Bulgarian Solitaire Variants

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Agenda

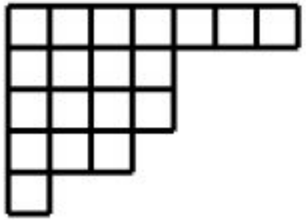
- Integer Partitions
- Bulgarian Solitaire
- Block Bulgarian Solitaire
- Minnesota Solitaire
- Future Directions

Integer Partitions and Young Diagrams

Definition 1.1. An *integer partition* of n, denoted $\lambda \vdash n$, is an unordered list of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that

$$\sum_{i=1}^{l} \lambda_i = n$$

where each λ_i is a positive integer less than or equal to n.



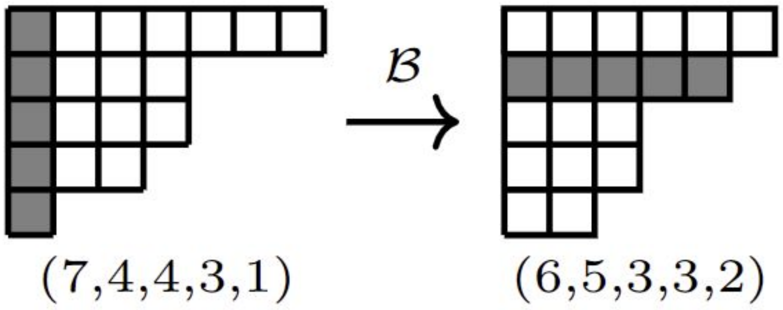
Young Diagram

The Bulgarian Solitaire Move

- On partitions

$$\mathcal{B}(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, l)$$

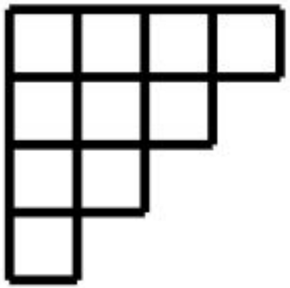
- On Young Diagrams



Key Results

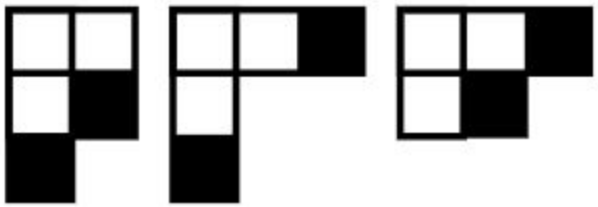
- (Gardner) Staircase partitions for triangular numbers
- (Brandt) Nearly staircase partitions for non-triangular numbers

n=10 (triangular)



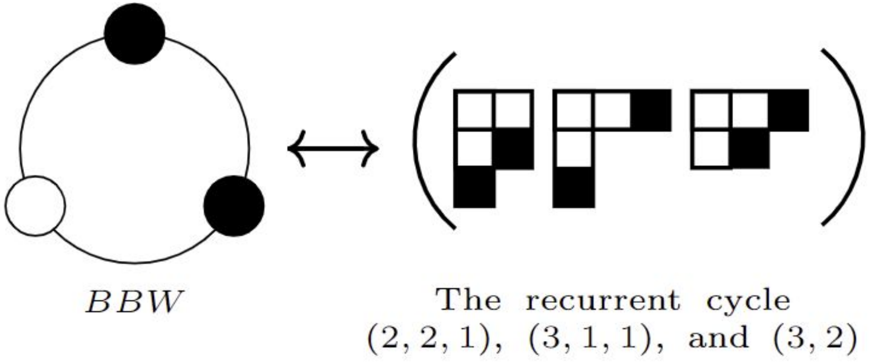
$$\Delta_b = (b, b-1, \ldots, 2, 1)$$

n=5 (not triangular)



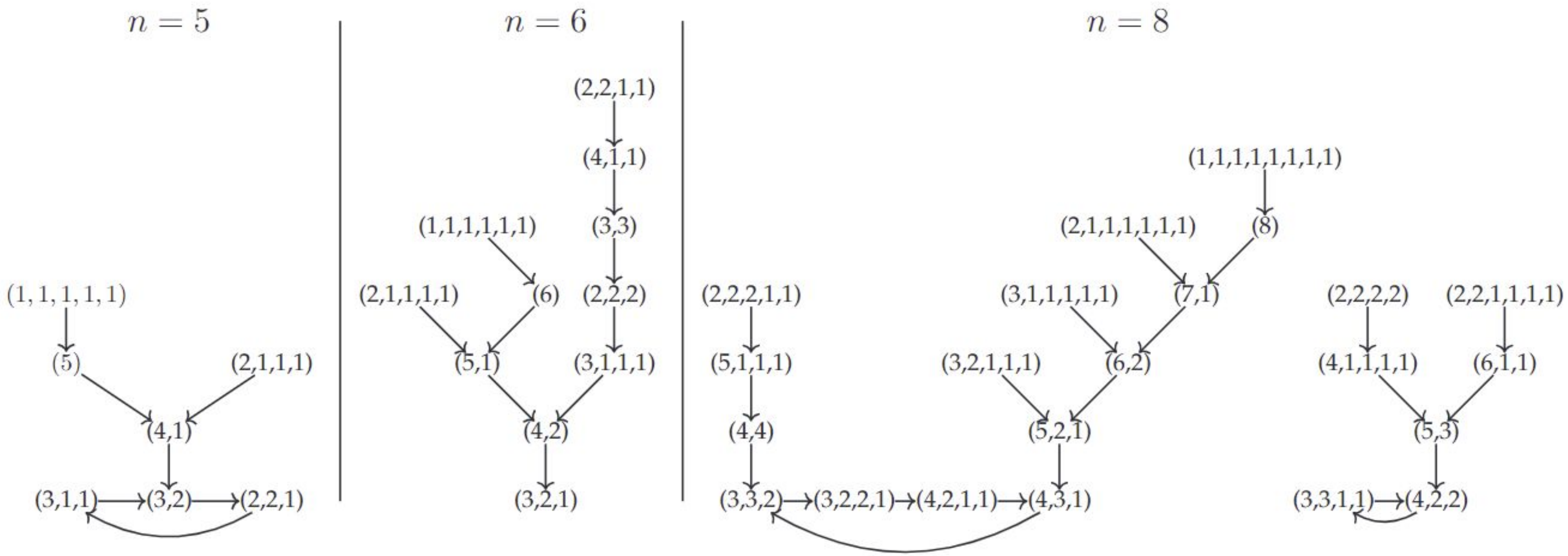
Brandt's Bijection

- Following Brandt's results, a way to understand recurrent cycles
- Bijection between recurrent cycle (and thus whole orbit) and a necklace



Bulgarian Solitaire Game Graph

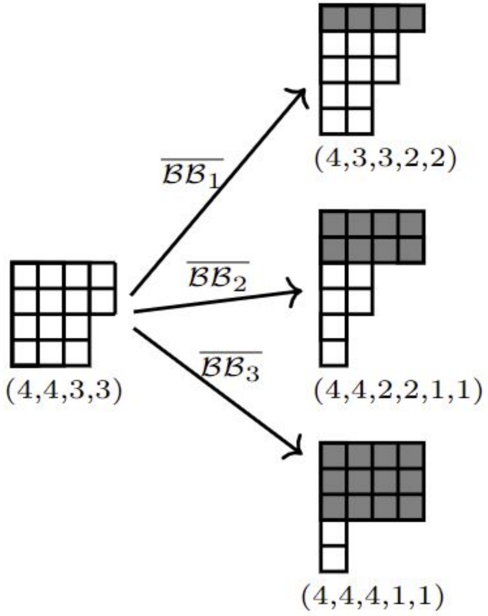
- Game Graph - display all orbits
  - Levels = distance from recurrent cycle



Block Bulgarian Solitaire

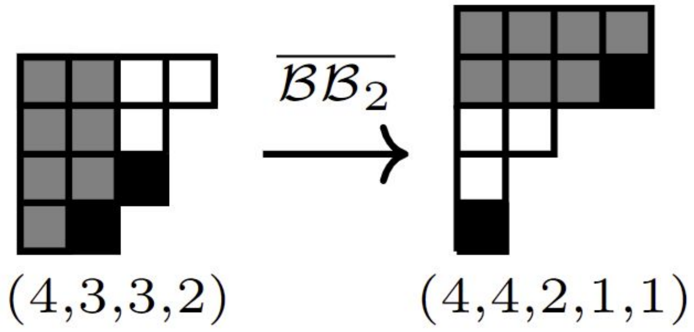
- Non-deterministic
- Take multiple cards from each pile (up to the size of the smallest pile), and create new piles
- BBS move

$$\overline{\mathcal{BB}}_i(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, l, l, \ldots, l)$$



Key Results

Lemma 3.2. *When possible, in a Bulgarian Solitaire recurrent cycle, playing the Block Bulgarian Solitaire 2-move ($\overline{\mathcal{B}\mathcal{B}_2}$) will switch you into a new Bulgarian Solitaire recurrent cycle that has the same number of black and white beads, just in a permutation of their order.*

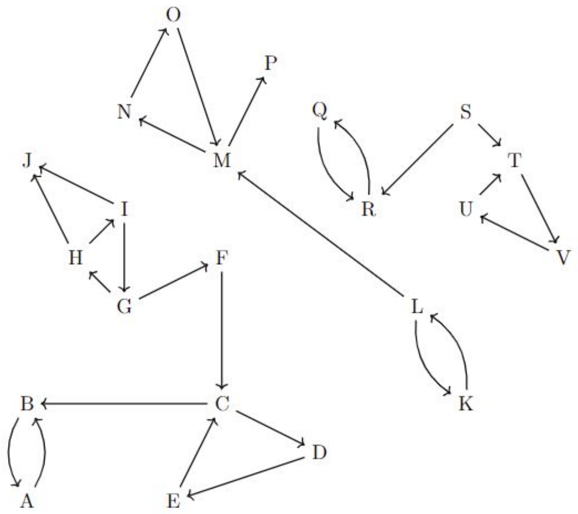


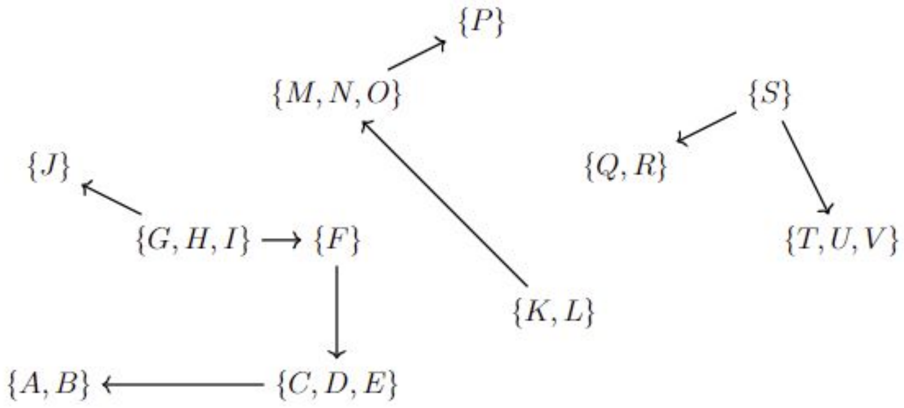
BBWWW BWBWW

Digraph Sinks

Definition 1.9. In a digraph, a *sink* is a vertex that has no edges leading out of it.

- A digraph of Strongly Connected Components inherits acyclic digraph structure
- In Bulgarian Solitaire and its variants, when referring to a *sink*, we actually mean a sink strongly connected component.





Key Results

Theorem 3.1. *Block Bulgarian Solitaire has a single sink containing exactly the elements of Bulgarian Solitaire recurrent cycles.*

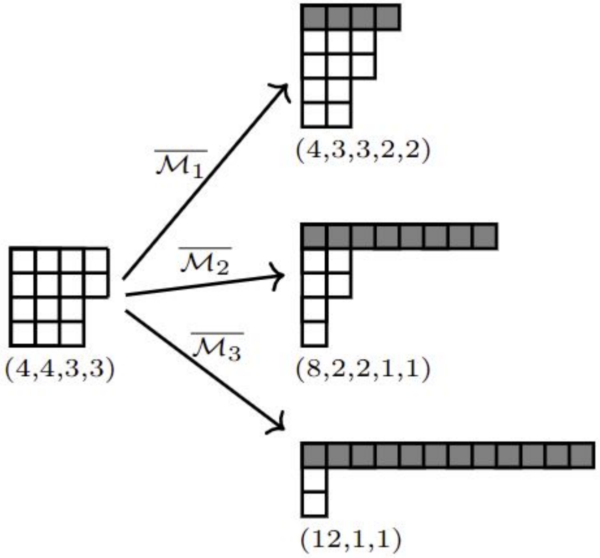
n	Block Bulgarian Solitaire Sink	Corresponding Necklaces
1	(1)	$B = W^2$
2	(2),(1,1)	BW, WB
3	(2,1)	$B^2 = W^3$
4	(2,2), (3,1), (2,1,1)	WBW, BWW, WWB
5	(2,2,1), (3,2), (3,1,1)	WBB, BBW, BWB
6	(3,2,1)	$B^3 = W^4$
7	(3,2,1,1), (3,2,2), (4,2,1), (3,3,1)	$WWWB$, $WWBW$, $BWWW$, $WBWW$
8	(4,2,1,1), (4,2,2), (4,3,1), (3,3,1,1), (3,3,2), (3,2,2,1)	$BWWB$, $BWBW$, $BBWW$, $WBWB$, $WBBW$, $WWBB$
9	(3,3,2,1), (4,3,2), (4,3,1,1), (4,2,2,1)	$WBBB$, $BBBW$, $BWBB$
10	(4,3,2,1)	$B^4 = W^5$

Table 2: Block Bulgarian Solitaire Sinks

Minnesota Solitaire

- Take multiple cards from each pile and form a single new pile
- Minnesota Solitaire move

$$\overline{\mathcal{M}_i}(\lambda) = (\lambda_1 - i, \lambda_2 - i, \ldots, \lambda_l - i, il)$$



Conjectures

Conjecture 3.4. *Minnesota Solitaire has a single sink containing every element of the Bulgarian Solitaire recurrent cycles for n.*

Conjecture 3.5. *For a nearly triangular $n = T_k - 1$, the Minnesota Solitaire sink contains exactly $2(k-1)$ elements.*

n	Minnesota Solitaire Sink	Corresponding Necklaces
1	(1)	$B = W^2$
2	(2),(1,1)	BW, WB
3	(2,1)	$B^2 = W^3$
4	(2,2), (3,1), (2,1,1), (4)	WBW, BWW, WWB, n/a
5	(2,2,1), (3,2), (3,1,1), (4,1)	WBB, BBW, BWB, n/a
6	(3,2,1)	$B^3 = W^4$
7	(3,2,1,1), (3,2,2), (4,2,1), (3,3,1), (4,3), (6,1), (5,2)	$WWWB$, $WWBW$, $BWWW$, $WBWW$ n/a, n/a, n/a
8	(4,2,1,1), (4,2,2), (4,3,1), (3,3,1,1) (3,3,2), (3,2,2,1), (7,1), (8), (4,4), (6,2), (5,2,1), (6,1,1), (5,3)	$BWWB$, $BWBW$, $BBWW$, $WBBW$ $WBBW$, $WWBB$, n/a, n/a n/a, n/a, n/a, n/a, n/a
9	(3,3,2,1), (4,3,2), (4,3,1,1), (4,2,2,1) (5,3,1), (6,2,1)	$WBBB$, $BBBW$, $BWBB$ n/a, n/a
10	(4,3,2,1)	$B^4 = W^5$

Table 4: Minnesota Solitaire Sinks

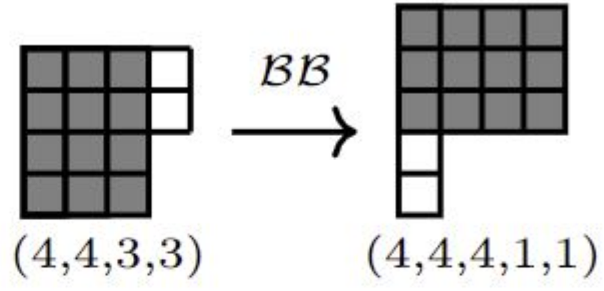
n	Sink Size	n	Sink Size
1	1	16	88
2	2	17	106
3	1	18	145
4	4	19	171
5	4	20	10
6	1	21	1
7	7	22	316
8	13	23	383
9	6	24	471
10	1	25	527
11	26	26	671
12	40	27	12
13	42	28	1
14	8	29	1133
15	1	30	1379

Table 5: Sizes of the Minnesota Solitaire Sink for $n \leqslant 30$

Maximal Block Bulgarian Solitaire

- Play the largest BBS move possible
- MBBS move

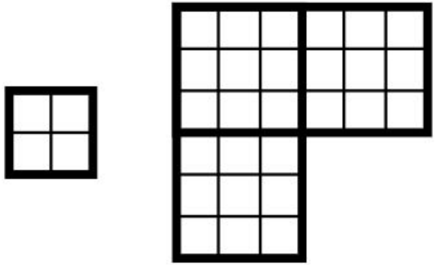
$$\mathcal{BB}(\lambda) = (l, l, \ldots, l, \lambda_1 - k, \lambda_2 - k, \ldots, \lambda_{l-1} - k)$$



Key Results

Theorem 4.1. *A partition $\lambda \vdash n$ is fixed under $\mathcal{BB}$ if and only if it is a square staircase.*

Corollary 4.2. *The number of fixed partitions of n under $\mathcal{BB}$ is equal to the number of ways n can be written as $a^2\binom{b+1}{2}$, for integers a and b.*



$\Lambda_{(2,1)}$ $\Lambda_{(3,2)}$

$$\Lambda_{(a,b)} = (ab, ab, \ldots, ab, a(b-1), a(b-1), \ldots, a, a, \ldots, a)$$

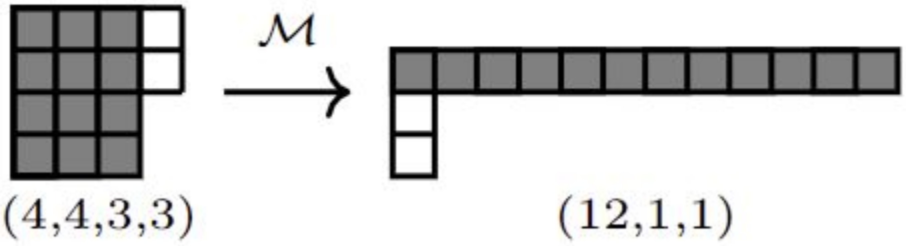
n	$\#\Lambda(a,b)$	n	$\#\Lambda(a,b)$	n	$\#\Lambda(a,b)$
1	1	11	0	36	2
2	0	12	1	144	2
3	1	13	0	44100	3
4	1	14	0	100800	2
5	0	15	1	11524800	3
6	1	16	1		
7	0	17	0		
8	0	18	0		
9	1	19	0		
10	1	20	0		

Table 7: The number of square staircase partitions for select n

Maximal Minnesota Solitaire

- Play the largest possible MS move
- Maximal MS move

$$\mathcal{M}(\lambda) = (\lambda_1 - k, \lambda_2 - k, \ldots, \lambda_l - k, kl)$$

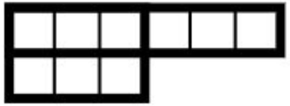


Key Results

Proposition 4.8. *The fixed partitions under $\mathcal{M}$ are the stretched staircase partitions.*
Lemma 4.9. *The number of stretched staircase partitions of n is equal to the number of triangular numbers that divide n.*

Theorem 4.10. *The number of fixed partitions of n under $\mathcal{M}$ is equal to the number of triangular numbers that divide n.*

$\Gamma_{(2,1)}$



$\Gamma_{(3,2)}$

$$\Gamma_{(a,b)} = (ab, a(b-1), a(b-2), \ldots, a)$$

n	$\#\Gamma_{(a,b)}$	n	$\#\Gamma_{(a,b)}$	n	$\#\Gamma_{(a,b)}$
1	1	11	1	21	3
2	1	12	3	22	1
3	2	13	1	23	1
4	1	14	1	24	3
5	1	15	3	25	1
6	3	16	1	26	1
7	1	17	1	27	2
8	1	18	3	28	2
9	2	19	1	29	1
10	2	20	2	30	5

Table 10: Number of fixed partitions under $\mathcal{M}$ for n up to 30

Future Directions

Conjecture 3.4. *Minnesota Solitaire has a single sink containing every element of the Bulgarian Solitaire recurrent cycles for* n.

Conjecture 3.5. *For a nearly triangular* $n = T_k - 1$, *the Minnesota Solitaire sink contains exactly* $2(k-1)$ *elements.*

- Formulas for the number of Square Staircase Partitions and Stretched Staircase Partitions
- Understand Maximal Minnesota/Block Bulgarian Solitaire Recurrent Cycles

Questions?

References

Jørgen Brandt. Cycles of partitions. Proceedings of the American Mathematical Society, pages 483–486, 1982.

Martin Gardner. Mathematical games: Tasks you cannot help finishing no matter how hard you try to block them. Scientific American, 249:12–21, 1983.

Gus Wiseman. Sequences counting and ranking integer partitions by the differences of their successive parts. On-Line Encyclopedia of Integer Sequences A007862, 2019.