Bulgarian Solitaire Variants

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University of Minnesota Latin Honors Thesis

Abstract

We present recent findings about the game of Bulgarian Solitaire, introduce some deterministic and non-deterministic variants, and prove several analogous results in these new games.

Bulgarian Solitaire

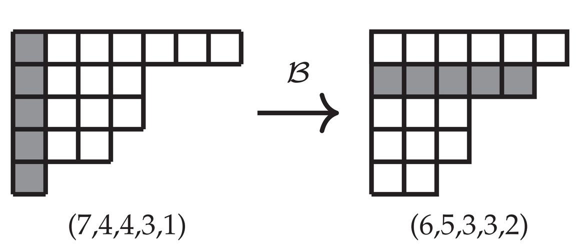
Bulgarian solitaire is a discrete dynamical system on the set of all partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$$

of a fixed number n. The system sends λ to

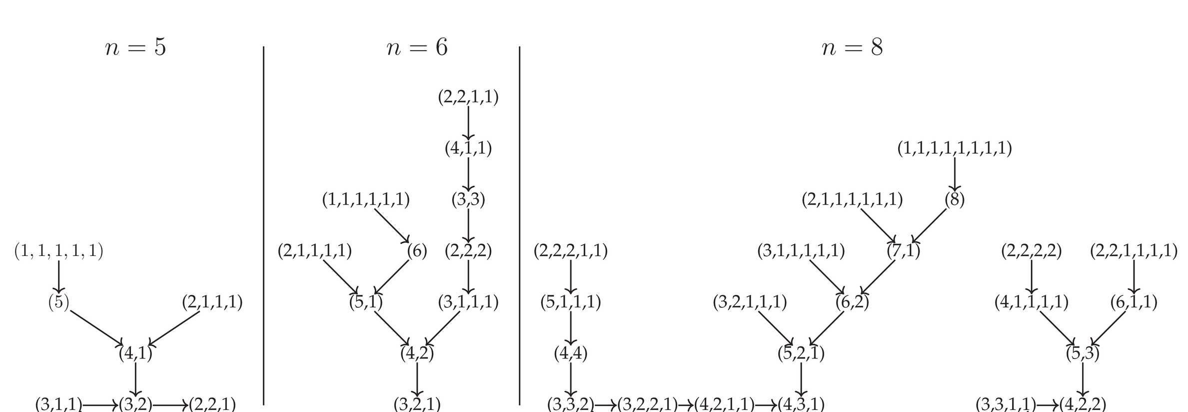
$$\mathcal{B}(\lambda) := ...(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, l)$$

The Bulgarian Solitaire move on Young Diagrams is to take the leftmost column and place it as a row.



Game Graphs

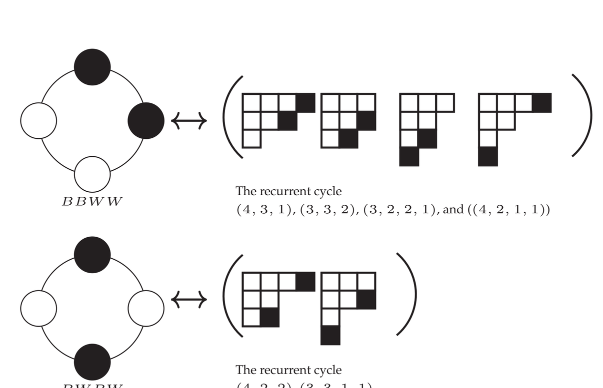
The Bulgarian Solitaire game graphs for $n = 5, 6, 8$



The game graph for $n = 8$ has two connected components, meaning that unlike $n = 5$ or $n = 6$, the recurrent cycle you reach depends on which partition you start with.

Key Results

Brandt proved in [1] that the orbits for n biject with the necklaces of length b with a black beads and $b - a$ white beads. The bijection for $n = 8$ is



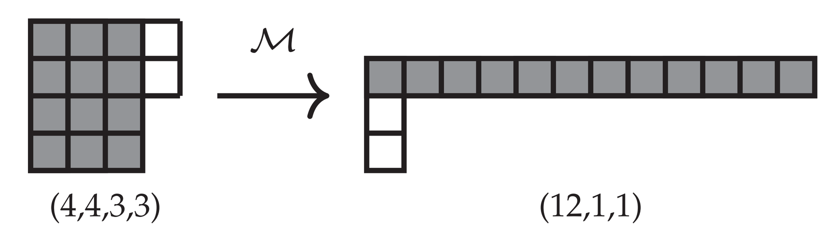
Variants

Maximal Minnesota Solitaire

The **Maximal Minnesota Solitaire move**, $\mathcal{M}$ plays the largest possible Minnesota Solitaire move (as defined in the next column): for a given partition $\lambda \vdash n$ with smallest pile of size k,

$$\mathcal{M}(\lambda) = (\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_l - k, kl)$$

On Young Diagrams, it looks like this:

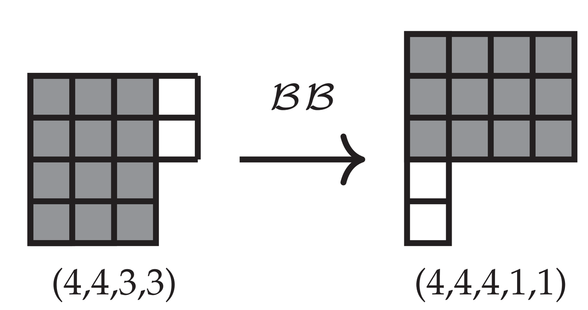


Maximal Block Bulgarian Solitaire

The **Maximal Block Bulgarian Solitaire move**, $\mathcal{BB}$ plays the largest possible Block Bulgarian Solitaire move (as defined in the next column): for $\lambda \vdash n$ with length l and smallest pile k, we have

$$\mathcal{BB}(\lambda) = (l, l, \ldots, l, \lambda_1 - k, \lambda_2 - k, \ldots, \lambda_{l-1} - k)$$

where there are k copies of l in the new partition (reordered in nonincreasing order as necessary). On Young Diagrams, it looks like this:



Key Results

Theorem. *The fixed partitions under $\mathcal{M}$ are the partitions of the form*

$$\Gamma_{(a,b)} = (ab, a(b-1), a(b-2), \ldots, a)$$

*which are called **stretched staircase partitions**.*

Theorem. *The fixed partitions under $\mathcal{BB}$ are the partitions of the form*

$$\Lambda_{(a,b)} = (ab, ab, \ldots, ab, a(b-1), a(b-1), \ldots, a, a, \ldots, a)$$

*which are called **square staircase partitions***

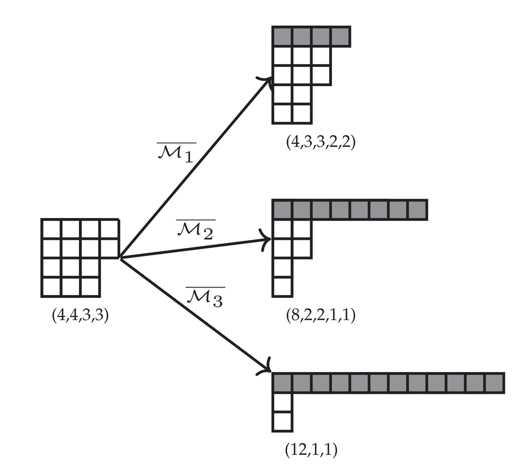
Non-Deterministic Variants

Minnesota Solitaire

The **Minnesota Solitaire Moves**, $\overline{\mathcal{M}_i}$: For a partition $\lambda \vdash n$, with smallest pile of size k,

$$\overline{\mathcal{M}_i}(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, il)$$

where $0 < i \leq k$ (and we place the pile of size il where it belongs in weakly descending order). On Young Diagrams, the non-deterministic moves are

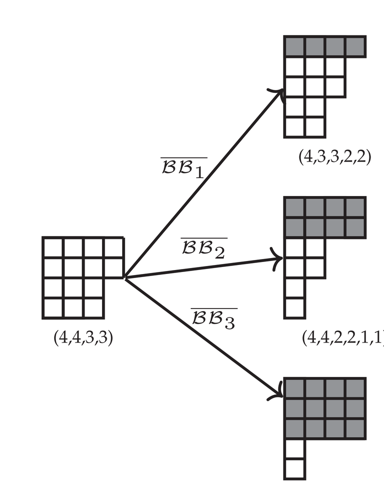


Block Bulgarian Solitaire

The **Block Bulgarian Solitaire Moves**, $\overline{\mathcal{BB}_i}$: For a partition $\lambda \vdash n$, with smallest nonzero pile of size k,

$$\overline{\mathcal{BB}_i}(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, l, l, \ldots, l)$$

where $0 < i \leq k$ (and we place the i piles of size l where they belong in weakly descending order). On Young Diagrams, these moves are:



Properties

Both of these non-deterministic variants can be thought of as extensions of Bulgarian Solitaire, since in both games, playing the 1-move ($\overline{\mathcal{M}_1}$ or $\overline{\mathcal{BB}_1}$) is just the Bulgarian Solitaire move.

Theorem. *Block Bulgarian Solitaire has a unique sink strongly-connected component, containing exactly the elements of Bulgarian Solitaire recurrent cycles.*

References

[1] Jørgen Brandt. Cycles of Partitions. Proceedings of the American Mathematical Society, JSTOR, 483–486, 1982

Directed Graphs and Sinks

Definition

- Our Bulgarian Solitaire variants used digraphs (=**directed graphs**) with arcs for the moves.
- A **Strongly Connected Component** of a digraph is a maximal subgraph such that for any two of its vertices u, v, there are directed paths u to v and v to u within the subgraph.
- The strongly connected components themselves inherit an **acyclic** digraph structure. In this acyclic digraph, call a **sink** a strongly connected component that has no edges leading out of it to another strongly connected component.
- When referring to the sink of a Bulgarian Solitaire Variant game graph, we actually mean the sink of the digraph of strongly connected components. For example, in the Bulgarian Solitaire game graph for $n = 5$, the sink is $\{(3,1,1), (3,2), (2,2,1)\}$, for $n = 6$, the sink is $\{(3,2,1)\}$, and $n = 8$ has two sinks: $\{(3,3,2), (3,2,2,1), (4,2,1,1), (4,3,1)\}$ and $\{(3,3,1,1), (4,2,2)\}$.

Future Directions

Minnesota Solitaire Sinks

Based on computational data, it appears that Minnesota Solitaire has a single sink containing all of the Bulgarian Solitaire recurrent cycles.

Maximal Game Recurrent Cycles

The most important area for continued study with these games is the question of how to determine the recurrent cycles in Maximal Minnesota and Maximal Block Bulgarian Solitaire. See [1] for the analogous work in Bulgarian Solitaire.

Acknowledgements

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