove such a result?

# List of the 17 wallpaper patterns



# A plate seen in Kyoto

How can we identify each pattern on the plate with one which of the 17 patterns?

# Exploring wallpaper patterns on the internet



The web site of Professor Helmer Aslaksen (National University of Singapore) has much information and many links:
`http://www.math.nus.edu.sg/aslaksen/teaching/math-art-arch.html`

Some different pictures of the 17 wallpaper patterns:
`http://mathmuse.sci.ibaraki.ac.jp/pattrn/PatternE.html`
`http://www2.spsu.edu/math/tile/symm/types/index.htm`

The software 'kali' allows us to draw wallpaper patterns:
`http://www.geom.uiuc.edu/java/Kali/program.html`

# Equivalence of crystals

We say that crystals are the equivalent if they have the 'same' symmetry group. What this means is that if we take a crystal and enlarge it in some direction, or translate it, or rotate it, or reflect it, provided that no new symmetries are introduced and none are lost, we regard the crystals as the same.

Another formulation of this is that if $G_1$ and $G_2$ are two isometry groups of crystals, we say they are the equivalent if there is an 'affine transformation' $a$ so that $aG_1a^{-1} = G_2$.

It is not at all obvious that the following statement is true.

**Theorem**

*Two crystals are equivalent if and only if their groups of isometries are isomorphic.*

# Meaning of the classification of crystals

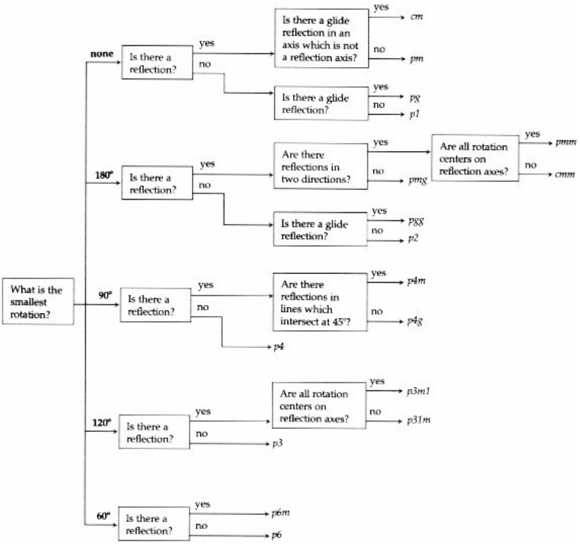
We can now say what it means to say that there are 17 wallpaper patterns. It means that every wallpaper pattern is equivalent to one of the 17 on the list, and these are all inequivalent.

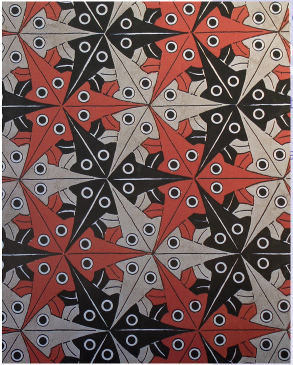
In 3 dimensions it is particularly useful to have a list of the inequivalent crystal structures. This is the basis of x-ray crystallography. There are either 230 or 217 crystal structures in 3 dimensions, depending on how they are counted.

Once we have a list of possible crystals, it is important to be able to identify a given crystal on the list.

Obtaining the list is the same as classifying the groups.

# Flowchart for distinguishing wallpaper patterns



We will now describe some of the mathematics behind the classification of crystals.

# The translation subgroup

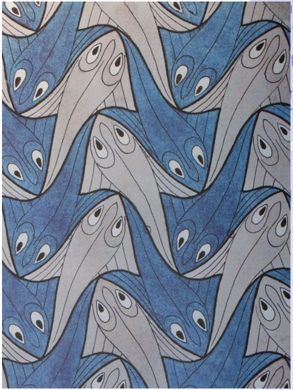
We classify the possible crystals by classifying their groups of isometries. To do this, we look carefully at the structure of the group.

A subgroup of a group is a subset which is a group in its own right under the given multiplication. This means that products and inverses of elements in the subset remain in the subset.

The spacegroup $G$ of a crystal has a subgroup $T$ called the translation subgroup. It is the subgroup whose elements are the translations which preserve the crystal.

# Example: the wallpaper pattern pg

The isometries which preserve wallpaper pattern pg consist only of translations and glide reflections. If we apply all possible translations in $T$ to a point in the pattern (e.g. the nose of a grey fish which looks right) we get a lattice in $\mathbb{E}^2$ with rectangular regions. Let $g$ be the glide reflection which moves the nose of a grey fish to the nose of the grey fish above it, and $t$ the translation which moves the nose of a grey fish to the nose of the grey fish two above it. Then $g^2 = t$. Every element of $G$ has the form $s$ or $gs$ where $s$ is some translation.



# The point group

Let us write
$$gT = \{gs \mid s \in T\}$$
for the set of elements in $G$ which have the form $gs$ for some translation. These subsets are called the cosets of $T$ in $G$.

## Example

With pg we have
$$G = T \cup gT$$
and there are two cosets: $T = 1T$ and $gT$.

We can make the cosets of $T$ into a group by defining $(aT)(bT) = abT$, and we use the symbol $G/T$ to denote this group: the quotient group of $G$ by $T$. It is called the point group associated to the crystal.

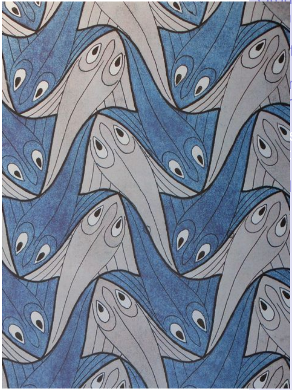
# Example: the wallpaper pattern pg continued

In the case of pg, $G/T$ has two elements: an identity element $T$ and another element $gT$.
The square of this second element is the identity, since

$$(gT)^2 = g^2T = tT = T.$$

Notice that the pattern pg does not have any isometry whose square is the identity!

Although the point group does not always consist of isometries of the pattern, it is still a fact that there is another way in which the point group does act on Euclidean space, preserving the translation lattice.



# Strategy for classifying space groups

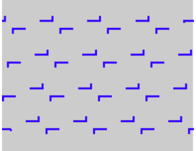
- Classify the possible point groups.
- For each point group, classify the possible ways it can act on the translation subgroup.
- For each action of the point group, classify the possible ways it can fit together with the translation subgroup to produce a spacegroup.

# The group $C_2 = \{1, g\}$ with two elements

$g$ acts via $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

There is one pattern:

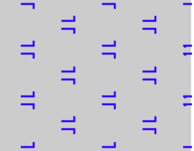
p2



$g$ acts via $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

There is one pattern:

cm

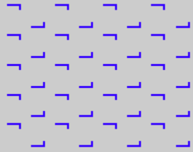


$g$ acts via $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

There are two patterns:

pm

pg



Check that the point groups are all $C_2$.

# Structure of the point group

The group of symmetries of a regular polygon with $n$ sides is called dihedral. The subgroup consisting of the rotations is called a cyclic group.

### Theorem

*(Leonardo da Vinci) Every finite group of isometries of* $\mathbb{E}^2$ *is either cyclic or dihedral.*

### Theorem

*Every rotation of* $\mathbb{E}^2$ *which is an isometry of a lattice has order 1, 2, 3, 4 or 6.*

### Corollary

*There is no crystal with the symmetry of the icosahedron.*

# Fitting the translation subgroup and point group together

Classifying the ways the point group and translation subgroup can fit together is quite an advanced topic, which can be studied as part of the cohomology of groups.

# Conclusions

Wallpaper patterns may seem straightforward, but the mathematics used to explain their properties is advanced.

The mathematics enables us to understand the symmetry of the patterns better, to the extent that we can classify them.

The interaction of mathematics and art is itself rather beautiful.

The mathematics of group theory leads us to further structures with even more remarkable symmetry than what we have just been studying.

# Thanks!!

**Thank you for your attention!**

Slides for this talk will be available at:
`http://www.math.umn.edu/~webb`

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