CRITICAL GROUPS OF MCKAY-CARTAN MATRICES

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Abstract. This thesis investigates the critical groups of McKay-Cartan matrices, a certain type of avalanche-finite matrix associated to a faithful representation γ of a finite group G. It computes the order of the critical group in terms of the character values of γ, and gives some restrictions on its subgroup structure. In addition, the existence of a certain Smith normal form over $\mathbb{Z}[t]$ is shown to imply a nice form for the critical group. This is used to compute the critical group for the reflection representation of $\mathfrak{S}_n$. In the case where $\mathrm{im}(\gamma) \subset SL(n, \mathbb{C})$ it discusses for which pairs (G, γ) the critical group is isomorphic to the abelianization of G, including explicit calculations demonstrating these isomorphisms for the finite subgroups of $SU(2)$ and $SO(3, \mathbb{R})$. In this case it also identifies a subset of the superstable configurations, answering a question posed in [3].

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Contents

1. Introduction

This thesis will study the *critical group*, $K(C) := \operatorname{coker}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell)$, which is defined for C an *avalanche-finite matrix*. Specifically it will investigate the structure of $K(C)$ when C is the *McKay-Cartan matrix* associated to a faithful representation γ of a finite group G. This matrix has entries $c_{ij} = n\delta_{ij} - m_{ij}$ for $1 \leq i, j \leq \ell$ where the m_{ij} are defined by

$$\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij}\chi_j$$

and where $\{1_G = \chi_0, \chi_1, ..., \chi_\ell\}$ is the set of irreducible complex characters of G; in this case we define $K(\gamma) := K(C)$. McKay-Cartan matrices were shown to be avalanche-finite in [3].

Later in Section 1 critical groups are introduced in the context of Laplacian matrices of graphs, which are themselves avalanche-finite matrices. Section 2 defines avalanche-finite matrices and McKay-Cartan matrices in general and gives certain basic results about the critical groups of the latter. Section 3 applies some results of Lorenzini in [8] to obtain the first main Theorem:

Theorem 1. *Let G be a finite group with faithful complex representation γ and critical group $K(\gamma)$. Let $e = c_0, c_1, ..., c_\ell$ be a set of conjugacy class representatives for G, then:*

i.
$$\prod_{i=1}^{\ell}(n - \chi_\gamma(c_i)) = |K(\gamma)| \cdot |G|$$

ii. *If χ_γ is real-valued, and $\chi_\gamma(c)$ is an integer character value achieved by m different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n - \chi_\gamma(c))\mathbb{Z})^{m-1}$.*

Section 4 derives explicit formulas for critical groups in the case of the reflection representation of $\mathfrak{S}_n$:

Theorem 2. *Let γ be the reflection representation of $\mathfrak{S}_n$ and let $p(k)$ denote the number of partitions of the integer k. Then*

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{\substack{1 \leq k \leq n \\ p(k)-p(k-1) \geq i}} k$$

Section 5 gives an alternative description of the critical group of a McKay-Cartan matrix in terms of the representation ring of the associated finite group and discusses two questions raised in [3], answering one of them:

Theorem 3. *Let $\gamma : G \hookrightarrow SL(n, \mathbb{C})$ be a faithful representation of G with McKay-Cartan matrix C. Then the standard basis vectors e_i corresponding to the non-trivial one-dimensional characters χ_i of G are superstable configurations for C.*

1.1. **Graph Laplacian matrices.** Let Γ be a loopless undirected graph on n vertices labelled $\{1, ..., n\}$. Let c_{ij} be the number of edges between vertices i and j, and let d_i denote the degree of vertex i. The *Laplacian matrix* $L(\Gamma)$ is an $n \times n$ integer matrix with entries:

$$L(\Gamma)_{ij} = \begin{cases} d_i & \text{if } i = j \\ -c_{ij} & \text{if } i \neq j \end{cases}$$

Many classical results in algebraic graph theory relate to the Laplacian matrix. Most famous is Kirchhoff's Matrix Tree Theorem:

Theorem 4 (Kirchhoff)**.** *Let Γ be an undirected graph with Laplacian matrix $L = L(\Gamma)$. Let $\kappa(\Gamma)$ denote the number of spanning trees of Γ. Then if L^{ij} denotes the matrix L with row i and column j removed, we have*

$$\det L^{ij} = \kappa(\Gamma)$$

for all $1 \leq i, j \leq n$.

In fact, finer results may be obtained. Results (i) and (ii) below are very classical (a proof can be found, for example, in [4, Corollary 6.5]). Part (iii) is clear from part (i) and the existence of Smith normal form over $\mathbb{Z}$ (see Section 1.3).

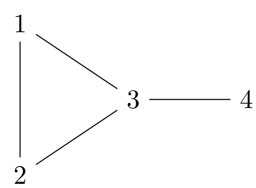
Proposition 1. *Let Γ be a connected graph on n vertices and $L = L(\Gamma)$ be its Laplacian matrix, then*

i. *L has rank $n - 1$, with kernel generated by $(1, ..., 1)^t$.*
ii. *Let $\lambda_1, ..., \lambda_{n-1}$ be the nonzero eigenvalues of L, then*

$$\kappa(\Gamma) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}$$

iii. coker$(\mathbb{Z}^n \xrightarrow{L} \mathbb{Z}^n)$ *is isomorphic to $\mathbb{Z} \oplus K(\Gamma)$ where $K(\Gamma)$ is a finite abelian group with $|K(\Gamma)| = \kappa(\Gamma)$.*

Example 1. Let Γ be the graph



Then

$$L(\Gamma) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

We find that the eigenvalues of $L(\Gamma)$ are 0,1,3,4 and its Smith normal form (see Section 1.3) is $\text{diag}(1, 1, 3, 0)$. Therefore $\kappa(\Gamma) = \frac{1 \cdot 3 \cdot 4}{4} = 3$ and $\text{coker}(L) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, so $K(\Gamma) \cong \mathbb{Z}/3\mathbb{Z}$. It is easy to see that Γ does in fact have 3 spanning trees and that $(1, 1, 1, 1)^t$ is the eigenvector corresponding to the eigenvalue 0.

1.2. **Critical groups of graphs.** The finite abelian group $K(\Gamma)$ from Proposition 1 is called the *critical group* of Γ and it provides a finer invariant for Γ than $\kappa(\Gamma)$, since there are many abelian groups of a given order. The group $K(\Gamma)$ encodes information about the *critical configurations* in the *abelian sandpile model*, a certain dynamical system on Γ. An introduction to this topic can be found in [7], and applications to areas such as economic models and energy minimization can be found, for example, in [1, 5].

1.3. **Smith normal form and the cokernel of a linear map.** Smith normal forms will be one of our primary tools for calculating critical groups. Several basic facts about Smith normal form are summarised in this section.

Let R be a ring and $A \in R^{n\times n}$ be a matrix. A matrix S is called the *Smith normal form* of A if:

- There exist invertible matrices $P, Q \in R^{n\times n}$ such that $S = PAQ$.
- S is a diagonal matrix $S = \mathrm{diag}(s_1, ..., s_n)$ with $s_i | s_{i+1}$ for $i = 1, ..., n-1$.

The Smith normal form of A, if it exists, is unique up to multiplication of the s_i by units in R.

Proposition 2. *Let $A \in R^{n\times n}$ be a matrix and suppose A has Smith normal form $S = \mathrm{diag}(s_1, ..., s_n)$. Then*

$$\mathrm{coker}(A : R^n \to R^n) \cong \bigoplus_{i=1}^{n} R/(s_i)$$

Proof. Clearly $\mathrm{coker}(S) = \bigoplus_{i=1}^{n} R/(s_i)$. The invertible maps P, Q provide the desired isomorphism. □

Proposition 3. *Let $A \in R^{n\times n}$ be a matrix. If R is a PID then A has a Smith normal form.*

Proof. By the classification of finitely generated modules over a PID, $\mathrm{coker}(A)$ is isomorphic to $\bigoplus_{i=1}^{n} R/(s_i)$ for some choice of the $s_i \in R$ with $(s_1) \supset \cdots \supset (s_n)$. It is easy to see that $S = \mathrm{diag}(s_1, ..., s_n)$ is the Smith normal form of A. □

2. Avalanche-finite matrices

The abelian sandpile model, which motivated our interest in the critical group $K(\Gamma)$ of a graph has recently been found to have a natural generalization to integer matrices other than graph Laplacians. These more general matrices are called *avalanche-finite matrices*. As before, we define the *critical group* $K(C)$ of such a matrix C as

$$K(C) := \mathrm{coker}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell)$$

And, as before, several important classes of configurations (the *superstable* and the *critical* configurations) form distinguished sets of coset representatives in this group.

Definition 1. A matrix $C = (c_{ij})$ in $\mathbb{Z}^{\ell\times\ell}$ with $c_{ij} \le 0$ for all $i \ne j$ is called a *Z-matrix*.

Given a Z-matrix C, we call the elements $v = (v_1, ..., v_\ell)^t \in \mathbb{N}^\ell$ *chip configurations*, and we define a dynamical system on the set of such configurations as follows:

- A configuration v is *stable* if $v_i < c_{ii}$ for $i = 1, ..., \ell$.

- If v is unstable, then choose some i so that $v_i \geq c_{ii}$ and form a new configuration $v' = (v'_1, ..., v'_\ell)^t$ where $v'_j = v_j - c_{ij}$ for $j = 1, ..., \ell$. The result v' is called the *C-toppling* of v at position i.

Definition 2. A Z-matrix is called an *avalanche-finite matrix* if every chip configuration can be brought to a stable one by a sequence of such topplings.

The *superstable configurations* of an avalanche-finite matrix are those which are not only stable, but cannot even be toppled at several positions at once, leaving the accounting to the end. It is known that the superstable configurations form a set of coset representatives for $K(C)$ (see [11, Theorem 13.4]).

Definition 3. Let $u \in \mathbb{N}^\ell$ be a chip configuration for an avalanche-finite matrix C. Then u is called a *superstable* configuration if $z \in \mathbb{N}^\ell$ and $u - C^t z \in \mathbb{N}^\ell$ together imply that $z = \mathbf{0}$.

In this thesis we will be interested in understanding the critical groups and superstable representations of a subset of avalanche-finite matrices called McKay-Cartan matrices.

2.1. **McKay-Cartan matrices.** Let G be a finite group with irreducible complex characters $\chi_0, ..., \chi_\ell$. We will always let χ_0 denote the character of the trivial representation. Let γ be a faithful (not necessarily irreducible) n-dimensional representation of G with character χ_γ. Let $M \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$ be the matrix whose entries are defined by the equations

$$\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

Definition 4. Let γ be an n-dimensional faithful complex representation of a finite group G, and let M be defined as above. The *extended McKay-Cartan matrix* $\widetilde{C} := nI - M$ and the *McKay-Cartan matrix* C is the submatrix of $\widetilde{C}$ obtained by removing the row and column corresponding to the trivial character χ_0.

Proposition 4 (A special case of Proposition 5.3 in [3])**.** *Let γ be a faithful complex representation of a finite group G.*

i. *A full set of orthogonal eigenvectors for $\widetilde{C}$ is given by the set of columns of the character table of G:*
$$\delta^{(g)} = (\chi_0(g), ..., \chi_\ell(g))^t$$
as g ranges over a collection of conjugacy class representatives for G.

ii. *The corresponding eigenvalues are given by*
$$\widetilde{C}\delta^{(g)} = (n - \chi_\gamma(g)) \cdot \delta^{(g)}$$

iii. *The vector $\delta^{(e)} = (\chi_0(e), ..., \chi_\ell(e))^t$ spans the nullspace of both $\widetilde{C}$ and $\widetilde{C}^t$. In particular, this implies that these matrices have rank ℓ.*

Proof. The equation $M\delta^{(g)} = \chi_\gamma(g)\delta^{(g)}$ follows immediately from evaluating both sides of the defining equations of M at g. Then

$$\widetilde{C}\delta^{(g)} = (nI - M)\delta^{(g)} = (n - \chi_\gamma(g))\delta^{(g)}$$

This proves (ii). The orthogonality in (i) follows from the second orthogonality relation for irreducible characters.

For (iii), note that since γ is faithful, $\chi_\gamma(g) \neq n$ for $n \neq e$. Therefore (ii) gives that the nullspace of $\widetilde{C}$ is 1-dimensional, spanned by $\delta^{(e)}$. This shows that $\widetilde{C}$, and therefore $\widetilde{C}^t$, has rank ℓ. Now, the i-th entry in $M^t\delta^{(e)}$ is

$$\sum_{j=0}^{\ell} m_{ji}\chi_j(e) = \sum_j (\dim \chi_j \cdot \langle \chi_j\chi_\gamma, \chi_i\rangle) = \langle (\bigoplus_j \chi_j^{\oplus \dim \chi_j}) \cdot \chi_\gamma, \chi_i\rangle$$
$$= \langle \chi_{reg}\chi_\gamma, \chi_i\rangle = \frac{1}{|G|}\sum_{g\in G} \chi_{reg}(g)\chi_\gamma(g)\overline{\chi_i}(g) = \frac{1}{|G|}(|G|\chi_\gamma(e)\chi_i(e)) = \chi_\gamma(e)\chi_i(e)$$

Therefore $\widetilde{C}^t\delta^{(e)} = (n - \chi_\gamma(e))\delta^{(e)} = 0$. □

Finally, we note that McKay-Cartan matrices are indeed avalanche-finite, motivating our study of their cokernels.

Theorem 5 ([3], Theorem 1.2)**.** *The McKay-Cartan matrix C associated to a faithful representation γ of a finite group G is an avalanche-finite matrix.*

Definition 5 ([3], Definition 5.11)**.** Given a faithful complex representation γ of a finite group G with McKay-Cartan and extended McKay-Cartan matrices $C, \widetilde{C}$, we define its *critical group* in either of the following equivalent ways:

$$K(\gamma) := \mathrm{coker}(C^t) = K(C)$$
$$\mathbb{Z} \oplus K(\gamma) := \mathrm{coker}(\widetilde{C}^t)$$

Proposition 5 ([3], Proposition 6.12)**.** *The critical group $K(\gamma)$ is unchanged by*

a. *adding copies of the trivial representation to γ.*
b. *precomposing with a group automorphism $\sigma : G \to G$.*

Proof. For (a), replacing χ_γ with $\chi_\gamma + d \cdot \chi_0$ replaces M with $M' = M + dI$ and therefore replaces $\widetilde{C}$ with $(n + d)I - (M + dI) = nI - M = \widetilde{C}$.

For (b), precomposition by σ permutes $\{\chi_0, ..., \chi_\ell\}$, this corresponds to a permutation of the basis for the map $\widetilde{C}$, which does not affect the cokernel up to isomorphism. □

Example 2. Let $G = \mathfrak{S}_4$ and consider the reflection representation γ of G (this is the action by permutation matrices on $\mathbb{C}^4$ with the copy of the trivial representation removed; the associated partition is $(3, 1)$). The character table is

	e	(12)	(123)	(1234)	(12)(34)
χ_0	1	1	1	1	1
χ_1	1	-1	1	-1	1
χ_γ	3	1	0	-1	-1
χ_3	3	-1	0	1	-1
χ_4	2	0	-1	0	2

The matrices $M, \widetilde{C}$ and C associated to γ are

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \widetilde{C} = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix} C = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

To calculate $K(\gamma)$, we calculate that C has Smith normal form diag$(1,1,1,4)$, or equivalently that $\widetilde{C}$ has Smith form diag$(1,1,1,4,0)$. Thus $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z}$. Theorem 2 will give the structure of the critical group for the reflection representations for all $\mathfrak{S}_n$.

3. The structure of critical groups for group representations

The following general results of Lorenzini about $(\ell+1)\times(\ell+1)$-integer matrices of rank ℓ will be useful.

Proposition 6 ([8], Propositions 2.1 and 2.3)**.** *Let M be any $(\ell+1)\times(\ell+1)$-integer matrix of rank ℓ with characteristic polynomial $char_M(x) = x\prod_{i=1}^{\ell}(x-\lambda_i)$. Let R be an integer vector in lowest terms generating the kernel of M, and let R' be the corresponding vector for M^t. Let H be the torsion subgroup of the cokernel of $\mathbb{Z}^{\ell+1} \xrightarrow{M} \mathbb{Z}^{\ell+1}$.*

i.
$$\prod_{i=1}^{\ell} \lambda_i = \pm|H|(R\cdot R')$$

ii. *Let $\lambda \neq \pm 1, 0$ be an integer eigenvalue of M, and $\mu(\lambda)$ be the maximal number of linearly independent eigenvectors for the eigenvalue λ. If M is symmetric and R has at least one entry with value ± 1 then C contains a subgroup isomorphic to $(\mathbb{Z}/\lambda\mathbb{Z})^{\mu(\lambda)-1}$.*

Translating this proposition into the context of critical groups of group representations allows us to prove Theorem 1.

Theorem 1. *Let G be a finite group with faithful complex representation γ and critical group $K(\gamma)$. Let $e = c_0, c_1, ..., c_\ell$ be a set of conjugacy class representatives for G, then:*

i.
$$\prod_{i=1}^{\ell}(n-\chi_\gamma(c_i)) = |K(\gamma)|\cdot|G|$$

ii. *If χ_γ is real-valued, and $\chi_\gamma(c)$ is an integer character value achieved by m different conjugacy classes, then $K(\gamma)$ contains a subgroup isomorphic to $(\mathbb{Z}/(n-\chi_\gamma(c))\mathbb{Z})^{m-1}$.*

Proof. Part (i) follows from Proposition 6 (i) with $M = \widetilde{C}$: the eigenvalues of $\widetilde{C}$ are given in Proposition 4 (ii), and we have that in this case $R = R' = (\chi_0(e), ..., \chi_\ell(e))^t$ by Proposition 4 (iii). Therefore

$$R\cdot R' = \sum_{i=0}^{\ell}(\dim \chi_i)^2 = |G|$$

The product on the left hand side is positive since the complex conjugate of a term $n-\chi_\gamma(c)$ also appears in the product as $n-\chi_\gamma(c')$ where c' is conjugate to c^{-1}.

For part (ii), notice that $\mu(n-\chi_\gamma(c)) = m$ since the eigenvectors of $\widetilde{C}$ are orthogonal by Proposition 4 (i). If χ_γ is real valued then

$$\begin{aligned}\langle \chi_j, \chi_\gamma \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \overline{\chi_\gamma(g)\chi_i(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_j(g)\chi_\gamma(g)\overline{\chi_i(g)} = \langle \chi_j\chi_\gamma, \chi_i \rangle\end{aligned}$$

so $\widetilde{C}$ is symmetric in this case. Finally, $\chi_\gamma(c) < n$, so we can rule out the cases $\lambda = 0, -1$ in Proposition 6 (ii). Applying this proposition then gives the desired result. □

Example 3. Let γ be the defining permutation representation of the alternating group $\mathfrak{A}_n$ for $n \geq 5$, which clearly has a real-valued character. This group always has a *split class*, a conjugacy class of $\mathfrak{S}_n$ which splits into two classes in $\mathfrak{A}_n$. This means that χ_γ has the same value on representatives c_1, c_2 of two different conjugacy classes, with $c_1, c_2 \neq e$. Thus $\widetilde{C}$ has a repeated eigenvalue $\lambda = n - \chi_\gamma(c_1) \geq 2$. By Theorem 1 (ii), this implies that $K(\gamma)$ has a subgroup isomorphic to $\mathbb{Z}/\lambda\mathbb{Z}$.

Corollary 1. *In the context of Theorem 1, if χ_γ is $\mathbb{Q}$-valued, then $Syl_p(K(\gamma)) = 0$ unless $p \leq 2n$.*

Proof. The fact that χ_γ is $\mathbb{Q}$-valued implies that it is $\mathbb{Z}$-valued since a sum of roots of unity is rational if and only if it is integral. Thus $-n \leq \chi_\gamma(c_i) < n$ for all i, and so the product on the left hand side of part (i) of the theorem is a product of integers which are at most $2n$. □

Example 4. The requirement that χ_γ is $\mathbb{Q}$-valued in Corollary 1 is necessary. If $G = \langle g | g^m = 1\rangle$ is a cyclic group and γ is given by

$$g \mapsto \begin{pmatrix} e^{2\pi i/m} & 0 \\ 0 & e^{-2\pi i/m} \end{pmatrix}$$

then $K(\gamma) \cong \mathbb{Z}/m\mathbb{Z}$ (see Appendix B.1). In this case $|K(\gamma)|$ may have prime divisors up to m which are larger than $2n = 4$.

Corollary 2. *For any faithful n-dimensional representation γ of G*

$$\frac{\prod_{i=1}^{\ell}(n - \chi_\gamma(c_i))}{|G|}$$

lies in $\mathbb{Z}$.

Proof. By Theorem 1 (i), this is, up to sign, the order of $K(\gamma)$. □

Theorem 1 (i) gives us the order of the abelian group $K(\gamma)$ and part (ii) gives some restrictions on the structure of this group, however this is not enough to uniquely specify $K(\gamma)$ in general. Below we analyse the Smith normal form of $tI - \widetilde{C}$ over $\mathbb{Q}[t]$ which will allow us to explicitly determine the critical group corresponding to the irreducible reflection representation of $\mathfrak{S}_n$ in Section 4.

Proposition 7. *Let G be a nontrivial finite group whose irreducible characters are $\mathbb{Z}$-valued (for example, a Weyl group) and let γ be a faithful complex representation of G. Let $m(\lambda)$ denote the multiplicity of an eigenvalue λ of $\widetilde{C}$. Then, up to multiplication of the rows by nonzero rational numbers, the Smith normal form over $\mathbb{Q}[t]$ of $tI - \widetilde{C}$ is $\mathrm{diag}(\alpha_{\ell+1}, ..., \alpha_1)$ where*

$$\alpha_i = \prod_{\substack{\lambda \\ m(\lambda) \geq i}} (t - \lambda)$$

In addition, $\alpha_{\ell+1} = 1$ and $t|\alpha_i$ if and only if $i = 1$.

Proof. Recall that two matrices A, B with entries in a common field F are similar (over F) if and only if $tI - A$ and $tI - B$ have the same Smith normal form over $F[t]$. Let T be the character table of G, the $(\ell+1) \times (\ell+1)$-matrix whose rows are the irreducible character values. By Proposition 4, T is the matrix of eigenvectors for $\widetilde{C}$, therefore $D := T^{-1}\widetilde{C}T$ is the diagonalization of $\widetilde{C}$. Since T has integer entries we know T^{-1} has rational entries; this implies that $tI - \widetilde{C}$ and $tI - D$ have the same Smith normal form over $\mathbb{Q}[t]$.

Now, $tI - D = \mathrm{diag}(t - \lambda_0, ..., t - \lambda_\ell)$. Since $\mathbb{Q}[t]$ is a Euclidean domain, 2×2 principal minors can be transformed

$$\begin{pmatrix} f(t) & 0 \\ 0 & g(t) \end{pmatrix} \to \begin{pmatrix} \gcd(f,g) & 0 \\ 0 & \mathrm{lcm}(f,g) \end{pmatrix}$$

using elementary row operations without affecting the rest of the matrix. This is essentially the same algorithm used for computing integer Smith normal forms. Repeated applications of these local transformations gives the desired Smith normal form.

Finally, since γ is faithful, χ_γ takes on at least two distinct values, and therefore $\widetilde{C}$ has at least two distinct eigenvalues. Thus $\{\lambda | m(\lambda) = \ell+1\}$ is empty, so $\alpha_{\ell+1} = 1$. By Proposition 4, $m(0) = 1$ so $t|\alpha_i$ if and only if $i = 1$. □

Corollary 3. *In the context of Proposition 7, if in addition $tI - \widetilde{C}$ has a Smith normal form [1] over $\mathbb{Z}[t]$ then*

i. *The Smith normal form over $\mathbb{Z}[t]$ is $\mathrm{diag}(\alpha_{\ell+1}, ..., \alpha_1)$ up to multiplication of each α_i by ± 1.*
ii. *The critical group is*

$$K(\gamma) \cong \bigoplus_{i=2}^{\ell+1} (\mathbb{Z}/|\alpha_i(0)|\mathbb{Z})$$

Proof. Let $S = \mathrm{diag}(s_{\ell+1}, ..., s_1)$ be the Smith normal form of $tI - \widetilde{C}$ over $\mathbb{Z}[t]$. It is a basic fact about Smith normal forms that $\det S$ is equal to $\det(tI - \widetilde{C})$ up to an invertible multiple in $\mathbb{Z}[t]$, that is, up to ± 1. Since $\det(tI - \widetilde{C})$ is monic, so must be $\det(S)$, and therefore each of the s_i is monic. Now, S is also a Smith normal form for $tI - \widetilde{C}$ over $\mathbb{Q}[t]$, therefore the s_i agree with the α_i up to nonzero rational multiple. Since both are monic, we see that in fact $s_i = \pm\alpha_i$.

For (ii), evaluating at $t = 0$ gives that $\mathrm{diag}(\alpha_{\ell+1}(0), ..., \alpha_1(0))$ is the Smith normal form for $\widetilde{C}$. By the proposition, $\alpha_i(0) = 0$ if and only if $i = 1$, which gives the desired result. □

[1]This is not guaranteed, since $\mathbb{Z}[t]$ is not a PID.

4. Critical groups for the reflection representation of $\mathfrak{S}_n$

The following proposition gives a formula for the Kronecker product of a Schur function s_λ with the Schur function $s_{(n-1,1)}$. This can be reinterpreted as a formula for the rows of $\widetilde{C}$ associated to the reflection representation of $\mathfrak{S}_n$.

Proposition 8 ([2], proof of Proposition 4.1)**.** *Let n be a positive integer and $\lambda \vdash n$ then*

$$s_{(n-1,1)} * s_\lambda = C(\lambda)s_\lambda + \sum s_\mu \tag{1}$$

where $C(\lambda) = |\{i|\lambda_i > \lambda_{i+1}, 1 \le i \le l(\lambda)-1\}|$ and the sum is over all partitions different from λ that can be obtained by removing one box and then adding a box to λ.

Theorem 6 ([6], special case of Theorem 1.2)**.** *Let Y denote Young's lattice, and let U and D be the up and down maps in $\mathbb{Z}Y$, the free abelian group with basis Y. Then $UD - tI$ has a Smith normal form over $\mathbb{Z}[t]$.*

We can now prove Theorem 2.

Theorem 2. *Let γ be the reflection representation of $\mathfrak{S}_n$ and let $\widetilde{C}$ be the associated extended McKay-Cartan matrix. Let $p(k)$ denote the number of partitions of the integer k. Then*

$$K(\gamma) \cong \bigoplus_{i=2}^{p(n)} \mathbb{Z}/q_i\mathbb{Z}$$

where

$$q_i = \prod_{\substack{1\le k\le n \\ p(k)-p(k-1)\ge i}} k$$

Proof. The right hand side of Equation (1) can easily be seen to be the sum of the Schur functions indexed by the partitions appearing in $(UD - I)\lambda$ since $|C(\lambda)|$ is one less than the number of corners in λ (the set $C(\lambda)$ excludes the last corner). Therefore $tI - \widetilde{C}$ has a Smith normal form over $\mathbb{Z}[t]$ by Theorem 6. Thus, by Corollary 3,

$$K(\gamma) \cong \bigoplus_{i=2}^{\ell+1} (\mathbb{Z}/|\alpha_i(0)|\mathbb{Z})$$

where

$$\alpha_i(t) = \prod_{\substack{k \\ m(k)\ge i}} (t-k)$$

Thus we just need to understand the multiplicities $m(k)$ of eigenvalues k of $\widetilde{C}$. Write $k = n - f$, then $m(k)$ is the number of conjugacy classes whose elements have exactly f fixed points. This in turn is the number of partitions of n with f parts of size 1, which is the number of partitions of $n - f$ with no fixed points. This is seen to be $p(k) - p(k-1)$ by an easy bijection. Therefore $|\alpha_i(0)| = q_i$. □

Remark. The existence of Smith normal forms over $\mathbb{Z}[t]$ for the operators $UD - tI$ in the differential posets Y^r for $r > 1$ was recently proven by Nie [10], generalizing Theorem 6. Of particular interest in the context of critical groups is the case $r = 2$, since the elements of Y^2 correspond to irreducible representations of the Weyl group of type B_n, just as Y

indexes the irreducibles of $\mathfrak{S}_n$. One might have hoped that $UD - cI$ would again have encoded $\gamma \otimes (-)$, as in Proposition 8, for some irreducible representation γ, thus allowing us to compute the critical group corresponding to this representation. Some computations for small values of n, however, show that this is not the case.

5. An alternative expression for critical groups

This section presents an alternative equivalent definition of $K(\gamma)$ in terms of the representation ring $R(G)$. In Section 5.1, we describe a relationship between $K(\gamma)$ and the Pontryagin dual $\widehat{G}$ of the group G. This section primarily follows the exposition from Section 6 of [3], but also answers Question 6.17 from that paper.

Recall that the *representation ring* $R(G)$ is the commutative $\mathbb{Z}$-algebra with basis $\chi_0, ..., \chi_\ell$, the set of irreducible characters of G. The additive structure is free and the multiplicative structure reflects pointwise equality of functions $G \to \mathbb{C}$:

$$\chi_i \chi_j = \sum_{k=0}^{\ell} c_k \chi_k$$

where the two sides agree as functions. The *degree function* $\deg : R(G) \to \mathbb{Z}$ is the $\mathbb{Z}$-algebra morphism defined by

$$\chi_i \mapsto \chi_i(e) = \dim \chi_i$$

Definition 6. Let γ be a faithful complex n-dimensional representation of G, define the quotient ring

$$R(\gamma) := R(G)/(n - \chi_\gamma)$$

Since $(n - \chi_\gamma) \subset \ker(\deg)$, the degree function descends to an algebra morphism $\deg : R(\gamma) \to \mathbb{Z}$. Define an ideal $I(\gamma)$ within $R(\gamma)$ by

$$I(\gamma) := \ker(\deg : R(\gamma) \to \mathbb{Z})$$

Proposition 9 ([3], Proposition 5.20)**.** *Let γ be a faithful complex representation of G, then there are additive group isomorphisms*

$$R(\gamma) \cong \operatorname{coker}(\widetilde{C}) \qquad (\cong \mathbb{Z} \oplus K(\gamma))$$
$$I(\gamma) \cong \operatorname{coker}(C) \qquad (\cong K(\gamma))$$

which therefore give $\mathbb{Z} \oplus K(\gamma)$ the structure of a $\mathbb{Z}$-algebra and $K(\gamma)$ the structure of a ring-without-unit (rng).

5.1. Critical groups for special linear representations.

Recall that the *Pontryagin dual* $\widehat{G}$ of the group G is the group of homomorphisms $G \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, such a homomorphism factors through the abelianization G^{ab} by the universal property of abelianization. Thus $\widehat{G} \cong \widehat{G^{ab}}$; for finite groups G this implies that $\widehat{G}$ is isomorphic to G^{ab}.

Definition 7. Let γ be a complex representation of G. Define an element $\det_\gamma \in \widehat{G}$ by $\det_\gamma = \det(\gamma(-))$.

The following theorem relates $K(\gamma)$ and $\widehat{G}$ in the case where γ is special-linear, that is, when $\operatorname{im}(\gamma) \subset SL_n(\mathbb{C})$.

Theorem 7 ([3], Theorem 6.2)**.** *For a faithful representation $\gamma : G \hookrightarrow SL_n(\mathbb{C})$ of a finite group G, the homomorphism*

$$\mathbb{Z}^{\ell+1} \xrightarrow{\pi} \widehat{G}$$
$$\chi_i \mapsto \det_{\chi_i}$$

induces a surjective homomorphism of abelian groups $K(\gamma) \twoheadrightarrow \widehat{G}$.

Corollary 4. *Let $\gamma : G \hookrightarrow SL_n(\mathbb{C})$ be a faithful representation of a finite group G and $e = c_0, c_1, ..., c_\ell$ be a set of conjugacy class representatives. Then*

$$\left| \prod_{i=1}^{\ell} (n - \chi_\gamma(c_i)) \right| \geq |G| \cdot |G^{ab}|$$

with equality if and only if $K(\gamma) \cong \widehat{G}$.

Proof of Corollary. By Theorem 1 (i), the left hand side is equal to $|G| \cdot |K(\gamma)|$. Since there is a surjection $K(\gamma) \twoheadrightarrow \widehat{G} \cong G^{ab}$, the corollary follows. □

Question 1. [3, Question 6.11] *Which finite groups G have a faithful representation $\gamma : G \hookrightarrow SL_n(\mathbb{C})$ such that the surjection π from Theorem 7 is an isomorphism $K(\gamma) \cong \widehat{G}$?*

Theorem 8 (Part (i) is Theorem 6.13 of [3])**.** *The following finite groups and faithful representations $\gamma : G \hookrightarrow SL_n(\mathbb{C})$ give isomorphisms $K(\gamma) \cong \widehat{G}$:*

i. *Finite subgroups G of $SL_2(\mathbb{C})$ with the natural representation.*
ii. *The following finite subgroups of $SO_3(\mathbb{R})$ with the natural representation, and no others:*
   - *Cyclic groups $G = \langle g | g^m = 1 \rangle$ where g acts on $\mathbb{R}^3$ by fixing the x_3-axis and rotating the x_1, x_2-plane through $2\pi/m$.*
   - *Dihedral groups $I_2(n)$ for n odd.*
   - *The alternating group $\mathfrak{A}_4$ acting as the group of rotational symmetries of a regular tetrahedron.*
   - *The alternating group $\mathfrak{A}_5$ acting as the group of rotational symmetries of a regular dodecahedron or icosahedron.*

Proof. Appendices B and C contain explicit calculations of $K(\gamma)$ when γ is the natural representation of one of these groups. The $SL_2(\mathbb{C})$ case is proved uniformly in Theorem 6.13 of [3] by identifying $K(\gamma)$ as the fundamental group of a finite root system and relating $|\widehat{G}|$ to the *index of connection* for the root system. □

Remark. Benkart, Klivans, and Reiner show in Corollary 6.10 of [3] that whenever $K(\gamma) \cong \widehat{G}$ as groups, the rng structure on $K(\gamma)$ is trivial. This fact is pointed out in the explicit calculations of $K(\gamma)$ in Appendices B and C below. Also notice that part (ii) includes all finite subgroups of $SO(3, \mathbb{R})$ except $I_2(n)$ for n even and $\mathfrak{S}_4$.

Our main tool for showing that $K(\gamma)$ is *not* isomorphic to $\widehat{G}$ for a given choice of γ and G is via Theorem 1. If repeated character values are known to exist, then Part (ii) of Theorem 1 can be used to show that $K(\gamma)$ has certain subgroups. If G^{ab} does not have an isomorphic subgroup, then $K(\gamma) \ncong \widehat{G}$.

Example 5. Let γ be the permutation representation of $\mathfrak{A}_n$ for $n \geq 5$. Since the permutations in $\mathfrak{A}_n$ are even, $\mathrm{im}(\gamma) \subset SL_n(\mathbb{C})$. By Example 3, $K(\gamma)$ has a subgroup isomorphic to $\mathbb{Z}/\lambda\mathbb{Z}$ for some $\lambda \geq 2$. However $\mathfrak{A}_n^{ab} = 0$, so $K(\gamma) \ncong \widehat{\mathfrak{A}_n}$.

Question 2. *Are the pairs (G, γ) listed in Theorem 8 the only pairs of a finite group G and a faithful representation $\gamma : G \hookrightarrow SL_n(\mathbb{C})$ such that $K(\gamma) \cong \widehat{G}$, up to equivalence of G-representations and up to adding or removing copies of the trivial representation?*

Remark. In [3, Proposition 6.19], Benkart, Klivans, and Reiner include a proof due to S. Koplewitz that if G is abelian and γ contains no copies of the trivial representation then $K(\gamma) \cong \widehat{G}$ if and only if $G \cong \mathbb{Z}/m\mathbb{Z}$ with $G \subset SL_2(\mathbb{C})$ as in type $\widetilde{A}_{m-1}$ of the McKay correspondence (see Appendix B.1).

5.2. **Superstable configurations.** We now recall and prove the last main result, which answers Question 6.17 in [3] in the affirmative:

Theorem 3. *Let $\gamma : G \hookrightarrow SL(n, \mathbb{C})$ be a faithful representation of G with McKay-Cartan matrix C. Then the basis vectors corresponding to the non-trivial one-dimensional characters χ_i of G are superstable configurations.*

Proof. Let e_i be the standard basis vector corresponding to a one-dimensional character χ_i. Suppose $e_i - C^t z \in \mathbb{N}^\ell$ for some $z \in N^\ell$ with $z \neq \mathbf{0}$. Let $y = C^t z$; we have inequalities $y_j \leq 0$ for $j \neq i$ and $y_i \leq 1$. We also have that

$$\sum_{k=1}^{\ell} y_k \cdot \dim(\chi_k) < 0$$

To see this, consider the possibilities giving $\sum_{k=1}^{\ell} y_k \cdot \dim(\chi_k) \geq 0$. They are

(1) $y = \mathbf{0}$,
(2) $y_i = 1$ and $y_j = 0$ for $j \neq i$, or
(3) $y_i = 1$, $y_{i'} = -1$ for some i' with $\dim(\chi_{i'}) = 1$ and $y_j = 0$ for $j \neq i, i'$.

Each of these is impossible since

(1) C^t is invertible and $z \neq \mathbf{0}$,
(2) $e_i \notin \mathrm{im}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell)$, and
(3) $e_i - e_{i'} \notin \mathrm{im}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell)$ since otherwise e_i would be equal to $e_{i'}$ in $K(\gamma)$, contradicting the fact that the map $K(\gamma) \twoheadrightarrow \widehat{G}$ from Theorem 7 is well-defined.

Now, $y = C^t z = (nI - (M')^t)z$, where M' is the submatrix of M with 0-th row and column removed, so for $j = 1, ..., \ell$:

$$\begin{aligned} y_j &= nz_j - \sum_{k=1}^{\ell} m_{kj} z_k = nz_j - \sum_{k=1}^{\ell} z_k \cdot \langle \chi_\gamma \chi_k, \chi_j \rangle \\ &= nz_j - \sum_{k=1}^{\ell} \langle \chi_\gamma \chi_k^{\oplus z_k}, \chi_j \rangle \\ &= nz_j - \langle \chi_\gamma \otimes \bigoplus_{k=1}^{\ell} \chi_k^{\oplus z_k}, \chi_j \rangle \end{aligned}$$

Let $\psi = \chi_\gamma \otimes \bigoplus_{k=1}^{\ell} \chi_k^{\oplus z_k}$ and multiply y_j by $\dim(\chi_j)$ and sum over all j to get

$$0 > \sum_{j=1}^{\ell} y_j \cdot \dim(\chi_j) = n \sum_{j=1}^{\ell} z_j \dim(\chi_j) - \sum_{j=1}^{\ell} \langle \chi_j, \psi \rangle \dim(\chi_j)$$

But this is a contradiction since

$$n \sum_{j=1}^{\ell} z_j \dim(\chi_j) = \dim(\psi)$$

but

$$\sum_{j=1}^{\ell} \langle \chi_j, \psi \rangle \dim(\chi_j) \leq \sum_{j=0}^{\ell} \langle \chi_j, \psi \rangle \dim(\chi_j) = \dim(\psi)$$

Therefore no such z exists and e_i is superstable. □

Remark. Given $\gamma : G \hookrightarrow SL(n, \mathbb{C})$, Theorem 3, along with the fact that $\mathbf{0}$ is always superstable, identifies $|\widehat{G}|$ of the superstable configurations for C. Since the superstable configurations form a set of coset representatives for $K(\gamma)$, Question 1 reduces to determining for which pairs (G, γ) these are *all* of the superstables.

Appendix A. The McKay correspondence

It is possible to nicely classify the finite subgroups of $SU(2)$ and their irreducible representations. Using the standard double cover $SU(2) \to SO(3,\mathbb{R})$ we can also obtain a classification of the finite subgroups of $SO(3,\mathbb{R})$.

Definition 8. Let G be a finite group with irreducible complex characters $\chi_0, ..., \chi_\ell$ and let χ_γ be the character of a faithful not-necessarily-irreducible representation of G. The *McKay graph* of G with respect to γ has vertex set $\{\chi_0, ..., \chi_\ell\}$ and weighted directed edges $\chi_i \xrightarrow{m_{ij}} \chi_j$ if $\chi_i \cdot \chi_\gamma = \sum_{k=0}^{\ell} m_{ij}\chi_j$.

The classification of the finite subgroups of $SU(2)$ makes use of their McKay graphs with respect to the natural faithful representation (inclusion of a finite subgroup G into $SU(2)$). This classification, often called the *McKay correspondence* asserts that these graphs are exactly the *affine Dynkin diagrams*. A proof can be found, for example, in [13]. The resulting classification is given in the table below.

Type	$G \subset SU(2)$	$K(\gamma) \cong \widehat{G}$	Affine diagram labeled by $\delta^{(e)}$
$\widetilde{A}_{n-1}$	$\langle g \mid g^n = 1\rangle$	$\mathbb{Z}/n\mathbb{Z}$	1 1 — 1 — ··· — 1 — 1
$\widetilde{D}_{n+2}$	$\langle r,s,t \mid r^2 = s^2 = t^n = rst\rangle$	$\begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } n \text{ even} \end{cases}$	1 1 2 — 2 - ··· - 2 — 2 1 1
$\widetilde{E}_6$	$\langle r,s,t \mid r^2 = s^3 = t^3 = rst\rangle$	$\mathbb{Z}/3\mathbb{Z}$	1 2 1 — 2 — 3 — 2 — 1
$\widetilde{E}_7$	$\langle r,s,t \mid r^2 = s^3 = t^4 = rst\rangle$	$\mathbb{Z}/2\mathbb{Z}$	2 1 — 2 — 3 — 4 — 3 — 2 — 1
$\widetilde{E}_8$	$\langle r,s,t \mid r^2 = s^3 = t^5 = rst\rangle$	0	3 1 − 2 − 3 − 4 − 5 − 6 − 4 − 2

Recall from the basic theory of Lie groups that there is a standard double cover $SU(2) \twoheadrightarrow SO(3,\mathbb{R})$. Therefore any finite subgroup of $SO(3,\mathbb{R})$ may be pulled back to a finite subgroup of $SU(2)$ of twice the order. The classification above therefore allows us to classify all finite subgroups of $SO(3,\mathbb{R})$:

Type	$G \subset SO(3,\mathbb{R})$	$\widehat{G}$
$\widetilde{A}_{n-1}$	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$
$\widetilde{D}_{n+2}$	Dihedral group $I_2(n)$ of order $2n$	$\begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } n \text{ is even} \end{cases}$
$\widetilde{E}_6$	$\mathfrak{A}_4$	$\mathbb{Z}/3\mathbb{Z}$
$\widetilde{E}_7$	$\mathfrak{S}_4$	$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{E}_8$	$\mathfrak{A}_5$	0

Appendix B. Calculation of $K(\gamma)$ for finite subgroups of $SU(2)$

All two-dimensional faithful representations $G \hookrightarrow SU(2)$ are irreducible when G is of type $\widetilde{D}_{n+2}$ or $\widetilde{E}_6, \widetilde{E}_7$ or $\widetilde{E}_8$, therefore in these cases the following proposition will be useful:

Proposition 10. *Let G be a finite group, and $G \to GL(V)$ an irreducible representation of G with real-valued character. Then $V \otimes V$ contains exactly one copy of the trivial representation.*

Proof. By assumption $\chi_V(g)$ is real for all $g \in G$. Since V is irreducible, we have

$$1 = \langle \chi_V, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_V(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_V^2(g) = \langle \chi_0, \chi_V^2 \rangle$$

Thus $V \otimes V$ contains exactly one copy of the trivial representation. □

Corollary 5. *If G is finite and $G \to SU(V)$ is a two-dimensional irreducible representation, then $V \otimes V$ contains exactly one copy of the trivial representation.*

Proof. All elements of $SU(2)$ have real trace, so the Proposition applies. □

B.1. **Type $\widetilde{A}_{n-1}$.** The group $G = \mathbb{Z}/n\mathbb{Z} = \langle g | g^n = e \rangle$ corresponds to the affine Dynkin diagram of type $\widetilde{A}_{n-1}$. Take the faithful representation γ given by

$$g \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{n-1} \end{pmatrix}$$

where $\zeta = e^{2\pi i/n}$ is an n-th root of unity. Since G is abelian, it is easy to see that χ_i sending $g \mapsto \zeta^i$ for $i = 0, ..., n-1$ form a complete set of irreducible representations. Thus $R(G)$ is generated by χ_1 as a $\mathbb{Z}$-algebra. If we let x denote χ_1, we have:

$$R(G) \cong \mathbb{Z}[x]/(x^n - 1)$$

Now, clearly $\chi_\gamma = \chi_1 + \chi_{n-1} = x + x^{n-1}$ and so

$$\begin{aligned}\mathbb{Z} \oplus K(\gamma) &\cong \mathbb{Z}[x]/(x^n - 1, x^{n-1} + x - 2) \\ &\cong \mathbb{Z}[x]/(x^{n-1} + x - 2, x^2 - 2x + 1) \\ &\cong \mathbb{Z}[x]/(x^2 - 2x + 1, n(x-1)) \\ &\cong \mathbb{Z}[u]/(u^2, nu)\end{aligned}$$

where $u = x - 1$. Thus

$$K(\gamma) \cong \mathbb{Z}/n\mathbb{Z} = G^{ab}$$

and $K(\gamma)$ has trivial rng structure since $u^2 = 0$.

B.2. **Type $\widetilde{D}_{n+2}$.** The group BD_{4n} of order $4n$ corresponds to the affine Dynkin diagram of type $\widetilde{D}_{n+2}$. This group is given by

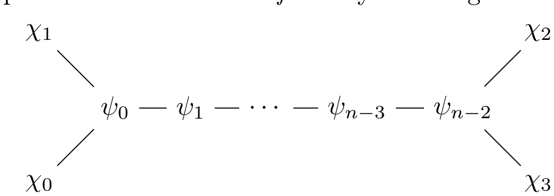
$$BD_{4n} = \langle g, h | g^{2n} = h^4 = 1, g^n = h^2, hgh^{-1} = g^{-1} \rangle$$

and for $j = 1, ..., n-1$ has 2-dimensional irreducible representations V_j defined by

$$V_j(g) = \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix} \qquad\qquad V_j(h) = \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix}$$

where $\zeta = e^{\pi i/n}$ is a $2n$-th root of unity. It is easy to check that these representations are pairwise non-isomorphic and irreducible. We know BD_{4n} has four 1-dimensional characters, so the fact that $2^2(n-1) + 1^2 \cdot 4 = 4n$ shows that the V_j are all of the higher dimensional irreducibles. Some of the V_j are faithful.

Throughout the calculations below, $\chi_0, \chi_1, \chi_2, \chi_3$ are the one dimensional irreducible characters of the group of type $\widetilde{D}_{n+2}$ and $\psi_0, ..., \psi_{n-2}$ are the two dimensional irreducibles; they will be used to refer to the irreducible representations and their characters interchangeably. Let χ_0 always refer to the trivial representation, and fix a faithful irreducible special-unitary representation ψ_0 with respect to which we construct McKay graphs. The labels of the other representations will be *defined* by this diagram:



The fact that ψ_0 is adjacent to χ_0 follows from Proposition 10.

B.2.1. *Case 1: n is odd.* In this case we know that $\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{Z}/4\mathbb{Z}$.

Proposition 11. *The character χ_1 has order two in Hom$(G, \mathbb{C}^\times) \cong \mathbb{Z}/4\mathbb{Z}$.*

Proof. We have $\psi_0 \otimes \psi_0 = \chi_0 \oplus \chi_1 \oplus \psi_1$. Notice from above that all of the 2-dimensional irreducibles have real trace. This forces χ_1 to be real-valued. In particular it must have order 1 or 2 in $\mathrm{Hom}(G, \mathbb{C}^\times)$. Since it is not equal to χ_0, it must have order 2. □

Proposition 12. *When n is odd, $R(G)$ is generated by ψ_0 and χ_2 as a $\mathbb{Z}$-algebra.*

Proof. We need to check that $S := \mathbb{Z}[\psi_0, \chi_2]$ contains each of the other irreducible characters. Since χ_2 generates $\mathrm{Hom}(G, \mathbb{C}^\times)$ we have $\chi_0, \chi_1, \chi_2, \chi_3 \in S$. We will now show that S contains each of the ψ_i by induction:

Suppose by induction that S contains $\psi_0, \psi_1, ..., \psi_{i-1}$ for some $2 \leq i \leq n-2$. Then S contains $\psi_0 \cdot \psi_{i-1} = \psi_{i-2} + \psi_i$. Since we know by induction that $\psi_{i-2} \in S$, so is ψ_i. For the base case, note that $\psi_0^2 - \chi_0 - \chi_1 = \psi_1$, so $\psi_1 \in S$. □

Now we will determine the relations satisfied by the generators $x = \chi_2$ and $y = \psi_0$ of $R(G)$. Since χ_2 generates $\mathrm{Hom}(G, \mathbb{C}^\times)$, we have $x^4 = 1$. Next notice that

$$\{1, x, x^2, x^3, y, y^2, ..., y^{n-1}\}$$

is linearly independent. This is because for each $i = 1, ..., n-1$, y^i is the smallest power of y to contain ψ_{i-1}, and each power of x is one of the characters χ_j. Therefore the linear independence of these powers follows from the linear independence of irreducible characters. These elements span $R(G)$, so we need to find relations expressing xy and y^n as a linear combination of these elements.

From the McKay quiver, we can read off the relation

$$xy = \psi_{n-2}$$

Thus we need to express ψ_{n-2} as a linear combination of powers of x and y. Notice that for $1 \leq i \leq n-3$, we have $y\psi_i = \psi_{i-1} + \psi_{i+1}$. Thus we get a recurrence

$$\psi_{i+1} = y\psi_i - \psi_{i-1} \tag{2}$$

with the initial conditions $\psi_0 = y$ and $\psi_1 = y^2 - (1 + x^2)$. Solving this as a linear recurrence yields

$$\psi_i = \left(\frac{y + \sqrt{y^2 - 2(1+x^2)}}{2}\right)^{i+1} + \left(\frac{y - \sqrt{y^2 - 2(1+x^2)}}{2}\right)^{i+1} \tag{3}$$

Which can be proven by induction. The right hand side of (3) is always polynomial, and it can be expressed in the basis $\{1, x, x^2, x^3, y, ..., y^{n-1}\}$ since $x^4 = 1$ and $x^2 y = y$. Call this resulting polynomial $a_i(x, y)$. Therefore we have the relation

$$xy = a_{n-2}(x, y)$$

Finally, we have the relation

$$ya_{n-2}(x, y) = a_{n-3}(x, y) + x + x^3$$

coming from the right side of the McKay quiver. Thus

$$R(G) \cong \mathbb{Z}[x, y]/(x^4 - 1, xy - a_{n-2}(x, y), ya_{n-2}(x, y) - a_{n-3}(x, y) - x - x^3)$$

Now we want to compute the critical group $K(\psi_0)$. To do this notice the effect of setting $y = 2$ on the last two relations.

$$a_i(x, 2) := \begin{cases} 2 & \text{if } i \text{ is even} \\ 3 - x^2 & \text{if } i \equiv 1 \mod 4 \\ 1 + x^2 & \text{if } i \equiv 3 \mod 4 \end{cases}$$

Which can also be seen by induction. When n is odd, we have

$$\mathbb{Z} \oplus K(\psi_0) \cong \begin{cases} \mathbb{Z}[x]/(x^4-1, x^2-2x+1) & \text{or} \\ \mathbb{Z}[x]/(x^4-1, x^2+2x-3, x^3+2x^2+x-4) \end{cases}$$

making the substitution $u = x-1$ in either case gives

$$\mathbb{Z} \oplus K(\psi_0) \cong \mathbb{Z}[u]/(4u, u^2)$$

so $K(\psi_0) \cong \mathbb{Z}/4\mathbb{Z}$ with trivial rng structure.

B.2.2. *Case 2: n is even.* In this case we know that $\mathrm{Hom}(G, \mathbb{C}^\times) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Proposition 13. *When n is even, $R(G)$ is generated by ψ_0, χ_1, and χ_2 as a $\mathbb{Z}$-algebra.*

Proof. We now require two of the three non-trivial one-dimensional characters in order to generate the others. The rest of the argument is the same is in the previous case. □

Let $x = \chi_2, y = \psi_0, z = \chi_1$. Similar to the previous case, the set $\{1, x, z, xz, y, ..., y^{n-1}\}$ spans $R(G)$. Since $\mathrm{Hom}(G, \mathbb{C}^\times) \cong (\mathbb{Z}/2\mathbb{Z})^2$ in this case, we know $x^2 = z^2 = 1$. We still need relations expressing xy, yz and y^n in terms of this basis. From the McKay quiver we can read off:

$$yz = y$$
$$xy = \psi_{n-2}$$
$$y\psi_{n-2} = \psi_{n-3} + z + xz$$

The formulas expressing ψ_i in terms of x, y from above still hold, so

$$R(G) \cong \mathbb{Z}[x,y,z]/(x^2-1, z^2-1, yz-y, xy-a_{n-2}(x,y), ya_{n-2}(x,y)-a_{n-3}(x,y)-z-xz)$$

Letting $y = 2$ this gives

$$\mathbb{Z} \oplus K(\psi_0) \cong \mathbb{Z}[x,z]/(x^2-1, z^2-1, 2x-2, 2z-2, x+z+xz-3)$$

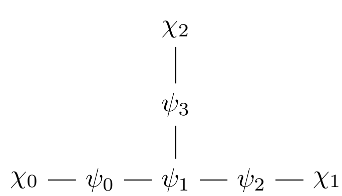
The substitutions $u = x-1$ and $v = y-1$ give

$$\mathbb{Z} \oplus K(\psi_0) \cong \mathbb{Z}[u,v]/(u^2, v^2, 2u, 2v, uv)$$

and so $K(\psi_0) \cong (\mathbb{Z}/2\mathbb{Z})^2$ with trivial rng structure.

B.3. **Types $\widetilde{E}_6, \widetilde{E}_6$ and $\widetilde{E}_8$.** In each of the three types below, fix a faithful two-dimensional special unitary representation ψ_0 (which is necessarily irreducible). The McKay graph with respect to ψ_0 is shown. The fact that ψ_0 and χ_0 are adjacent follows from Proposition 10, the labels of the other representations are defined by this diagram (up to conjugation).

B.3.1. *Type $\widetilde{E}_6$.*



Proposition 14. *$R(G)$ is generated by $y = \psi_0$ and $x = \chi_1$ as a $\mathbb{Z}$-algebra.*

Proof. We can easily read the following from the fact that $\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{Z}/3\mathbb{Z}$ and the McKay graph:

$$\begin{aligned}\chi_2 &= x^2\\ \psi_1 &= y^2 - 1\\ \psi_2 &= xy\\ \psi_3 &= x^2 y\end{aligned}$$

□

Thus $R(G)$ has a basis $\{1, x, y, xy, x^2, y^2, x^2y\}$ so we need relations expressing y^3 and xy^2 in this basis (we know $x^3 = 1$). From the McKay graph:

$$\begin{aligned}y^3 &= y(\chi_0 + \psi_1) = 2y + \psi_2 + \psi_3 = 2y + xy + x^2y\\ xy^2 &= y(\psi_2) = \psi_1 + \chi_1 = y^2 + x - 1\end{aligned}$$

Thus

$$R(G) \cong \mathbb{Z}[x, y]/(x^3 - 1, y^3 - x^2y - xy - 2y, xy^2 - y^2 - x + 1)$$

And so

$$\begin{aligned}\mathbb{Z} \oplus K(\psi_0) &\cong \mathbb{Z}[x]/(x^3 - 1, 2x^2 + 2x - 4, 3x - 3)\\ &\cong \mathbb{Z}[u]/(3u, u^2)\end{aligned}$$

where $u = x - 1$. Thus $K(\psi_0) \cong \mathbb{Z}/3\mathbb{Z}$ with trivial rng structure.

B.3.2. *Type* $\widetilde{E}_7$.

$$\begin{array}{ccccccccccccc} & & & & & & \psi_5 & & & & & & \\ & & & & & & | & & & & & & \\ \chi_0 & — & \psi_0 & — & \psi_1 & — & \psi_2 & — & \psi_3 & — & \psi_4 & — & \chi_1\end{array}$$

Proposition 15. *$R(G)$ is generated by $y = \psi_0$ and $x = \chi_1$ as a $\mathbb{Z}$-algebra.*

Proof. We can read these off from the McKay graph:

$$\begin{aligned}\psi_1 &= y^2 - 1\\ \psi_2 &= y\psi_1 - \psi_1 = y^3 - 2y\\ \psi_3 &= xy^2 - x\\ \psi_4 &= xy\\ \psi_5 &= y\psi_2 - \psi_1 - \psi_3 = y^4 - 3y^2 - xy^2 + x + 1\end{aligned}$$

□

Thus $R(G)$ has a basis $\{1, x, y, xy, y^2, y^3, xy^2, y^4\}$, so we need relations expressing y^5 in this basis (we clearly have $x^2 = 1$). We have

$$\begin{aligned} y^5 &= y^3(1+\psi_1) = y^2(2y+\psi_2) \\ &= y(2+3\psi_1+\psi_3+\psi_5) \\ &= 5y+5\psi_2+\psi_4 = 5y+5(y^3-2y)+xy \\ &= 5y^3+xy-5y \end{aligned}$$

therefore

$$R(G) \cong \mathbb{Z}[x,y]/(x^2-1, y^5-5y^3-xy+5y)$$

and so

$$\begin{aligned} \mathbb{Z} \oplus K(\psi_0) &\cong \mathbb{Z}[x]/(x^2-1, 2x-2) \\ &\cong \mathbb{Z}[u]/(2u, u^2) \end{aligned}$$

so $K(\psi_0) \cong \mathbb{Z}/2\mathbb{Z}$ with trivial rng structure.

B.3.3. *Type* $\widetilde{E}_8$.

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & \psi_7 & & & & \\ & & & & & & & & & & | & & & & \\ \chi_0 & — & \psi_0 & — & \psi_1 & — & \psi_2 & — & \psi_3 & — & \psi_4 & — & \psi_5 & — & \psi_6 \end{array}$$

We know by Theorem 6.13 in [3] that $K(\psi_0) \cong G^{ab} = 0$. Therefore its rng structure is trivial.

Appendix C. Calculation of $K(\gamma)$ for finite subgroups of $SO(3,\mathbb{R})$

C.1. **Type** $\widetilde{A}_{n-1}$**.** The cyclic group $\mathbb{Z}/n\mathbb{Z} = \langle g | g^n = 1\rangle$ is faithfully represented inside $SO(3,\mathbb{R})$ with g fixing the x_3-axis and rotating the x_1, x_2-plane by $2\pi/n$. After removing the copy of the trivial representation, this representation is isomorphic to the representation of $\mathbb{Z}/n\mathbb{Z}$ inside $SU(2)$. Since adding a copy of the trivial representation does not affect $K(\gamma)$, the calculation from Appendix B shows that $K(\gamma) \cong \mathbb{Z}/n\mathbb{Z} \cong \widehat{G}$.

C.2. **Type** $\widetilde{D}_{n+2}$**.** Let $I_2(n)$ denote the dihedral group of order $2n$. We have a presentation

$$I_2(n) = \langle g, h | g^m = h^2 = (hg)^2 = e\rangle$$

The character table of $I_2(n)$ depends on the parity of n:

$G = I_2(n)$, n odd	e	g	h
χ_0	1	1	1
χ_h	1	1	-1
ψ_k $(k = 1, 2, ..., \frac{n-1}{2})$	2	$\zeta_n^k + \zeta_n^{-k}$	0

$G = I_2(n)$, n even	e	g	h
χ_0	1	1	1
χ_g	1	-1	1
χ_h	1	1	-1
$\chi_g \cdot \chi_h$	1	-1	-1
ψ_k $(k = 1, 2, ..., \frac{n-2}{2})$	2	$\zeta_n^k + \zeta_n^{-k}$	0

where $\zeta_n = e^{2\pi i/n}$ is an n-th root of unity.

Let x denote χ_h, y denote χ_g and z_k denote ψ_k. The following relations hold for both n even and n odd:

$$\begin{aligned} x^2 &= 1 \\ xz_k &= z_k \\ z_k z_\ell &= z_{k+\ell} + z_{k-\ell}, \text{if } k+\ell \leq \frac{n-1}{2} \text{ and where } z_0 = 1+x \end{aligned}$$

This last relation, with $\ell = 1$ gives $z_{k+1} = z_k z_1 - z_{k-1}$. Induction shows that x, z_1 generate the rest of the z_k. Thus $R(G)$ is generated by x, z_1 for n odd and x, y, z_1 for n even. We have the following additional relations depending on the parity of n:

$$\begin{aligned} y^2 &= 1, \text{if } n \text{ is even} \\ yz_k &= z_{\frac{n}{2}-k}, \text{if } n \text{ is even} \\ z_1 z_{(n-2)/2} &= y + xy, \text{if } n \text{ is even} \\ z_1 z_{\frac{n-1}{2}} &= z_{\frac{n-1}{2}} + z_{\frac{n-3}{2}}, \text{if } n \text{ is odd} \end{aligned}$$

Now, the faithful representation $\gamma : G \hookrightarrow SO(3, \mathbb{R})$ has character $\chi_\gamma = x + z_1$, so $R(\gamma) = R(G)/(3 - (x + z_1))$.

C.2.1. *n is odd.* The relation $z_1 z_k = z_{k+1} + z_{k-1}$ shows that $\{1, x, z_1, z_1^2, ..., z_1^{(n-1)/2}\}$ forms a basis for $R(G)$. The argument essentially the same as in the type $\widetilde{D}_{n+2}$ case for subgroups of $SU(2)$. We already have the relations $x^2 = 1$ and $xz_1 = z_1$, therefore the only remaining relation expresses $z_1^{(n+1)/2}$ in terms of this basis.

Let $a_i(x, z_1)$ denote the expression for z_i in the above basis. The last remaining relation in $R(G)$ is then

$$z_1 a_{(n-1)/2}(x, z_1) = a_{(n-1)/2}(x, z_1) + a_{(n-3)/2}(x, z_1)$$

Thus

$$R(G) \cong \mathbb{Z}[x, z_1]/(x^2 - 1, xz_1 - z_1, z_1 a_{(n-1)/2}(x, z_1) - a_{(n-1)/2}(x, z_1) - a_{(n-3)/2}(x, z_1))$$

Now, we can express the relations for $R(\gamma)$ in terms of only x by substituting $z_1 = 3 - x$. In particular this gives $x(3 - x) = 3 - x$ and so $4(x - 1) = 0$ in $R(\gamma)$. We now need to compute $a_i(x, 3 - x)$ in order to determine the last relation. We have $a_0(x, z_1) = 1 + x$ and $a_1(x, z_1) = z_1$ and will use the relations $4(x - 1) = 0$ and $x^2 = 1$ to reduce at each step:

$$\begin{aligned} a_0(x, 3-x) &= 1 + x \\ a_1(x, 3-x) &= 3 - x \\ a_2(x, 3-x) &= (3-x)a_1(x, 3-x) - a_0(x, 3-x) = 3x - x^2 = 3x - 1 = 3 - x \\ a_3(x, 3-x) &= (3-x)a_2(x, 3-x) - a_1(x, 3-x) = 9 - 7x = 1 + x \end{aligned}$$

Thus by induction we see that in general

$$a_i(x, 3-x) = \begin{cases} 1 + x & \text{if } i \text{ is even} \\ 3 - x & \text{if } i \text{ is odd} \end{cases}$$

Therefore in $R(\gamma)$ the last relation becomes

$$\begin{cases} (3-x)^2 = (3-x) + (1+x) & \text{if } (n-1)/2 \text{ is odd} \\ (3-x)(1+x) = (3-x) + (1+x) & \text{if } (n-1)/2 \text{ is even} \end{cases}$$

Both of these relations reduce to $2(x-1) = 0$. Thus

$$R(\gamma) \cong \mathbb{Z}[x]/(x^2-1, 4(x-1), 2(x-1)) = \mathbb{Z}[x]/(x^2-1, 2(x-1))$$

and letting $u = x-1$ gives

$$R(\gamma) \cong \mathbb{Z}[u]/(u^2, 2u)$$

so $K(\gamma) \cong \mathbb{Z}/2\mathbb{Z} = \widehat{G}$ with trivial rng structure.

C.2.2. *n is even.* Similarly to the previous case, we see that $\{1, x, y, xy, z_1, z_1^2, ..., z_1^{(n-2)/2}\}$ is a basis for $R(G)$ and that the relations $x^2 = 1, xz_1 = z_1$, and $y^2 = 1$ hold. The remaining relations are $y + xy + a_{(n-4)/2}(x, z_1) = z_1 a_{n/2}(x, z_1)$, $yz_1 = a_{(n-2)/2}(x, z_1)$, and $ya_2(x, z_1) = a_{(n-4)/2}(x, z_1)$. Thus $R(G)$ is

$$\frac{\mathbb{Z}[x, y, z_1]}{(x^2-1, y^2-1, xz_1 - z_1, y + xy + a_{(n-4)/2} - z_1 a_{n/2}, yz_1 - a_{(n-2)/2}, ya_2 - a_{(n-4)/2})}$$

When we now impose the relation $3 - (x + z_1) = 0$, this allows us to substitute $z_1 = 3 - x$ in each relation. As before, $xz_1 - z_1 = 0$ becomes $4(x-1) = 0$. We also get

$$(3-x)y = \begin{cases} 3-x & \text{if } (n-2)/2 \text{ is odd} \\ 1+x & \text{if } (n-2)/2 \text{ is even} \end{cases}$$

$$(1+x)y = \begin{cases} 1+x & \text{if } (n-2)/2 \text{ is even} \\ 3-x & \text{if } (n-2)/2 \text{ is odd} \end{cases}$$

Adding these gives $4y = 4$, regardless of the parity of $(n-2)/2$. Finally, the relation $y + xy = z_1 a_{n/2}(x, z_1)$ becomes

$$y + xy = \begin{cases} 1+x & \text{if } (n-2)/2 \text{ is odd} \\ 3-x & \text{if } (n-2)/2 \text{ is even} \end{cases}$$

Let $u = x - 1$ and $v = y - 1$ we get

$$R(\gamma) \cong \begin{cases} \mathbb{Z}[u, v]/(4u, 4v, u^2 + 2u, v^2 + 2v, uv + 2v) & \text{if } (n-2)/2 \text{ is odd} \\ \mathbb{Z}[u, v]/(4u, 4v, u^2 + 2u, v^2 + 2v, uv + 2u + 2v) & \text{if } (n-2)/2 \text{ is even} \end{cases}$$

Thus in either case $I(\gamma) = (\mathbb{Z}/4\mathbb{Z})u + (\mathbb{Z}/4\mathbb{Z})v$ so

$$K(\gamma) \cong (\mathbb{Z}/4\mathbb{Z})^2 \ncong (\mathbb{Z}/2\mathbb{Z})^2 = \widehat{G}$$

Notice that the rng structure of $I(\gamma)$ is nontrivial.

C.3. **Types $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$.**

C.3.1. *Type* $\widetilde{E}_6$. The alternating group $\mathfrak{A}_4$ has a faithful representation γ inside $SO(3,\mathbb{R})$; this is the permutation representation with the copy of the trivial representation removed. The character table for $G = \mathfrak{A}_4$ is

	e	(123)	(132)	(12)(34)
χ_0	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_γ	3	0	0	-1

Clearly $R(G)$ is generated by χ_1 and χ_γ as a $\mathbb{Z}$-algebra. We can see from the character table that

$$\chi_1^3 = 1$$
$$\chi_1\chi_\gamma = \chi_\gamma$$
$$\chi_\gamma^2 = 2\chi_\gamma + \chi_1^2 + \chi_1 + 1$$

Therefore if we let $x = \chi_1$ and $y = \chi_\gamma$ we have

$$R(G) \cong \mathbb{Z}[x,y]/(x^3-1, xy-y, y^2-2y-x^2-x-1)$$

Now, modding out by the ideal $(y-3)$ gives

$$R(\gamma) \cong \mathbb{Z}[x]/(x^3-1, 3(x-1), x^2+x-2) \cong \mathbb{Z}[x]/(3(x-1),(x-1)^2)$$

Letting $u = x-1$ we have

$$R(\gamma) \cong \mathbb{Z}[u]/(3u, u^2)$$

and so $K(\gamma) \cong \mathbb{Z}/3\mathbb{Z} = \widehat{G}$ with trivial rng structure.

C.3.2. *Type* $\widetilde{E}_7$. The symmetric group $\mathfrak{S}_4$ has a faithful representation γ inside $SO(3,\mathbb{R})$; this is the permutation representation with the copy of the trivial representation removed. The character table for $G = \mathfrak{S}_4$ is

	e	(12)	(123)	(1234)	(12)(34)
χ_0	1	1	1	1	1
χ_1	1	-1	1	-1	1
χ_γ	3	1	0	-1	-1
χ_3	3	-1	0	1	-1
χ_4	2	0	-1	0	2

It is easy to check that $R(G)$ is generated by $x = \chi_1$ and $y = \chi_\gamma$ as a $\mathbb{Z}$-algebra. We have relations

$$x^2 = 1$$
$$xy^2 = y^2 + x - 1y^3 \qquad\qquad = 2y^2 + xy + 2y + x - 1$$

Modding out by the ideal $(y-3)$ gives

$$R(\gamma) \cong \mathbb{Z}[x]/(x^2-1, 4(x-1))$$

Letting $u = x-1$ gives

$$R(\gamma) \cong \mathbb{Z}[u]/(u^2+2u, 4u)$$

Thus $K(\gamma) \cong \mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} = \widehat{G}$. The rng $K(\gamma)$ has nontrivial structure since $u^2 \neq 0$.

C.3.3. *Type* $\widetilde{E}_8$. The alternating group $\mathfrak{A}_5$ is isomorphic to the group of rotational symmetries of a regular dodecahedron or icosahedron, giving a faithful representation γ of $G = \mathfrak{A}_5$ into $SO(3, \mathbb{R})$. The character table for $G = \mathfrak{A}_5$ is

	e	(123)	(12)(34)	(12345)	(13452)
χ_0	1	1	1	1	1
χ_1	4	1	0	-1	-1
χ_2	5	-1	1	0	0
χ_γ	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_4	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

By decomposing χ_γ^j into irreducibles for $j = 2, 3, 4$ one can see that $R(G)$ is generated by χ_γ as a $\mathbb{Z}$-algebra. Therefore, letting $x = \chi_\gamma$, $R(G)$ is a quotient of $\mathbb{Z}[x]$, and $R(\gamma)$ is a quotient of $\mathbb{Z}[x]/(x-3) \cong \mathbb{Z}$, so it must in fact be equal. Thus $K(\gamma) = 0 = \widehat{G}$, with trivial rng structure.

References

[1] M. Baker and F. Shokrieh, Chip-firing games, potential theory on graphs, and spanning trees. *J. Combin. Theory* Ser. A **120** (2013), no. 1, 164-182.

[2] C. Ballantine and R. Orellana, A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_\lambda$. *Sm. Lothar. Combin.* 54A (2005/07).

[3] G. Benkart, C. Klivans, and V. Reiner, Chip firing on Dynkin diagrams and McKay quivers. arXiv:1601.06849 [math.CO], 2016.

[4] N. Biggs, Algebraic graph theory, 2nd edition. Cambridge University Press, Cambridge, 1993.

[5] N. Biggs, Chip-firing and the critical group of a graph, *J. Algebraic Combin.* **9** (1999)(1), 25-45.

[6] T.W.X. Cai and R. Stanley, The Smith normal form of a matrix associated with Young's lattice. *Proc. Amer. Math. Soc.* **143** (2015), no.11.

[7] L. Levine and J. Propp, What is a sandpile? *Notices Amer. Math. Soc.* **57** (2010), 976-979.

[8] D. Lorenzini, Smith normal form and Laplacians. *J. of Combinatorial Theory*, Series B 98 (2008), no. 6, 1271-1300.

[9] A. Miller and V. Reiner, Differential posets and Smith normal forms. *Order* **26** (2009), no. 3, 197-228.

[10] Z. Nie, On nice extensions and Smith normal form, in preparation.

[11] A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals. *Trans. Amer. Math.* **356** (2004), no.8, 3109-3142.

[12] R. Stanley, Smith normal form in combinatorics. arXiv:1602.00166v2 [math.CO], 2016.

[13] R. Steinberg, Finite subgroups of SU_2, Dynkin diagrams and affine Coxeter elements. *Pacific J. Math.* **118** (1985), no. 2, 587-598.