Coherence of f-Monotone Paths on Zonotopes.

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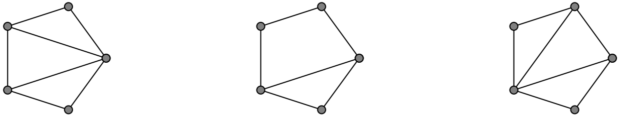
An Analogy: The Secondary Polytope

Definition (Polytope)

A polytope is a convex hull of finitely many points in $\mathbb{R}^d$. Combinatorially a polytope can be defined by its face lattice.

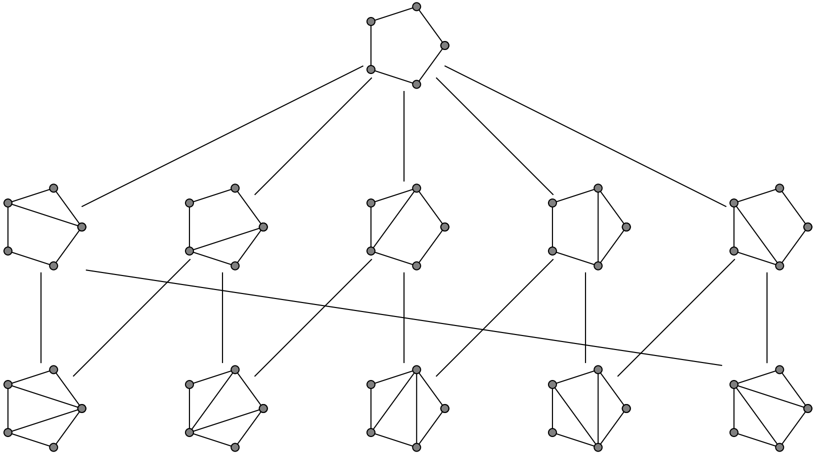
Definition (Polyhedral Subdivision)

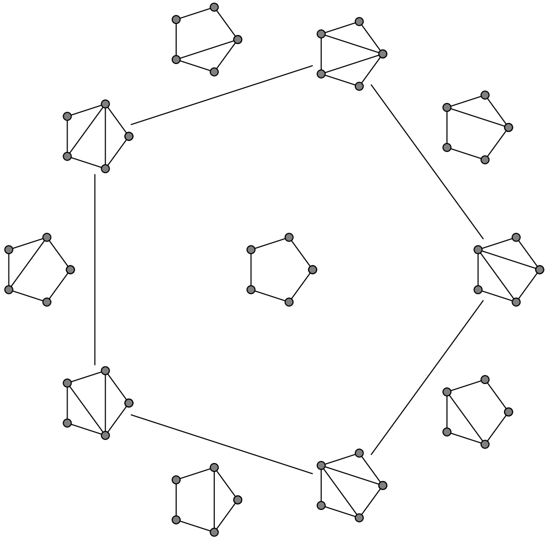
A polyhedral subdivision is a decomposition of P into subpolytopes. A subdivision is a triangulation when each subpolytope is a simplex.



Remark

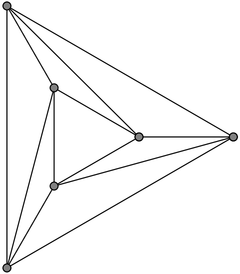
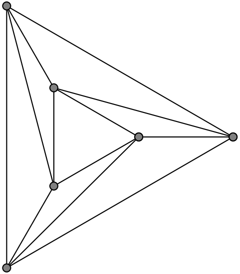
Subdivisions of P form a poset called the refinement poset of P.





Remark

In this example, the refinement poset is the face lattice of a polytope.

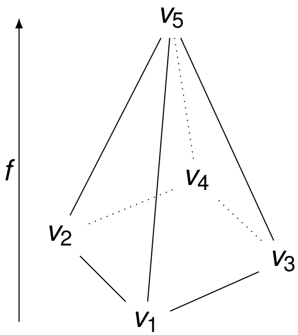
 

- Some bad triangulations are not *regular* or are *incoherent*.
- Coherence is a linear inequality condition.
- $\Sigma(P)$ is an example of a *Fiber Polytope*.

Theorem (GKZ)

The refinement poset of all regular subdivisions of P is the face lattice of a polytope $\Sigma(P)$.

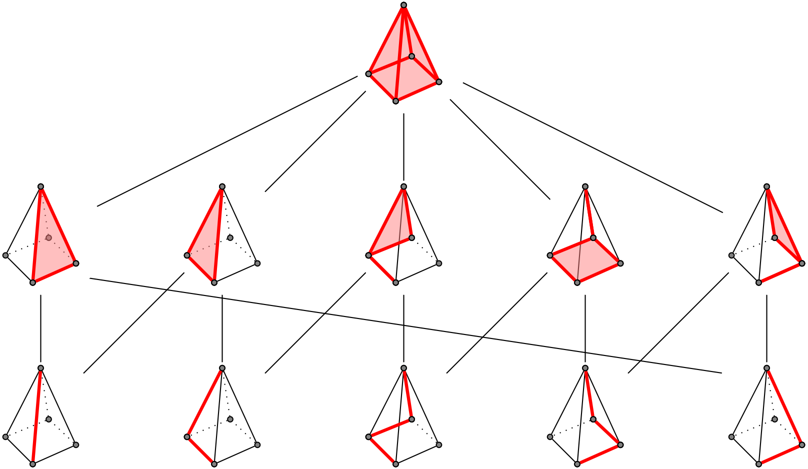
Our Work: Monotone Paths



- Our version of triangulations are *f-monotone edge paths* of P .
- f must be *generic*, non-constant on each edge of P.
- The *refinement poset* consists of cellular strings.

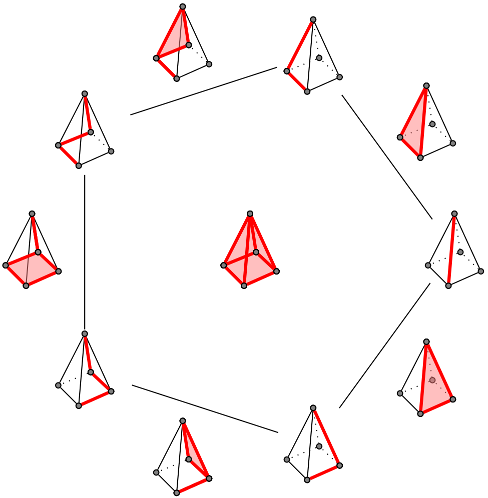
Definition

An f-monotone edge path is a path from the f-minimal vertex $-z$ to the f-maximal vertex z along the edges of P.



Definition

- *The vertices graph $G_2(P, f)$ is formed from all elements on the bottom level levels of the refinement poset.*
- *In this example every f-monotone path is coherent.*

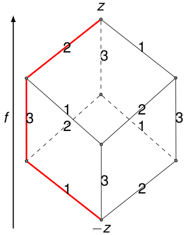


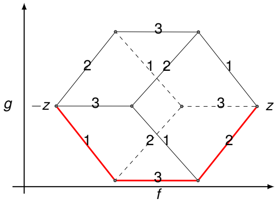
Question

When does P have incoherent f-monotone paths?

Definition (Coherent)

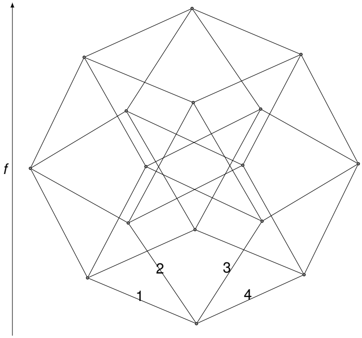
An f-monotone path γ is coherent if there exists $g \in \left(\mathbb{R}^d\right)^*$ making γ the lower face of the polytope $P = \mathrm{Conv}\left\{(f(p_i), g(p_i))\right\} \subset \mathbb{R}^2$.

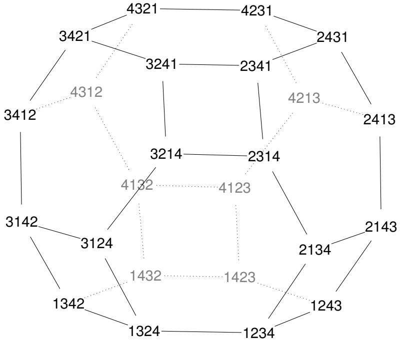




Remark

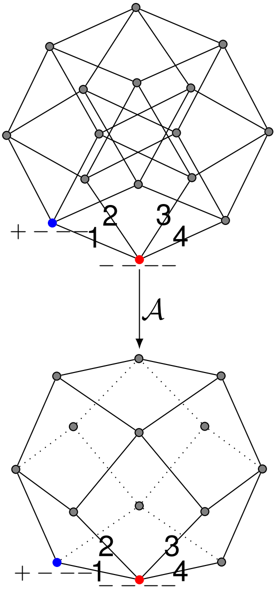
The refinement poset of coherent cellular strings is the fiber polytope $\Sigma(P, f)$.





Theorem (Billera & Sturmfels)

Every f-monotone path of a cube is coherent.

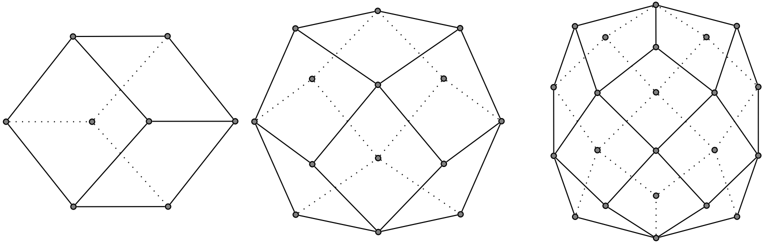


Definition

- *A zonotope is the image of the n-cube in $\mathbb{R}^d$ under a projection $\mathcal{A} : C_n \to \mathbb{R}^d$ specified by a $d \times n$ matrix*

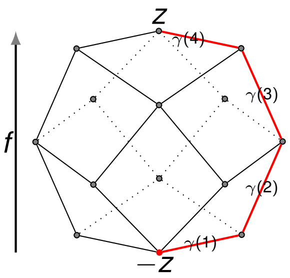
$$\mathcal{A} = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{pmatrix}$$

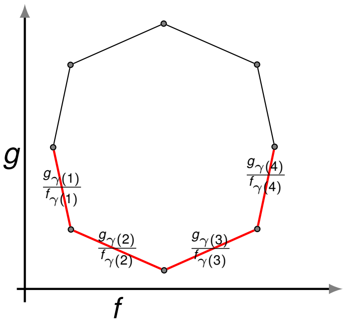
- *The zonotope $Z(\mathcal{A}) = \sum[-a_i, +a_i]$ is the Minkowski of the columns of $\mathcal{A}$.*
- *The vertices of $Z(\mathcal{A})$ are sign vectors*



Proposition

- *Every f-monotone path of $Z(\mathcal{A})$ is of length n.*
- *The function f is generic when $f(a_i) > 0$ for all i.*
- *The choice of f corresponds to the choice of a f-minimal vertex $-z$.*
- *But not all vertices are symmetric, so we will have to consider multiple options for z.*
- *The corank of Z is $n - d$.*





Proposition

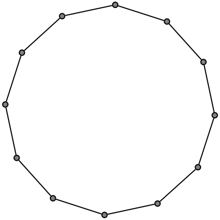
A f-monotone path γ is coherent if there exists a $g \in \left(\mathbb{R}^d\right)^*$ so that:

$$\frac{g_{\gamma(1)}}{f_{\gamma(1)}} < \frac{g_{\gamma(2)}}{f_{\gamma(2)}} < \dots < \frac{g_{\gamma(n)}}{f_{\gamma(n)}}$$

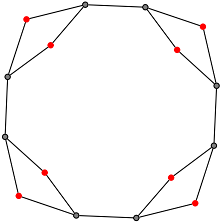
Corank 1

$$Z(4,3) = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$

$$Z(4,3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$



$-++\,++$



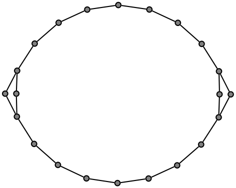
$++\,+\,++$

Remark

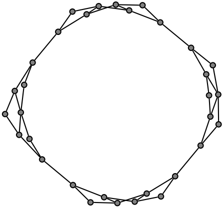
- *Every f-monotone path is coherent for* $-++\,++$.
- $++\,+\,++$ *has an incoherent f-monotone path for every f.*

Corank 2 (cyclic)

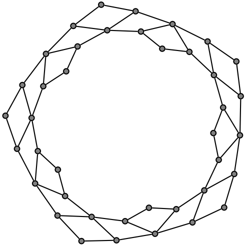
$$Z(5,3) = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \end{array}$$



$- + + + +$



$- - + + +$



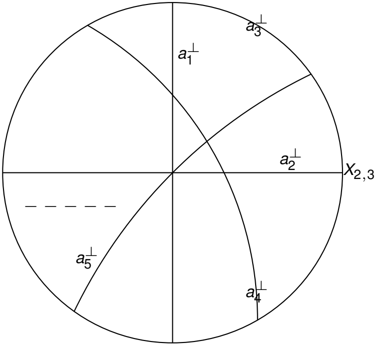
$+ + + + + +$

Remark

- *Has incoherent f-monotone path for every f.*
- $+ + + + + +$ *is an important geometric counterexample.*

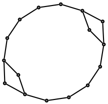
Definition (Pointed hyperplane arrangement)

The normal fan of the zonotope, is a hyperplane arrangement, $\mathcal{A} = \{a_1^\perp, \ldots, a_n^\perp\}$. The choice of a chamber c of $\mathcal{A}$ corresponds to the choice of f.

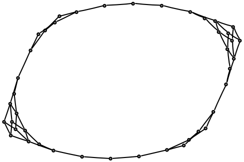


- Easy to draw under stereographic projection
- k-faces of $Z \iff d-k$ intersections of hyperplanes.
- $L_2(\mathcal{A})$ are the codimension 2 intersections of hyperplanes.

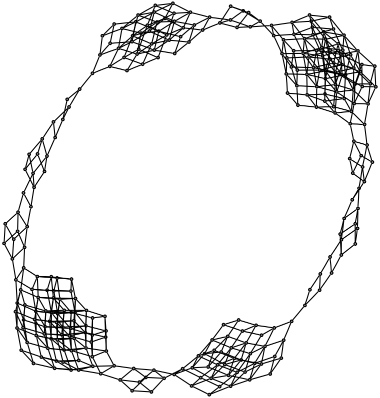
Reflection Arrangements



A_3



B_3



H_3

Remark

- *Does not depend on the choice of a base chamber c.*
- *Paths corresponds to reduced words.*

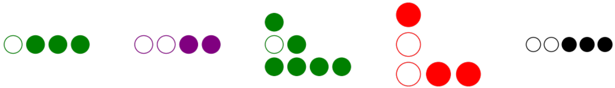
- Dual hyperplane configuration is a $(n-d)\times n$ matrix.
- Functions on $\mathcal{A}$ correspond to dependencies of $\mathcal{A}^*$.
- When $n-d$ is small, this makes things easy.

$$\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 0 \qquad \mathcal{A}^* = \begin{array}{c} \begin{array}{cccc} a_1^* & a_2^* & a_3^* & a_4^* \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Example

$$\begin{array}{ccc} ++++ & f(x,y,z) = x+y+z & a_1^*+a_2^*+a_3^*+3a_4^*=0 \\ -+++ & f(x,y,z) = -x+y+z & -a_1^*+a_2^*+a_3^*+a_4^*=0 \\ +++- & ? & ? \end{array}$$

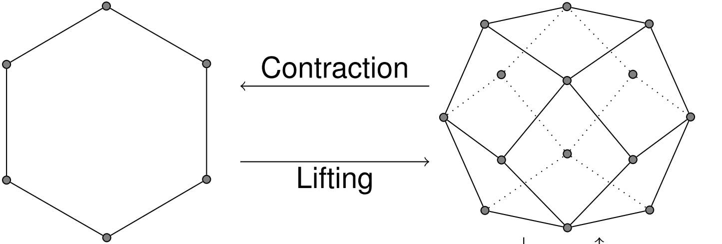
Affine Gale duals replace $(\mathcal{A}, f)$ with a picture.

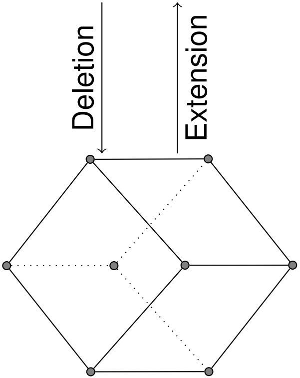


$$
\begin{tikzcd}
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \arrow[r, shift right=2, "\text{Lifting}"'] & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \arrow[l, shift right=2, "\text{Contraction}"'] \arrow[d, shift right=2, "\text{Deletion}"'] \\
 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \arrow[u, shift right=2, "\text{Extension}"']
\end{tikzcd}
$$

Proposition

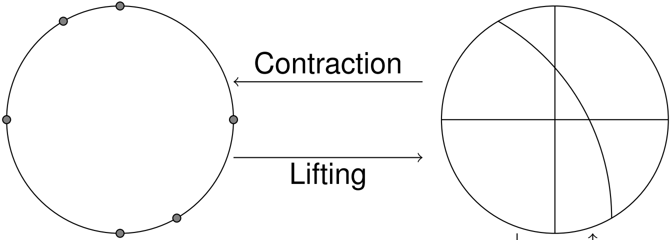
- *Extensions preserve dimension.*
- *Liftings preserve corank; if f is generic on $\mathcal{A}$ then there exists $\widehat{f}$ is generic on $\widehat{\mathcal{A}}$.*

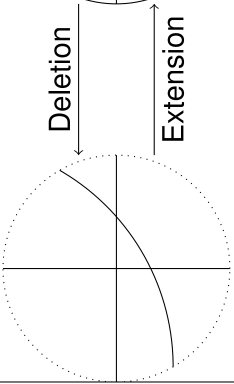




Proposition

If γ is an f-monotone path of $\mathcal{A}$ and $\widehat{\mathcal{A}}$ a single-element lifting of $\mathcal{A}$, then any $\widehat{\gamma}$ with $\widehat{\gamma}/(n+1) = \gamma$ is an $\widehat{f}$-monotone path of $\widehat{\mathcal{A}}$.



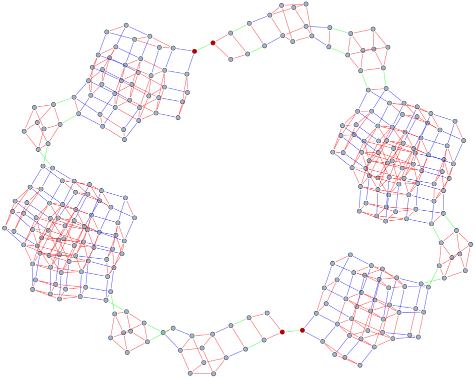


Proposition

If $\mathcal{A}^+$ is a single-element extension of $\mathcal{A}$, and γ^+ is an f-monotone path of $\mathcal{A}^+$ then any $\gamma\backslash(n+1)$ is an f-monotone path of $\mathcal{A}$.

Findings: Reflection Arrangements

$\mathcal{A}$	$\lvert\Gamma(\mathcal{A})\rvert$
H_3	152
D_4	2316
D_5	12985968
D_6	3705762080
F_4	2144892

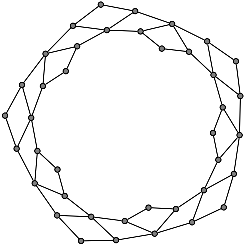


Proposition

H_3 has exactly 4 L_2-accessible nodes.

Findings: Diameter

There is an $(\mathcal{A}, f)$ pair with *no* L_2-accessible nodes.



Example

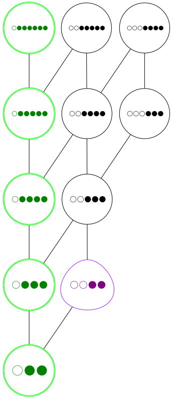
$Z(8,4)$, cyclic arrangement of 8 vectors in $\mathbb{R}^4$ has Diam $G_2(\mathcal{A}, c) = 30$ but $|L_2| = 28$ for $c = (-)^4(+)^4$.

Theorem

When $n - d = 1$ $G_2(\mathcal{A}, f)$ has diameter $|L_2|$ and always has an L_2-accessible node.

Findings: Classification of $(\mathcal{A}, f)$ in corank 1.

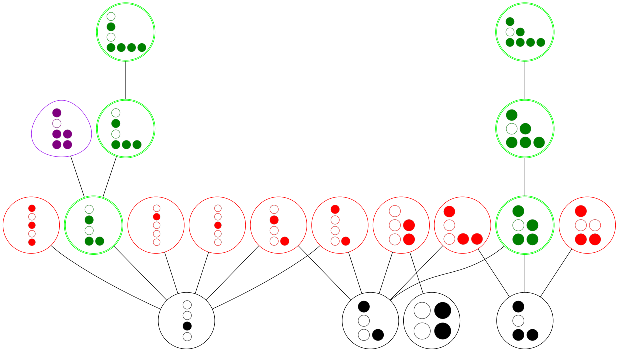
- The purple $(\mathcal{A}, f)$ pair is a *minimal obstruction*, all other $(\mathcal{A}, f)$ containing incoherent f-monotone paths are liftings of it.
- Really remarkable: Coherence depends only on the oriented matroid structure, not on the particular f.



Theorem

When $n - d = 1$ there is a unique family of all-coherent $(\mathcal{A}, f)$ pairs and all other $(\mathcal{A}, f)$ pairs have incoherent paths.

Findings: Classification of $(\mathcal{A}, f)$ in corank 2.



Theorem

When $n - d = 2$ there are two all-coherent families and 9 minimal obstructions. Of the 9 minimal obstructions 8 are single-element lifting of the corank 1 minimal obstruction.

Findings: Minimal obstructions for Cyclic Zonotopes

$$\mathcal{A}(n,d) = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & \vdots & & \vdots \\ t_1^{d-1} & t_2^{d-1} & \cdots & t_n^{d-1} \end{pmatrix} \end{array},$$

Theorem

When $d > 2$ and f realizing c, the monotone path graph

- *When $n - d = 1$, every f-monotone path of $(\mathcal{A}(n,d), f)$ is coherent when c is a reorientation of a certain hyperplane arrangement, and has incoherence f-monotone paths for all other c.*
- *When $n - d \geq 2$, $(\mathcal{A}(n,d), f)$ has incoherent galleries for every f.*

Lemma (4.17)

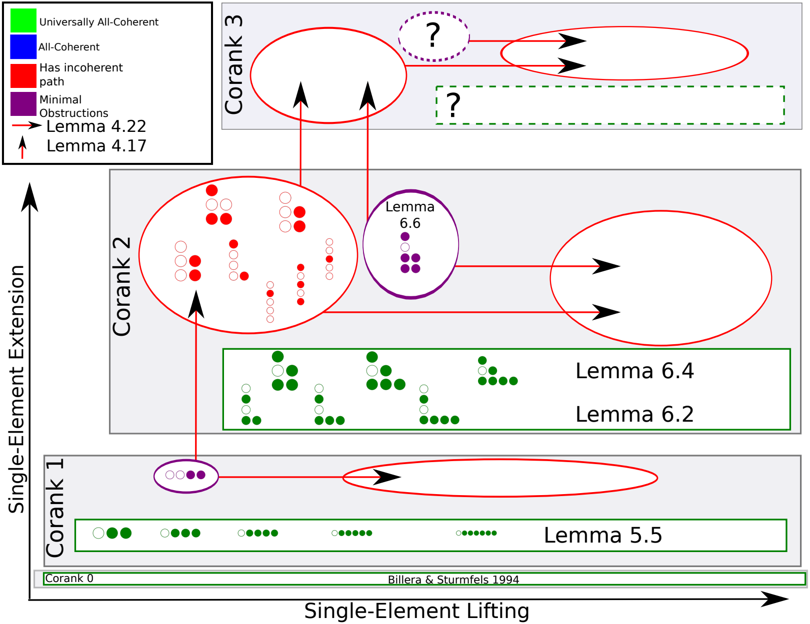
Suppose $\mathcal{A}^+ = \{a_i, \ldots, a_{n+1}\}$ is a single-element extension of $\mathcal{A}$ and f is a generic function on both $Z(\mathcal{A})$ and $Z(\mathcal{A}^+)$. If γ^+ is a coherent f-monotone path of $(\mathcal{A}^+, f)$ then $\gamma = \gamma^+ \backslash (n+1)$ is a coherent f-monotone path of $(\mathcal{A}, f)$.

Lemma (4.22)

Let $\mathcal{A}$ be a hyperplane arrangement and $\widehat{\mathcal{A}}$ a single element lifting of $\mathcal{A}$. Suppose

$$\widehat{\gamma_g} = (n+1, 1, 2, \ldots, n)$$
$$\widehat{\gamma_h} = (1, 2, \ldots, n, n+1)$$

are coherent $\widehat{f}$-monotone paths of $(Z(\widehat{\mathcal{A}}), \widehat{f})$ for some $\widehat{f}$. Then there is a generic functional f on $Z(\mathcal{A})$ for which γ is a coherent f-monotone path.


Universally All-Coherent
All-Coherent
Has incoherent path
Minimal Obstructions
Lemma 4.22
Lemma 4.17
Corank 3
?
?
Corank 2
Lemma 6.6
Lemma 6.4
Lemma 6.2
Corank 1
Lemma 5.5
Corank 0
Billera & Sturmfels 1994
Single-Element Extension
Single-Element Lifting

Questions?

Thank You.

Committee Members	
Victor Reiner	Alexander Voronov
Pavlo Pylyavskyy	Kevin Leder