On the Hurwitz Action in Finite Complex Reflection Groups

Maria Gilbert

Submitted under the supervision of Dr. Victor Reiner to the University Honors Program at the University of Minnesota-Twin Cities in partial fulfillment of the requirements for the degree of Bachelor of Science, magna cum laude in Mathematics.

May 11, 2018

1 Abstract

In the 2016 paper *Circuits and Hurwitz action in finite root systems* by Joel Brewster Lewis and Victor Reiner, a conjecture is made that "In a well-generated finite complex reflection group, two factorizations of a Coxeter element into reflections lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes". Here a SageMath/CoCalc program is used to demonstrate several cases of this conjecture by performing Hurwitz moves to determine the size of the Hurwitz orbit for the groups G4, G8, and G20. A separate section of code is used to calculate the number of reflection factorizations with a generating function from Chapuy and Stump. These numbers are compared, to test the conjecture.

2 Introduction

In a complex vector space V of dimension n, a reflection is a linear transformation that, when represented as a matrix, has all but one eigenvalue equal to 1 [2].

Definition 1. A *complex reflection group* **G** is a finite group generated by reflections on V. **G** of rank *n* is considered *well-generated* if it is generated by a subset of *n* reflections.

Definition 2. A finite *Coxeter group* is a group generated by reflections on a real vector space C. A *Coxeter element* **c** in G is an element of order $h = \frac{|\mathcal{R}|+|\mathcal{R}^*|}{n}$ where $\mathcal{R}$ is the set of all reflections in G and $\mathcal{R}^*$ is the set of all reflecting hyperplanes.

Definition 3. A *reflecting hyperplane* is the kernel of the transformation matrix of a reflection, i.e., what ends up in the same place after the reflection.

Definition 4. The *Hurwitz move* σ_i acts on a factorization $c = t_1...t_l$ by mapping $(t_1, ...t_i, t_{i+1}, ...t_l)$ to $(t_1, ..., t_{i+1}, t_{i+1}^{-1}t_it_{i+1}, ..., t_l)$. A *Hurwitz orbit* is a subset of reflection factorizations such that beginning with any one factorization, all the others in the orbit can be found by doing one or more

Hurwitz moves on the others.

In this paper I will discuss a SageMath program written by myself and another student, Xuan Liu, that tests the following conjecture [3]:

Conjecture 6.3 *In a well-generated finite complex reflection group, two factorizations of a Coxeter element into reflections lie in the same Hurwitz orbit if and only if they share the same multiset of conjugacy classes.*

3 Background

In [3], Conjecture 3 is proven for finite real reflection groups. In [1], Chapuy and Stump give the generating function for the number of Coxeter element reflection factorizations of each length in any irreducible well-generated complex reflection group.
Chapuy and Stump's main theorem reads as follows:

Let W be an irreducible, well-generated complex reflection group of rank n. Let c be a Coxeter element in W, let $\mathcal{R}$ be the set of all reflections in W, and let $\mathcal{R}^*$ be the set of all reflecting hyperplanes. Define

$$FAC_W(t) := \sum_{l \geq 0} \frac{t^l}{l!} \#\{(\tau_1, \tau_2, ...\tau_l) \in \mathcal{R}^l, \tau_1\tau_2...\tau_l = c\}$$

to be the exponential generating function of factorizations of c into a product of reflections. Then $FAC_W(t)$ is given by the formula

$$FAC_W(t) = \frac{1}{|W|}(e^{t|\mathcal{R}|/n} - e^{-t|\mathcal{R}^*|/n})^n.$$

[1]
The SageMath program presented in this paper calculates sizes of Hurwitz orbits.

For example, with the S_3 symmetric group:

$$S_3 = \{e, s, t, st, ts, sts\}$$

where *e* is the identity, and *s* and *t* are the reflections represented in the following notations and equivalent linear transformation matrices:

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$t = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Simple matrix multiplication can be used to find the remaining three elements of the group:

$$st = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$ts = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$sts = tst = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

From this full group, one Coxeter element with the property $s^2 = t^2 = e$ is

$$c = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = st$$

One factorization of c is (s, t). Starting with this, Hurwitz moves can be done repeatedly until no more new factorizations are found, at which point the total number of factorizations is the size of the Hurwitz orbit.

Two Hurwitz actions can be done on a factorization of this length:

$$(1) \qquad (s, t) \rightarrow (t, t^{-1}st) = (t, tst)$$

$$(2) \qquad (s, t) \rightarrow (s^{-1}ts, s) = (sts, s)$$

Because $s^2 = t^2 = e$, the equalities $s^{-1} = s$ and $t^{-1} = t$ hold. Next, Hurwitz moves can be used on each of these new factorizations.

$$(1) \qquad (t, tst) \to (tst, tstttst) = (tst, s) = (sts, s)$$

$$(2) \qquad (t, tst) \to (ttstt, t) = (s, t)$$

$$(1) \qquad (sts, s) \to (s, sstss) = (s, t)$$

$$(2) \qquad (sts, s) \to (tst, stsssts) = (sts, t)$$

Since all of the actions that can be done on the factorizations have been exhausted, every factorization in the Hurwitz orbit is here. There are three: $(s, t), (t, tst)$, and (sts, s).

Definition 5. A *Hurwitz map* is a graphical representation of reflection factorizations transformed by Hurwitz moves. A factorization is a vertex with edges connecting two factorizations if doing one Hurwitz move on one of them creates the other.

In this example, since there are only three factorizations, the Hurwitz map will look like a triangle. For a more interesting example, one can look at Figure 4 from Muhle and Ripoli [4].

In this figure, the reflections can be represented as:

$$t_1 = (12)$$

$$t_2 = (24)$$

$$t_3 = (23)$$

$$t_1^{-1}t_2t_1 = t_2^{-1}t_1t_2 = (14)$$

$$t_1^{-1}t_3t_1 = t_3^{-1}t_1t_3 = (13)$$

$$t_3^{-1}t_2t_3 = t_2^{-1}t_3t_2 = (34)$$

The figure shows the Hurwitz map of factorizations:

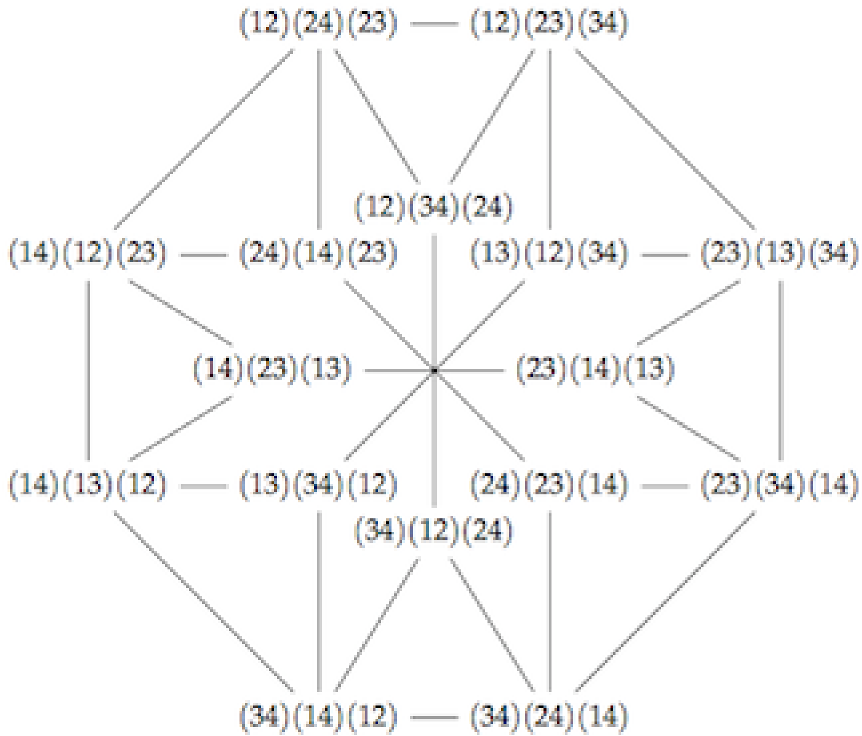

FIGURE 4. The Hurwitz graph of the long cycle $c = (1\ 2\ 3\ 4)$ in the symmetric group $\mathfrak{S}_4$ generated by its transpositions.

In this case, the map shows a Hurwitz orbit of 16 factorizations, each of length 3.

In the original exmaple in S_3, the size of the Hurwitz orbit can be compared to [1]. In symmetric groups such as this one, $|\mathcal{R}| = |\mathcal{R}^*| = \binom{n}{2}$. So in this example,

$$|\mathcal{R}| = |\mathcal{R}^*| = \binom{3}{2} = 3$$

.

So using the formula,

$$FAC_W(t) = \frac{1}{|W|}(e^{t|\mathcal{R}|/n} - e^{-t|\mathcal{R}^*|/n})^n = \frac{1}{3!}(e^{3t/2} - e^{-3t/2})^2$$

expands to

$$\frac{3t^2}{2!} + \frac{27t^4}{4!} + \frac{207t^6}{6!} + ...$$

so there are 3 factorizations of length 3, 27 of length 4, and 207 of length 6. Further terms of the series would tell the number of factorizations of longer lengths.

As shown, there are also 3 factorizations in the Hurwitz orbit of a Coxeter element in S_3, which is evidence of the conjecture. Of course, since S_3 is a real reflection group, the theorem that applies has already been proven. The SageMath program is designed to test the conjecture for complex groups.

4 Program and Results

Here the Sage program that tests Reiner and Lewis' conjecture for complex reflection groups will be explained.

The following code performs the Hurwitz action, switching two tuples and conjugating one tuple.

```
def hurwitz(t,k):
    tp = list(t)
    tp[k-1] = tp[k]
    tp[k] = ~t[k]*t[k-1]*t[k]
    return tuple(tp)
```

To find the Hurwitz orbit, two group of factorizations are created: one called "old" to keep track of those already found, and another called "new" to keep track of all factorizations, including the ones that have just resulted from Hurwitz moves. When all possible Hurwitz moves are done to all existing factorizations but no new factorizations result, the size of "old" will be the same as the size of "new", and the loop terminates, as the Hurwitz orbit is full.

```
def hurwitz_orbit_size(t):
    old = set()
    new = set([t])
    while len(old) < len(new):
        old = copy(new)
        for item in old:
            for k in range(1,len(t)):
                a = hurwitz(item,k)
                new.add(a)
    return(len(new))
```

For example, here is the code that was run on the group G_4:

First, the series from Chapuy and Stump's equation is calculated:

```
W=ReflectionGroup(8)
N=W.number_of_reflections();
Nstar=W.number_of_reflection_hyperplanes();
g=1/len(W)*(exp(N*x/2)-exp(-Nstar*x/2))^2;
g.taylor(x,0,10)
1793583/175*x^10 + 39852/7*x^9 + 19926/7*x^8 + 6318/5*x^7 + 2457/5*x^6
+ 162*x^5 + 45*x^4 + 9*x^3 + 3/2*x^2
```

The last four terms of the series simplify to:

$$\frac{19440}{5!}x^5 + \frac{1080}{4!}x^4 + \frac{54}{3!}x^3 + \frac{3}{2!}x^2$$

To demonstrate the conjecture, the Hurwitz orbit of length 2 will need to have 3 factorizations, that of length 3 will need to have 54 factorizations, that of length 4 will need to have 1080 factorizations, and that of length 5 will need to have 19440 factorizations.

First, the code is used to find the Hurwitz orbit for reflection factorizations of length 2 and 3:

```
s=list(W.simple_reflections())[0]
t=list(W.simple_reflections())[1]
fac2 = tuple([s,t])
hurwitz_orbit_size(fac2)
3
```

```
fac3=tuple([s^3,s^2,t])
hurwitz_orbit_size(fac3)
54
```

While each of these only have one Hurwitz orbit, for length 4 and length 5, each has four Hurwitz orbits. The sum of the sizes of each of their Hurwitz orbits should equal the number from the Chapuy and Stump series, if Reiner and Lewis' conjecture is true.

```
fac4a=tuple([s^3,s^2,t^3,t^2])
fac4b=tuple([s^2,s^2,s,t])
fac4c=tuple([s^3,s^3,s^3,t])
fac4d=tuple([s,t,t,t^3])
a=hurwitz_orbit_size(fac4a);a
b=hurwitz_orbit_size(fac4b);b
c=hurwitz_orbit_size(fac4c);c
d=hurwitz_orbit_size(fac4d);d
324
324
108
324
```

```
fac5a=tuple([s^2,s,s,s,t])
fac5b=tuple([s^3,s,s,t^2,t^3])
fac5c=tuple([s^2,s^3,t^3,t^3,t^3])
fac5d=tuple([s^2,s^2,s^2,s^3,t])
a=hurwitz_orbit_size(fac5a);a
b=hurwitz_orbit_size(fac5b);b
c=hurwitz_orbit_size(fac5c);c
d=hurwitz_orbit_size(fac5d);d
1620
9720
1620
6480
```

For length 3, $324 + 324 + 108 + 324 = 1080$, and for length 4, $1620 + 9720 + 1620 + 6480 = 19440$. For each of lengths 2, 3, 4, and 5, the sum of the factorizations is the same as the number from the series. Therefore, Reiner and Lewis' conjecture has been demonstrated to work for these lengths in this group.

The following table shows the sizes of the Hurwitz orbits of certain lengths in certain groups, as found using the code.

Data Collected from Code					
Group	Length 2 Hurwitz Orbit Totals	Length 3 Hurwitz Orbit Total	Length 4 Hurwitz Orbit Totals	Length 5 Hurwitz Orbit Totals	Length 6 Hurwitz Orbit Totals
G4	3	18	180	1320	11088
G5	4	48	960	14080	
G8	3	54	1080	19440	353808
G20	5	150	7500		

References

[1] Guillaume Chapuy and Christian Stump. “Counting Factorizations of Coxeter Elements into Products of Relfections”. In: *Journal of the London Mathematical Society* 90.3 (2014), pp. 919–939.

[2] Arjeh M. Cohen. “Finite Complex Reflection Groups”. In: *Annales scientifiques de lÉcole normale supérieure* 9.3 (1976), pp. 379–436.

[3] Joel Brewster Lewis and Victor Reiner. “Circuits and Hurwitz actions in finite root systems”. In: *New York Journal of Mathematics* 22 (2016), pp. 1457–1468.

[4] Henri Muhle and Vivien Ripoll. “Connectivity Properties of Factorization Posets in Generated Groups”. In: (2017).