The Minnesota Journal of Undergraduate Mathematics

Double covers of flower graphs and Tutte Polynomials

Kaustubh Verma

University of Minnesota

The Minnesota Journal of Undergraduate Mathematics

Volume 6 (2020-2021 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 6 (2020-2021 Academic Year)

Double covers of flower graphs and Tutte Polynomials

Kaustubh Verma

University of Minnesota

1. Introduction

For a graph G, its *Tutte polynomial* $T_G(x,y)$ is a two-variable polynomial with nonnegative coefficients, and an isomorphism invariant of G, having a remarkable number of interesting combinatorial interpretations via specialization of the variables x,y. For example, $T_G(x+1,1)$ gives a generating function that counts spanning forests in G according to their number of components, and in particular, $T_G(1,1)$ is the number of spanning trees of G; see, e.g., Brylawski and Oxley [2] and Bollobas [1, Page 345].

When one has a *covering map* $\pi : \tilde{G} \to G$, it is known that the spanning tree number $T_G(1,1)$ divides $T_{\tilde{G}}(1,1)$. Furthermore, when π is a 2-*fold* or *double* covering parametrized by a *voltage assignment* G_β on G (as explained in Section 2 below),one has an interpretation for the ratio via work of Chepuri et al. [6, Cor. 4.5]

$$\frac{T_{\tilde{G}}(1,1)}{T_G(1,1)} = \sum_{Z \in NVF(G_\beta)} 2^{\#(\text{minus cycles of } Z)-1}. \tag{1.1}$$

Here $NVF(G_\beta)$ is the set of *negative vector fields* for G_β, which are certain orientations of a subset of the edges of G defined in Chepuri et al. [6, §4], and reviewed in Section 3.1 below.

In general $T_G(x,y)$ does not divide $T_{\tilde{G}}(x,y)$. However, one can ask if other specializations besides $x = y = 1$ lead to divisibility. The following example is suggestive.

Example 1.1. Let C_b denote a cycle with b edges. We have that

$$T_{C_b}(x,y) = x^{b-1} + x^{b-2} + \ldots + x + y$$

There is a double covering $\pi : C_{2b} \to C_b$ that "wraps" C_{2b} twice around C_b. Note that here $T_{C_b}(x,y)$ does not divide $T_{C_{2b}}(x,y)$. However, specializing $y = 1$, one has

$$T_{C_b}(x,1) = x^{b-1} + x^{b-2} + \ldots + x + 1$$

and then one does have a divisibility $T_{C_b}(x,1)$ into $T_{C_{2b}}(x,1)$:

$$\frac{T_{C_{2b}}(x,1)}{T_{C_b}(x,1)} = \frac{x^{2b-1} + x^{2b-2} + \ldots + x + 1}{x^{b-1} + x^{b-2} + \ldots + x + 1} = x^b + 1.$$

The above example is an instance of our main result, Theorem 1.2 below, about double coverings of *flower graphs*. We show that for each flower graph F, and connected double covering graph $\tilde{F} \to F$ parametrized by a voltage assignment F_β, that every negative vector field Z for F_β has only one minus cycle (see Proposition 3.8 below). Hence (1.1) simplifies to the following:

$$\frac{T_{\tilde{F}}(1,1)}{T_F(1,1)} = \#NVF(F_\beta).$$

Motivated by this, for each such voltage assignment F_β on a flower F, we will define (Definition 3.9 below) a certain *base negative vector field* Z_0 in $NVF(F_\beta)$, and for any other negative vector field Z in $NVF(F_\beta)$, we will define a statistic comm(Z) which measures how far away Z is from Z_0. It is because of this that comm() is an abbreviation of common since in some sense we want to know how common Z is with Z_0. We will then prove the following main result.

Theorem 1.2. *Let F be a flower graph, and $\tilde{F} \to F$ a connected double covering graph corresponding to a voltage assignment F_β. Then*

$$\frac{T_{\tilde{F}}(x,1)}{T_F(x,1)} = \sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)}$$

One might wonder if arbitrary double coverings $\tilde{G} \to G$ have $T_G(x,1)$ dividing $T_{\tilde{G}}(x,1)$, but small examples of non-flower graphs show that this can fail.

Example 1.3. For the following voltage assignment G_β and double cover $\tilde{G} \to G$ in Figure 1

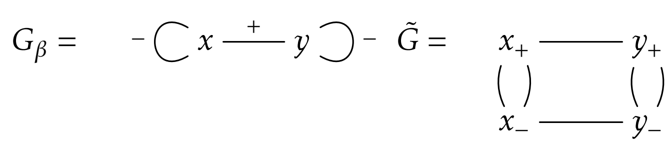


Figure 1. Non-Example

one calculates that $T_G(x,1) = x$ does not divide $T_{\tilde{G}}(x,1) = x^3 + 3x^2 + 3x + 5$.

This raises the following question:

> Do double coverings of flower graphs lie in some natural larger class of double covering graphs $\tilde{G} \to G$ for which $T_G(x,1)$ divides $T_{\tilde{G}}(x,1)$?

It would also be interesting to understand a closer connection between Theorem 1.2 and the results of Chepuri et al. [6], which we currently find mysterious.

The rest of this paper is structured as follows. Section 2 explains some background on graph coverings and discusses double coverings of flowers. Section 3 recalls the notation of negative vector fields. Section 4 gives the definition of Tutte polynomials and some facts about them which will be useful in later sections. Section 5 gives motivating examples of some simple graphs for which 1.2 holds. Section 6 outlines the strategy for proving Theorem 1.2. Section 7 proves parallel recurrences that reduce the main theorem from general double covers of flowers to those that have no *plus cycles* in their voltage assignment. Sections 8 then proves parallel recurrences that complete the proof by induction on the number of minus cycles in the flower.

2. Double Coverings of Flowers

In this section we will define the notion of flower graph. We will also introduce facts about the voltage assignments on flower graphs which will let us more easily characterize double coverings of flower graphs later in the paper.

2.1. Flowers.

This paper primarily deals with the following kind of graph.

Definition 2.1. An *n-petal flower graph F* will be defined as follows. Let C_a denote a cycle with a edges. Let $C_{a_1}, C_{a_2}, \ldots, C_{a_n}$ be cycles with vertex sets $V_1, V_2, \ldots, V_n$ respectively such that the vertex sets are pairwise disjoint from each other. Then we pick vertices $v_1 \in V_1, v_2 \in V_2, \ldots, v_n \in V_n$ and identify them all with a single vertex d which we will refer to as the *stem* vertex of the flower F. Thus we have "glued" cycles $C_{a_1}, C_{a_2}, \ldots, C_{a_n}$, at a single vertex stem vertex d, into a connected graph F, for which we will use the following notation:

$$F = \bigoplus_{1 \le i \le n} C_{a_i},$$

Example 2.2. In Figure 2 we have a 4-petal flower F with cycles of sizes $2, 3, 4$ and 4 (up to isomorphism).

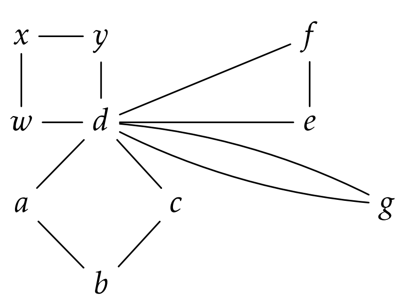


Figure 2. Four Petal Flower

2.2. **Double coverings and voltage assignments.** We will assume prior knowledge of the definition of graph coverings as is presented in [5, Page 274]. We will however list some of the important results cited in [5] and important implications which those results have for graph coverings.

Definition 2.3. A *double covering* $\pi : \tilde{G} \to G$ of a graph $G = (V(G), E(G))$ has

- for each vertex v in $V(G)$, two vertices $\{v_+, v_-\} = \pi^{-1}(v)$ in $V(\tilde{G})$, and
- for each edge e in $E(G)$ with endpoints v, w in G, either $\pi^{-1}(e)$ in $E(\tilde{G})$ is
  - a pair of *plus edges* with endpoints v_+, w_+ and v_-, w_-, or
  - a pair of *minus edges* with endpoints v_+, w_- and v_-, w_+.

The corresponding *voltage assignment* is the map $\beta : E(G) \to \{\pm 1\}$ indicating whether $\pi^{-1}(e)$ are plus edges (if $\beta(e) = +1$) or minus edges (if $\beta(e) = -1$).

Example 2.4. Consider the following flower F with voltage assignment β:

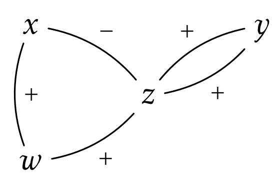


Figure 3. Two Petal Flower

Then the following is the resulting double covering graph $\tilde{F}$:

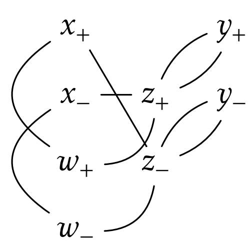


Figure 4. Double Cover of the Above Two Petal Flower

When considering double covering graphs of flower graphs, it will be convenient to put them in a standard form up to isomorphism, using the notion of *vertex-switching*.

Definition 2.5. Consider a double covering graph $\tilde{G} \to G$ resulting from voltage assignment G_γ, and a vertex $v \in G$. Now for every edge incident to vertex $v \in G$ if the edge is a plus edge under γ change it to a minus edge and if the edge is a minus edge under γ then change it to a plus edge. Say that this new voltage assignment $G_{\gamma'}$ is obtained from G_γ by *vertex-switching at v*.

Example 2.6. On the left is a voltage assignment G_β, and on the right is the result of switching at the vertex x:

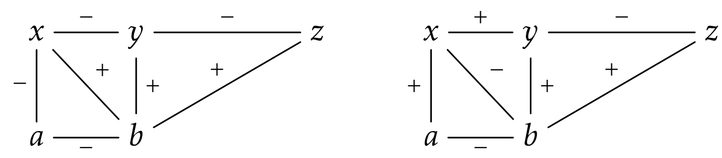


FIGURE 5. Vertex Switch Example

Proposition 2.7. [7, Page 404] If $G_{\gamma'}$ is obtained from G_γ by a vertex-switch, then the resulting double covering graphs $\tilde{G}' \to G$ and $\tilde{G} \to G$ are isomorphic.

The next proposition is well-known, but we include a proof for self-containment.

Proposition 2.8. For a graph G and spanning tree T, every voltage assignment G_γ is equivalent via vertex-switchings to one with all + on the edges of T.

Proof. Start at any given vertex v in G_γ. Let N_v denote the set of neighboring vertices of v in T. Then for any $w \in N_v$ such that the edge vw is $(-)$ in G_γ we apply a vertex change to w. Note that for any $w, z \in N_v$, wz cannot be an edge in T because otherwise v, w and z would form a cycle in T. Apply the aforementioned logic inductively to each layer of vertices until all edges in T have a (+) assignment. □

2.3. **Double Coverings for Flowers.** In this section we will derive a way to obtain all possible double coverings of a flower F, and a convenient way to label the vertices and edges of a flower F.

Note that for a cycle C_a, its spanning trees are all graphs obtained by removing a single edge. Similarly for an n-petal flower graph

$$F = \bigoplus_{1 \le i \le n} C_{a_i} \tag{2.1}$$

with cycles $C_{a_1}, \ldots, C_{a_n}$ sharing the stem vertex d, a spanning tree consists of a choice of spanning trees $S_1, S_2, \ldots, S_n$ within $C_{a_1}, \ldots, C_{a_n}$. Fix such a choice of a spanning tree for F, and let ρ_i be the unique edge of $C_{a_i} - S_i$ for $i = 1, 2, \ldots, n$.

Propositions 2.7 and 2.8 then immediately imply the following.

Proposition 2.9. All double coverings of the flower F in (2.1) up to isomorphism can be obtained from a voltage assignment F_β that assigns each ρ_i a (+) or (-), and all of the other edges in (+).

Furthermore, in this situation C_{a_i} is a (+) or (-) cycle of F_β depending upon whether ρ_i is assigned (+) or (-).

We now describe a systematic labeling of the vertices and the edges of F.

Definition 2.10. Let d be the stem vertex of F. For $1 \le i \le n$ we label the vertices of cycle C_{a_i} sequentially as $v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,a_i}$ in a clockwise direction with $v_{i,1} = d$. For $1 \le i \le n$ in cycle C_{a_i} we label the edge $v_{i,r}v_{i,r+1}$ as $e_{i,r}$ for $1 \le r \le a_i$ where we set $v_{i,a_i+1} = v_{i,1}$.

Example 2.11. Here is an example of labeling in accordance with definition 2.10:

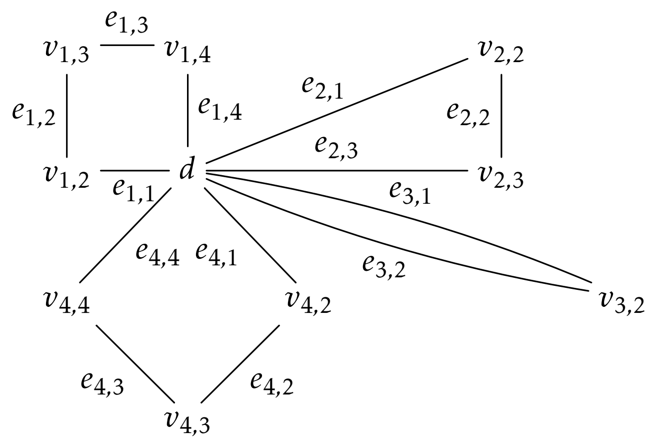


Figure 6. Labeling of Flower in Figure 2

3. Negative Vector Fields for Flowers

3.1. **Definitions.** Now we will define negative vector fields of a given graph G with a $\pm$ voltage assignment G_β.

Definition 3.1. For edge xy in G the orientation $\overrightarrow{xy}$ of edge xy will contribute 1 to the out-degree of vertex x and contribute 0 to the out-degree of vertex y. Note we regard $\overrightarrow{yx}$ and $\overleftarrow{xy}$ as the same orientations.

Definition 3.2. For any cycle in G if that cycle has an odd number of edges with a $(-)$ assignment in G_β then we will call that cycle a *minus cycle* of G_β. If a cycle of G has an even number of edges with a $(+)$ assignment in G_β then we will call it a *plus cycle* of G_β.

Example 3.3. Consider the following signed graphs from examples 5.2 and 2.4:

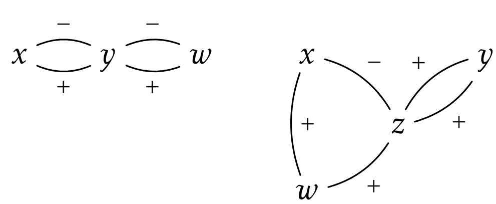


Figure 7. Examples of Plus and Minus Cycles

In the first flower the petal with vertex set $\{x, y\}$ is a minus cycle. In the second flower the petal with vertex set $\{z, y\}$ is a plus cycle.

Definition 3.4. (see Chepuri et al. [6, Defn. 4.4]) A *negative vector field* Z for G_β is a directed graph on the same vertex V set as $G = (V, E)$ with these properties:

(i) For each undirected edge $\{x, y\}$ in E, out of its two directed arcs $x \to y$ and $y \to x$, at most one (and possibly neither) will appear in Z.
(ii) Every vertex x in V has an out-degree of 1 in Z.
(iii) Directed cycles in Z are supported on *minus cycles* in G_β.

Combining Chepuri et al.'s results [6, Cor. 1.5, Cor. 4.5] gives the formula (1.1) quoted in the Introduction.

Example 3.5. We present in Figure 8 one negative vector field Z for the flower F with a given voltage assignment F_β from Example 2.2 on the right in Figure 8.

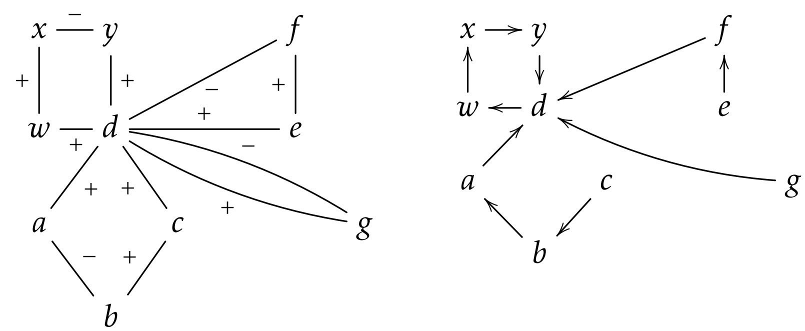


FIGURE 8. Negative Vector Field Example

In Figure 9 are two non-examples of negative vector fields of flower F under the same voltage assignment as in Figure 8.

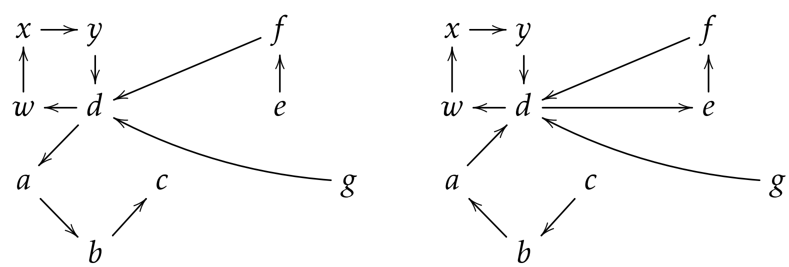


FIGURE 9. Negative Vector Field Non-Example

Both examples in Figure 9 do not qualify because in both d has an out-degree of two. Further in the left negative vector field c has an out-degree of 0 which is another disqualifying factor.

Example 3.6. Consider the voltage graph G_β and one of its vector fields Z_G in Figure 10.

Then we see from the definition 3.4 that although Z_G is a valid vector field of G_β, it is not a valid negative vector field of G_β since the loop on x in Z_G is a plus cycle in G_β and not a minus cycle. Hence Figure 10 is another non-example of negative vector fields.

Figure 10. Vector Field Non-Example

3.2. **Characterization.** In this section we will first characterize the negative vector fields of F_β. Then we will define our choice of base negative vector field Z_0, and the statistic $\mathrm{comm}(Z)$ for $Z \in NVF(F_\beta)$.

Definition 3.7. Given an edge $e_{i,r}$ with endpoints $\{v_{i,r} v_{i,r+1}\}$ and $Z \in NVF(F_\beta)$, say that Z directs $e_{i,r}$

- *clockwise* if it is directed $\overrightarrow{v_{i,r} v_{i,r+1}}$, and
- *anti-clockwise* if it is directed $\overleftarrow{v_{i,r} v_{i,r+1}}$.

Proposition 3.8. Every negative vector field Z in $NVF(F_\beta)$ has this form:

- There is a unique minus cycle in the flower ,which we denote as C_Z, that Z directs clockwise or counterclockwise.
- In each of the other remaining cycles $C_{a_j} \neq C_Z$, there is a unique edge that Z omits and does not direct, which we will call $\chi_{Z,j}$.
- The remaining vertices in each $C_{a_j} \neq C_Z$ have a unique directed path in Z to the stem vertex d that does not pass through the omitted edge $\chi_{Z,j}$.

In particular, the undirected graph for Z is a spanning unicyclic subgraph of F.

Proof. In Definition 3.4, use property (i) to encode Z as a function $V \xrightarrow{f} V$ defined by $f(x) = y$ if the unique directed arc in Z out of x is $x \to y$.

To see the first property in the proposition, let $C_Z = C_{a_i}$ be the unique cycle of F that contains the directed arc $d \to f(d)$. By iterating f, and using property (ii) in Definition 3.4, one concludes that the vertices

$$d, f(d), f^2(d), \ldots, f^{a_i-1}(d), f^{a_i}(d) = d$$

must run through the vertices of C_{a_i}. Thus $C_{a_i}(=: C_Z)$ is directed as a cycle by Z. Property (iii) in Definition 3.4 implies that C_Z must be a minus cycle in F_β.

For any vertex x not on C_Z, say in $C_{a_j} \neq C_Z$, if the directed arc $x \to f(x)$ is clockwise (resp. anti-clockwise), then one similarly sees that iterating $x, f(x), f^2(x), \ldots$ gives a sequence of clockwise (resp. anti-clockwise) edges inside C_{a_j} that eventually reaches the stem vertex d. This already accounts for arcs in Z that cover a spanning tree of edges within each $C_{a_j} \neq C_Z$, and all of the edges in C_Z. The number of such arcs therefore is $|V|$, which is the total number of arcs in Z by property (ii) in Definition 3.4. Hence there are no more arcs in Z, and we have described Z completely. □

Recall that a flower F must have at least one minus cycle in the voltage assignment F_β, in order for $\tilde{F} \to F$ to be connected. We will assume this is the case for the rest of the paper.

Thus without loss of generality we may assume that the first cycle C_{a_1} of F is a minus cycle in F_β.

Definition 3.9. Choose the *base negative vector field* $Z_0 \in NVF(F_\beta)$ as follows

- Choose $C_{Z_0} = C_{a_1}$, having all of its arcs directed clockwise.
- For each $C_{a_i} \neq C_{Z_0}$ choose $\chi_{N,j} = e_{j,1}$, so that $e_{j,2}, e_{j,3}, \ldots, e_{j,a_j}$ are all directed clockwise in Z.

Definition 3.10. Given Z in $NVF(F_\beta)$, define comm(Z) to be the cardinality of the set of directed arcs $x \to y$ in Z satisfying both of these two conditions:

- the exact same arc $x \to y$, with the same direction, appears in Z_0, and
- additionally, if $C_Z \neq C_{Z_0}(= C_{a_1})$, then $x \to y$ is not an arc in C_Z.

Example 3.11. The flower F_β from Example 2.2 is shown here on the left, along with the negative vector field $Z_0 \in NVF(G_\alpha)$ from Example 3.5 is shown in Figure 11:

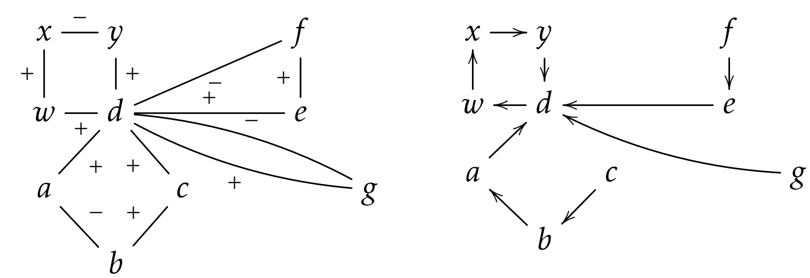


Figure 11. F_β and its Base Negative Vector Field

Now we set Z_0 as the base negative vector field of F_β since it meets the requirements of Definition 3.9. Now consider these two elements Z_1, Z_2 of $NVF(F_\beta)$ in Figure 12:

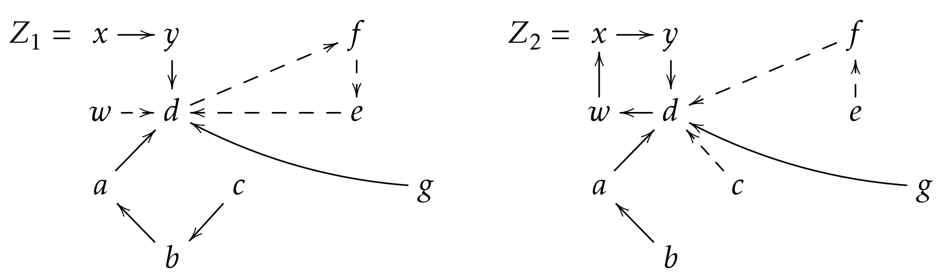


Figure 12. Negative Vector Field comm()

In the above we have put the edges which contribute to comm() as smooth and straight and the edges which don't contribute to comm() are dashed. Hence we have that:

$$\text{comm}(Z_1) = 2 + 1 + 3 = 6,$$

$$\text{comm}(Z_2) = 4 + 0 + 1 + 2 = 7.$$

4. Tutte polynomials

Throughout this paper we will use the following definition of the Tutte polynomial via a deletion and contraction recursion.

Definition 4.1. Define e to be a *bridge* of graph G if the graph $G-e$ has strictly more connected components than G. Define the Tutte polynomial $T_G(x,y)$ for a graph G recursively by $T_G(x,y) = 1$ if G has no edges, and otherwise if e is an edge of G, then

$$T_G(x,y) = \begin{cases} xT_{G-e}(x,y) & \text{if } e \text{ is a bridge} \\ yT_{G-e}(x,y) & \text{if } e \text{ is a loop} \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{if } e \text{ is neither a loop nor a bridge.} \end{cases}$$

Here $G-e$ and G/e denote the graphs obtained from G by deleting edge e and by contracting on edge e, respectively.

By iterating the recursion in Definition 4.1, one can calculate that for a cycle C_a having a edges that:

$$\begin{aligned} T_{C_a}(x,y) &= x^{a-1} + x^{a-2} + \cdots + x + y, \\ T_{C_a}(x,1) &= x^{a-1} + x^{a-2} + \cdots + x + 1 =: [a]_x, \end{aligned}$$

The following multiplicative property of $T_G(x,y)$ (see, e.g., [2, Page 128]) will be useful. Given graphs G_1, G_2 on disjoint vertex sets, along with a choice of a vertex v_i in G_i for $i = 1,2$, their sum $G_1 \oplus G_2$ is the quotient of the disjoint union $G_1 \sqcup G_2$ obtained by identifying the two vertices v_1, v_2 as a single vertex. Although $G_1 \oplus G_2$ depends up to graph isomorphism upon the choices of v_1, v_2, the following result says that its Tutte polynomial does not.

Proposition 4.2. For any graphs G_1 and G_2,

$$T_{G_1 \oplus G_2}(x,y) = T_{G_1}(x,y) \cdot T_{G_2}(x,y).$$

Consequently, since the definition of the n-petal flower $F = \bigoplus_{1 \le i \le n} C_{a_i}$ is consistent with iterating this $G_1 \oplus G_2$ construction, one sees that

$$\begin{aligned} T_F(x,y) &= \prod_{i=1}^{n} T_{C_{a_i}}(x,y), \\ T_F(x,1) &= \prod_{i=1}^{n} T_{C_{a_i}}(x,1) = \prod_{i=1}^{n} [a_i]_x, \end{aligned}$$

5. Motivating Examples

In this section we demonstrate some examples of Theorem 1.2.

Example 5.1. The cycle graph C_b has only two negative vector fields, one which we take to be Z_0 whose arcs all point clockwise, and another Z whose arcs all point counterclockwise. Thus

$$\sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)} = x^{\mathrm{comm}(Z_0)} + x^{\mathrm{comm}(Z)} = x^b + 1$$

in agreement with our previous calculation.

Example 5.2. Consider the following flower graph G with its voltage assignment β shown with the associated double covering graph $\tilde{G}$:

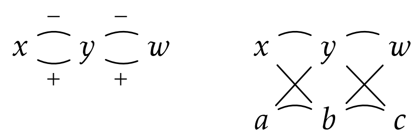


Figure 13. G_β and it Double Covering $\tilde{G}$

Then one can calculate that

$$\begin{aligned} T_G(x,1) &= (x+1)^2, \\ T_{\tilde{G}}(x,1) &= (x+1)^3(x^2+3), \\ \frac{T_{\tilde{G}}(x,1)}{T_G(x,1)} &= (x+1)(x^2+3) = x^3 + x^2 + 3x + 3. \end{aligned}$$

Choosing the following as the base negative vector field Z_0 in $NVF(G_\beta)$

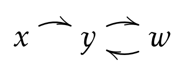


Figure 14. Base Negative Vector Field for G_β

then the elements Z of $NVF(G_\beta)$ and their values comm(Z) are tabulated in Table 1. Note that the right column of table 1 agrees with the prediction of Theorem 1.2 that

$$\sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)} = \frac{T_{\tilde{F}}(x,1)}{T_F(x,1)} = x^3 + x^2 + 3x + 3.$$

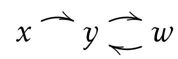
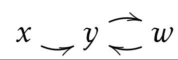
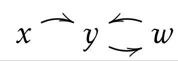
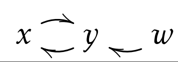
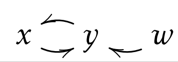
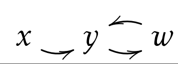
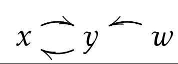
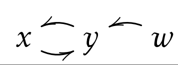
Z	comm(Z)
	3
	2
	1
	1
	1
	0
	0
	0

Table 1. Negative Vector Fields of G_β and Associated *comm*() Values

6. Strategy for Proof

We explain here our strategy for proving Theorem 1.2. Introduce these abbreviations for the left and right sides in the theorem:

- We define $t(F_\beta)$ as
$$t(F_\beta) := \frac{T_{\tilde{F}}(x,1)}{T_F(x,1)}$$

- We define $n(F_\beta)$ as
$$n(F_\beta) := \sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)}$$
.

So Theorem 1.2 asserts
$$t(F_\beta) = \frac{T_{\tilde{F}}(x,1)}{T_F(x,1)} = \sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)} = n(F_\beta)$$
for all voltage assignments F_β of flowers F that lead to connected double covering graphs $\tilde{F} \to F$. The connectedness condition is equivalent in this situation to saying that at least one of the cycles of F is a minus cycle in F_β.

- Section 7 deals with F_β having at least one plus cycle, say the one labeled C_{a_n}. This section prove two recurrences that hold when one obtains $F'_{\beta'}$ from F_β by "plucking" the petal C_{a_n}:

$$t(F_\beta) = [a_n]_x \cdot t(F'_{\beta'}), \tag{6.1}$$
$$n(F_\beta) = [a_n]_x \cdot n(F'_{\beta'}). \tag{6.2}$$

This reduces the proof to the case where F_β has no plus, only minus cycles.

- Section 8 deals with F_β a voltage assignment on a flower $F = \bigoplus_{1\le i\le n} C_{a_i}$ as in (2.1) having *all minus cycles*. This section prove two recurrences that let one "pluck" a minus petal C_{a_n} from such a flower, assuming C_{a_n} does not support the unique directed cycle in the base vector field Z_0:

$$t(F_\beta) = [a_n]_x \cdot t(F'_{\beta'}) + 2 \prod_{1\le i\le n-1} [a_i]_x, \tag{6.3}$$
$$n(F_\beta) = [a_n]_x \cdot n(F'_{\beta'}) + 2 \prod_{1\le i\le n-1} [a_i]_x. \tag{6.4}$$

Using Example 1.1 as the base case $n = 1$, this completes the proof of Theorem 1.2 via induction on n.

7. Removing a Plus Petal

7.1. **Proof of the recurrence** (6.1)**.** Assume that our flower $F = \bigoplus_{i=1}^{n} C_{a_i}$ has voltage assignment F_β with at least one plus cycle, labeled C_{a_n} without loss of generality. Let $F' := \bigoplus_{i=1}^{n-1} C_{a_i}$, so that

$$F = F' \oplus C_{a_n} \tag{7.1}$$

and let $F'_{\beta'}$ be the voltage assignment on F' such that the cycles $C_{a_1}, \ldots, C_{a_{n-1}}$ agree with their assignment in F_β.

Proposition 7.1. With the above definitions of F_β and $F'_{\beta'}$, the associated double coverings $\tilde{F} \to F$ and $\tilde{F}' \to F'$ satisfy

$$\tilde{F} \cong \tilde{F}' \oplus C_{a_n} \oplus C_{a_n}.$$

Proof. By Proposition 2.9, every edge of C_{a_n} is assigned $(+)$. Since (7.1), says that F' and C_{a_n} share only the vertex d within F, one similarly has in $\tilde{F}$ that that are two copies of $\tilde{C}_{a_n}$, one with all vertices of the form x_+ for vertices x in C_{a_n}, the other with all vertices of form x_-, the first copy sharing only the vertex d_+ with $\tilde{F}'$, and the second only sharing vertex d_- with $\tilde{F}'$. □

Example 7.2. Recall Example 2.4, where F_β and $\tilde{F}$ looked as follows. Here the role of C_{a_n} is the plus cycle in F_β on the vertex set $\{d, y\}$.

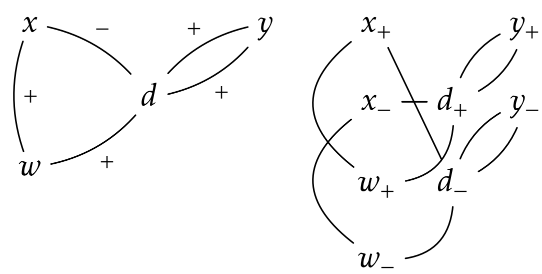


Figure 15. Plus Petal Example

Propositions 7.1 and 4.2 now allow us to derive the first recurrence in (6.1):

$$\begin{aligned} t(F_\beta) = \frac{T_{\tilde{F}}(x,1)}{T_F(x,1)} &= \frac{T_{\tilde{F}'\oplus C_{a_n}\oplus C_{a_n}}(x,1)}{T_{F'\oplus C_{a_n}}(x,1)} \\ &= \frac{T_{C_{a_n}}(x,1)^2 \cdot T_{\tilde{F}'}(x,1)}{T_{C_{a_n}}(x,1)\cdot T_{F'}(x,1)}, \\ &= T_{C_{a_n}}(x,1)\cdot t(F'_{\beta'}) = [a_n]_x \cdot t(F'_{\beta'}). \end{aligned}$$

7.2. **Proof of the recurrence** (6.2). Assume the same conventions on the flowers F, F' and on $F_\beta, F'_{\beta'}$ as in Section 7.1.

Proposition 7.3. Define a function by

$$\begin{aligned} f : NVF(F_\beta) &\longrightarrow NVF(F'_{\beta'}) \times \{0,1,\ldots,a_n-1\} \\ Z &\longmapsto f(Z) := (Z', j) \end{aligned}$$

where $\chi_{Z,n} = e_{n,a_n-j}$ and Z' is the restriction of the arcs of Z to F'. Then f is a bijection.

Furthermore, when $f(Z) = (Z', j)$ one has

$$\mathrm{comm}(Z) = \mathrm{comm}(Z') + j. \tag{7.2}$$

Proof. Proposition 3.8, shows that if Z lies in $NVF(F_\beta)$, then Z' will lie in $NVF(F'_{\beta'})$, and $\chi_{Z,n}$ must be an edge of the form e_{n,a_n-j} for some $j = 0,1,\ldots,a_n-1$. Thus the map f is well-defined. On the other hand, the same proposition shows that Z is completely determined if we know both Z' and the edge $\chi_{Z,n} = e_{n,a_n-j}$, so the map f is certainly injective, and any choice of such a Z' and edge e_{n,a_n-j} will give rise to a Z in $NVF(F_\beta)$. So the map is bijective.

One can check that (7.2) holds because Z has exactly j arcs of C_Z directed the same as they are directed in Z_0. □

Given Proposition 7.3, the recurrence in (6.2) follows:

$$\begin{aligned} n(F_\beta) &= \sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)} \\ &= \sum_{\substack{(Z',j): \\ Z' \in NVF(F'_{\beta'}) \\ j=0,1,\ldots,a_n-1}} x^{\mathrm{comm}(Z')+j} \\ &= \left(\sum_{Z' \in NVF(F'_{\beta'})} x^{\mathrm{comm}(Z')} \right) \left(\sum_{j=0}^{a_n-1} x^j \right) = [a_n]_x \cdot n(F'_{\beta'}) \end{aligned}$$

8. Removing a Minus Petal

8.1. **Proof of the recurrence** (6.3)**.** Assume that our flower $F = \bigoplus_{i=1}^{n} C_{a_i}$ has voltage assignment F_β such that all cycles are minus cycles. Let $F' = \bigoplus_{i=1}^{n-1} C_{a_i}$, so that $F = F' \oplus C_{a_n}$, and let $F'_{\beta'}$ be the voltage assignment on F' such that cycles $C_{a_1}, \ldots, C_{a_{n-1}}$ agree with their assignment on F_β. As before, let $\tilde{F} \to F$ and $\tilde{F}' \to F'$ be their associated double coverings.

The following property will be useful in computing $t(F_\beta)$. Say that the graph G' is obtained from G by *adding p series copies of an edge* if G' contains a path P of p edges shown in Figure 16.

$$v_0 \text{ —— } v_1 \text{ —— } \cdots \text{ —— } v_{p-1} \text{ —— } v_p$$

Figure 16. P-Series Extension of an Edge

Where each of the vertices $v_1, \ldots, v_{p-1}$ has degree two in G', and $G = G' - P$ is obtained from G' by deleting all of the edges in P.

Proposition 8.1. In this setting, $\tilde{F}$ is obtained from $\tilde{F}'$ by twice doing an extension of a_n copies of an edge in series. In both cases, the extension adds p series copies of an edge along a path whose endpoints are d_+, d_- in $\tilde{F}'$.

Proof. With labeling as in Definition 2.10,we have by proposition 2.9 that the two added paths are the ones shown in Figure 17 up to isomorphism.

$$d_- = v_{n,1,-} \text{ — } v_{n,2,+} \text{ — } v_{n,3,+} \text{ — } \ldots \text{ — } v_{n,a_n,+} \text{ — } v_{n,1,+} = d_+$$

$$d_+ = v_{n,1,+} \text{ — } v_{n,2,-} \text{ — } v_{n,3,-} \text{ — } \ldots \text{ — } v_{n,a_n,-} \text{ — } v_{n,1,-} = d_- \quad \square$$

Figure 17. Paths Added

Proposition 8.2. [3, Lemma 2.5] In the above setting with graph G,

$$T_{G'}(x,y) = [p]_x T_G(x,y) + T_{G'/P}(x,y).$$

where G'/P is obtained from G' by contracting all of the edges in P.

Corollary 8.3. *In the above setting,*

$$T_{\tilde{F}}(x,1) = [a_n]_x \cdot \left([a_n]_x \cdot T_{\tilde{F}'}(x,1) + 2 \prod_{1 \le i \le n-1} [a_i]_x^2 \right).$$

Proof. Since $\tilde{F}$ is obtained from $\tilde{F}'$ by adding two series paths P_1, P_2, each with a_n edges, we can apply Proposition 8.2 twice:

$$\begin{aligned} T_{\tilde{F}}(x,1) &= [a_n]_x T_{\tilde{F}-P_1}(x,1) + T_{\tilde{F}/P_1}(x,1) \\ &= [a_n]_x \left([a_n]_x T_{\tilde{F}-P_1-P_2}(x,1) + T_{(\tilde{F}-P_1)/P_2}(x,1) \right) + T_{\tilde{F}/P_1}(x,1) \end{aligned}$$

However, note that $\tilde{F} - P_1 - P_2 = \tilde{F}'$, while

$$\tilde{F}/P_1 \cong C_{a_n} \oplus \bigoplus_{i=1}^{n-1} \left(C_{a_i} \oplus C_{a_i} \right) \text{ and } (\tilde{F} - P_1)/P_2) \cong \bigoplus_{i=1}^{n-1} \left(C_{a_i} \oplus C_{a_i} \right)$$

Thus one can rewrite this last expression as

$$T_{\tilde{F}}(x,1) = [a_n]_x \left([a_n]_x T_{\tilde{F}'}(x,1) + \prod_{i=1}^{n-1} [a_i]_x^2 \right) + [a_n]_x \prod_{i=1}^{n-1} [a_i]_x^2$$

which is equivalent to the assertion in the corollary. Since $F \cong \bigoplus_{i=1}^{n} C_{a_i}$ and $F' \cong \bigoplus_{i=1}^{n-1} C_{a_i}$ gives (via Proposition 4.2) that

$$T_F(x,1) = \prod_{i=1}^{n} [a_i]_x = [a_n]_x \prod_{i=1}^{n-1} [a_i]_x = [a_n]_x T_{F'}(x,1).$$

□

One can now check recurrence (6.3):

$$\begin{aligned} t(F_\beta) = \frac{T_{\tilde{F}}(x,1)}{T_F(x,1)} &= \frac{[a_n]_x \cdot \left([a_n]_x \cdot T_{\tilde{F}'}(x,1) + 2 \prod_{i=1}^{n-1} [a_i]_x^2 \right)}{T_F(x,1)} \\ &= \frac{[a_n]_x^2 \cdot T_{\tilde{F}'}(x,1)}{T_F(x,1)} + \frac{2[a_n]_x \prod_{i=1}^{n-1} [a_i]_x^2}{T_F(x,1)} \\ &= \frac{[a_n]_x^2 \cdot T_{\tilde{F}'}(x,1)}{[a_n]_x T_{F'}(x,1)} + \frac{2[a_n]_x \prod_{i=1}^{n-1} [a_i]_x^2}{\prod_{i=1}^{n} [a_i]_x} = [a_n]_x \cdot t(F'_{\beta'}) + 2 \prod_{i=1}^{n-1} [a_i]_x. \end{aligned}$$

8.2. **Proof of the recurrence** (6.4). Assume the conventions on the flowers F, F' and $F_\beta, F'_{\beta'}$ as in Section 8.1.

Proposition 8.4. One has

$$\sum_{\substack{Z \in NVF(F_\beta):\\ C_Z \neq C_{a_n}}} x^{\mathrm{comm}(Z)} = [a_n]_x \cdot n(F'_{\beta'}), \tag{8.1}$$

$$\sum_{\substack{Z \in NVF(F_\beta):\\ C_Z = C_{a_n}}} x^{\mathrm{comm}(Z)} = 2\prod_{i=1}^{n-1}[a_i]_x, \tag{8.2}$$

Proof. To prove (8.1), note that one has a bijection

$$\begin{array}{rcl} \{Z \in NVF(F_\beta) : C_Z \neq C_{a_n}\} & \longrightarrow & NVF(F'_{\beta'}) \\ Z & \longmapsto & f(Z) = (Z', j) \end{array}$$

very similar to the one in Proposition 7.3, where Z' is the restriction of the arcs of Z to F', and $\chi_{Z,n} = e_{n,a_n - j}$, again satisfying $\mathrm{comm}(Z) = \mathrm{comm}(Z') + j$. The key point is that $C_Z \neq C_{a_n}$ ensures that the restriction of Z to C_{a_n} is not a directed cycle, but rather a spanning tree inside C_{a_n} directed toward the stem vertex d. Then (8.1) follows via a calculation similar to the one following Proposition 7.3.

To prove (8.2), note that if Z in $NVF(F_\beta)$ has $C_Z = C_{a_n}$, then it is completely determined by the choice of whether C_{a_n} is directed as a circuit in Z clockwise or anti-clockwise, along with the choice of the edges $\chi_{Z,i} = e_{i,a_i - j_i}$ for $i = 1, 2, \ldots, n-1$ which Z omits omits from each of the cycles $C_{a_1}, \ldots, C_{a_{n-1}}$. This gives a bijection between the set

$$\{Z \in NVF(F_\beta) : C_Z = C_{a_n} \text{ is oriented } \textit{clockwise} \text{ as a circuit in } Z\}$$

and the Cartesian product

$$\{0, 1, \ldots, a_1 - 1\} \times \{0, 1, \ldots, a_2 - 1\} \times \cdots \times \{0, 1, \ldots, a_{n-1} - 1\}$$

by sending Z to the sequence $(j_1, j_2, \ldots, j_{n-1})$. Furthermore, by Definition 3.10, one has $\mathrm{comm}(Z) = j_1 + j_2 + \cdots + j_{n-1}$. Hence the sum of $x^{\mathrm{comm}(Z)}$ over such Z is

$$\sum_{\substack{(j_1, j_2, \ldots, j_{n-1}):\\ 0 \leq j_i \leq a_i - 1}} x^{j_1 + j_2 + \cdots + j_{n-1}} = [a_1]_x [a_2]_x \cdots [a_{n-1}]_x = \prod_{i=1}^{n-1} [a_i]_x.$$

One has the same formula for the Z in $NVF(F_\beta)$ for which $C_Z = C_{a_n}$ is oriented *anti-clockwise* as a circuit in Z. Adding the two of these formulas gives (8.2). $\square$

Therefore recurrence (6.4) holds:

$$n(F_\beta) := \sum_{Z \in NVF(F_\beta)} x^{\mathrm{comm}(Z)} = \sum_{\substack{Z \in NVF(F_\beta): \\ C_Z \neq C_{a_n}}} x^{\mathrm{comm}(Z)} + \sum_{\substack{Z \in NVF(F_\beta): \\ C_Z = C_{a_n}}} x^{\mathrm{comm}(Z)}$$

$$= [a_n]_x \cdot n(F'_{\beta'}) + 2 \prod_{i=1}^{n-1} [a_i]_x.$$

9. Acknowledgments

I would like to thank Dr. Victor Reiner at the University of Minnesota for overseeing this research project.

References

[1] Béla Bollobás. Modern Graph Theory. Springer Science+Business Media, Inc., 233 Spring Street, New York, NY 10013, USA, 1998.
[2] Brylawski, Thomas; Oxley, James The Tutte polynomial and its applications. Matroid applications, 123–225, Encyclopedia Math. Appl., 40, Cambridge Univ. Press, Cambridge, 1992.
[3] C. Merino, M. Ramírez-Ibáñez, and G. Rodríguez-Sánchez. The Tutte polynomial of some matroids. Int. J. Comb. 2012, Art. ID 430859, 40 pp.
[4] James Oxley. Matroid Theory. Oxford University Press, 2011.
[5] Jonathan Gross and Thomas Tucker. Generating All Graph Coverings by Permutation Voltage Assignments. Discrete Mathematics, 18(3): 273-283, 1977.
[6] S. Chepuri, C.J. Dowd, A. Hardt, G. Michel, S. W. Zhang, and V. Zhang. Arborescences of Covering Graphs arXiv:1912.01060.
[7] Thomas Zaslavsky. Characterizations of Signed Graphs. Journal of Graph Theory, 5: 401-406, 1981.

Student biographies

Kaustubh Verma: (verma112@umn.edu) is a 2021 graduate of the University of Minnesota with a BS in Mathematics. He is pursuing a Masters degree in Mathematics from the University of Wisconsin Madison starting in Fall 2021. He subsequently plans to pursue a PhD in mathematics.