Combinatorics of Bulgarian Solitaire

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Presentation Outline

1. Preliminary
2. Introduction
3. Past results
4. Data and Results
5. Further study

Outline

Preliminary: Integer Partitions

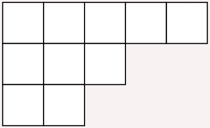
Definition 1.1

A partition of a positive integer n is a way to write it as a sum of integers regardless of the order.
Let $\mathcal{P}(n)$ be the set of partitions of n and $p(n) = |\mathcal{P}(n)|$.

Example of integer partitions:

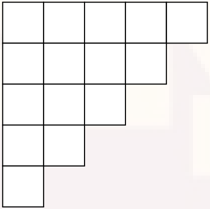
$$\mathcal{P}(4) = \{1+1+1+1, 2+1+1, 2+2, 3+1, 4\}$$

The *Young diagram* is a visualization of an integer partition. For example, the partition $10 = 5+3+2$ is drawn as



Preliminary: Integer Partitions

The $m-$staircase partition of $n = \binom{m+1}{2}$ is denoted by $\Delta_m = (m, m-1, \ldots, 1, 0)$. For example Δ_5 is



Outline

Introduction: The Bulgarian Solitaire game

- Introduced by Martin Gardner in 1983.
- Original game:
  - Start with $45 = 1 + 2 + \ldots + 9$ cards divided into a number of piles.
  - Bulgarian Solitaire rule: pick one card from each pile and form a new pile.
  - Stop when pile sizes are not changed.
- The game terminates after a finite number of moves, into 9 piles of size from 1 to 9.
- Same convergent behaviour if starting with any triangular number of cards.

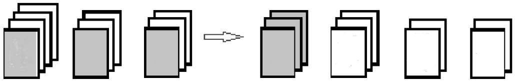


Figure 1: Bulgarian Solitaire on decks of cards

Introduction: The Bulgarian Solitaire rule

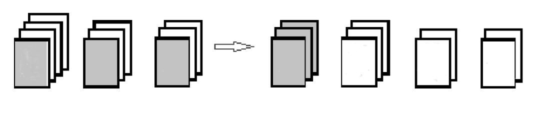


Figure 2: Bulgarian Solitaire on decks of cards

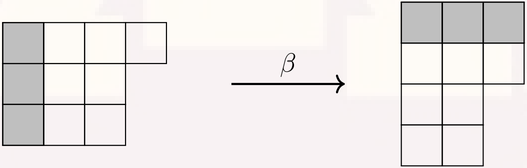


Figure 3: Bulgarian Solitaire on Young diagram $\beta((4, 3, 3)) = (3, 3, 2, 2)$

Introduction: Bulgarian Solitaire game graph

Definition 2.1

The Bulgarian Solitaire game graph on the set of partition $\mathcal{P}(n)$ is a directed graph:

- *Nodes: partitions of n.*
- *Edges: $\lambda \to \beta(\lambda)$.*

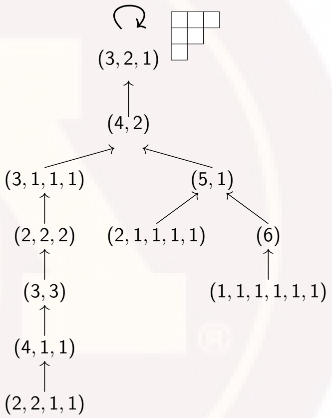


Figure 4: Bulgarian Solitaire game graph for $n = 6$.

Introduction: Bulgarian Solitaire game graph

What about starting with non-triangular number n?

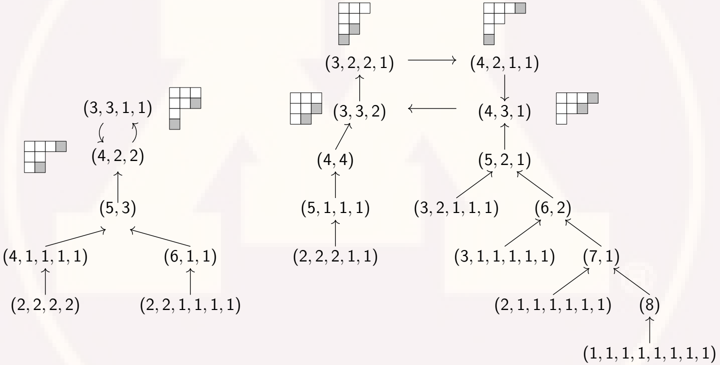


Figure 5: Bulgarian Solitaire game graph for $n = 8$.

Introduction: Bulgarian Solitaire game graph

Definition 2.2

We denote $\psi(\lambda)$ to be the Bulgarian Solitaire orbit that contains λ, that is, λ, μ lie in the same BS orbit $\psi(\lambda) = \psi(\mu)$ if there exists integers $a, b \geq 0$ for which $\beta^a(\lambda) = \beta^b(\mu)$.

Example.

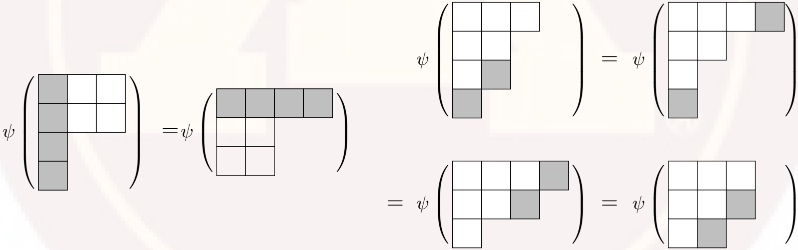


Figure 6: Bulgarian Solitaire orbit representation.

Introduction: Bulgarian Solitaire game graph

Definition 2.3 (recurrent cycle)

Each orbit of the Bulgarian Solitaire system on $\mathcal{P}(n)$ has a unique recurrent cycle $\mathcal{C}$, that is, if $\lambda \in \mathcal{C}$, then $\beta^t(\lambda) \in \mathcal{C}$ for any t.

Example.

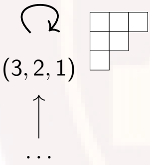


Figure 7: Recurrent cycle for triangular number $n = 6$.

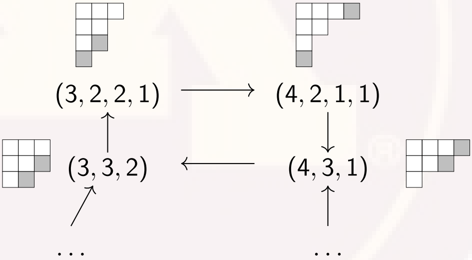


Figure 8: Recurrent cycle for non-triangular number $n = 8$.

Introduction: necklaces

Definition 2.4

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a finite sequence of letters $\{B, W\}$. Define the cyclic rotation ω by

$$\omega(\alpha_j) = \alpha_{(j+1)\ mod\ n}$$

A *necklace* N of black and white beads is an equivalence class of sequences of letters $\{B, W\}$ under cyclic rotation ω. We call N a *primitive necklace* if it cannot be written as a concatenation $N = P^k = PP \cdots P$ of copies of another necklace P. We will reserve P for *primitive necklaces.*

Let $\mathcal{N}$ be the collection of all classes necklaces with at least 1 white bead.

Example 2.5

Primitive necklaces: W and $BWW = WBW = WWB$

Non-primitive necklace: $N = (BW)^2 = BWBW = (WB)^2 = WBWB$.

Introduction: BS orbits and necklaces

Bijection $\mathcal{O} : \mathcal{N} \longrightarrow \mathcal{BS}$ by Brandt, 1982 [1]. Let $\mathcal{O}_N = \mathcal{O}(N)$ for necklace class $N \in \mathcal{N}$.

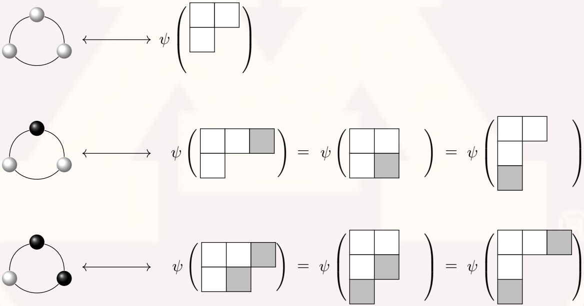


Figure 9: The map $\mathcal{O}$ for primitive necklaces of length 3: WWW, BWW, BBW.

Introduction: BS orbits and necklaces

Theorem 2.6 (Brandt 1982 [1], Drensky 2015 [2])

Uniquely express

$$n = \binom{m}{2} + r \text{ for some } 0 \leq r \leq m-1$$

and let $\lambda \in \mathcal{P}(n)$. Then the orbits of the Bulgarian Solitaire system on $\mathcal{P}(n)$ biject with necklaces N with $b(N) = r$ black beads and $w(N) = m - r$ white beads. The partitions λ within the recurrent cycle of orbit $\mathcal{O}_N$ consist of a staircase partition along with an extra square in each row indexed by a black bead from necklace in N.

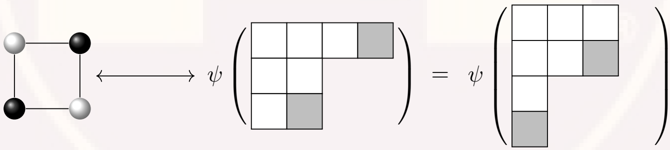


Figure 10: The map $\mathcal{O}$ for non-primitive necklaces of length 4. The recurrent set in $\mathcal{O}_{(BW)^2}$ has only 2 elements, shown above.

Introduction

Definition 2.7 (distance to cycle)

For any primitive necklace $P \in \mathcal{N}$ and any $\lambda \in \mathcal{O}_{P^k}$, we denote by $D_{P^k}(\lambda)$ the minimum number of moves to reach the recurrent cycle starting from λ, that is, define $D_{P^k} : \mathcal{O}_{P^k} \to \mathbb{N}$ by

$$D_{P^k}(\lambda) = \min\{d \in \mathbb{N} : \beta^d(\lambda) \in \mathcal{C}_{P^k}\}$$

The recurrent cycle is of level 0, and $D_{P^k}^{-1}(d)$ is the set of partitions of level d.

Outline

Past results

The following theorem is for triangular number $n = \binom{k+1}{2}$ with corresponding necklace W^k:

Theorem 3.1 (Eriksson, Jonsson, 2017 [3])

In the limit as k grows, the sequence of level sizes $(D_{W^k}^{-1}(1), D_{W^k}^{-1}(2), \ldots)$ converges to the subsequence of evenly-indexed Fibonacci numbers $(F_{2d})_{d=0}^{\infty}$, with the generating function

$$H_W(x) = \frac{(1-x)^2}{1-3x+x^2}$$
$$= 1 + x + 3x^2 + 8x^3 + 21x^4 + 55x^4 + \ldots.$$

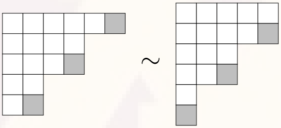
We wish to generalize this result for arbitrary number n.

Outline

Data and results: recall of necklaces

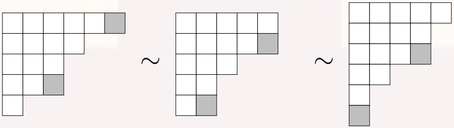
Some examples to recall of the necklaces and their corresponding partitions:

$(BW)^3 \longleftrightarrow$



$\longleftrightarrow (6, 4, 4, 2, 2) \sim (5, 5, 3, 3, 1, 1)$

$(BWW)^2 \longleftrightarrow$



$\longleftrightarrow (6, 4, 3, 3, 1) \sim (5, 5, 3, 2, 2) \sim (5, 4, 4, 2, 1, 1)$

Data: orbit sizes

N	$\lvert\mathcal{O}_N\rvert$
W	1
W^2	3
W^3	11
W^4	42

N	$\lvert\mathcal{O}_N\rvert$
BW	2
$(BW)^2$	7
$(BW)^3$	26
$(BW)^4$	97

N	$\lvert\mathcal{O}_N\rvert$
BWW	5
$(BWW)^2$	$25 = 5^2$
$(BWW)^3$	$125 = 5^3$
$(BWW)^4$	$625 = 5^4$

N	$\lvert\mathcal{O}_N\rvert$
BBW	7
$(BBW)^2$	$35 = 7 \cdot 5$
$(BBW)^3$	$175 = 7 \cdot 5^2$
$(BBW)^4$	$875 = 7 \cdot 5^3$

Table 1: Orbit sizes for orbits $\mathcal{O}_{P^k}$ of primitive necklaces P of length $1, 2, 3$.

Result: primitive necklaces of length 3

Theorem 4.1 (Pham 2022$^+$)

For each $k = 1, 2, 3, \ldots$, one has

$$\left|\mathcal{O}_{(BWW)^k}\right| = 5^k,$$
$$\left|\mathcal{O}_{(BBW)^k}\right| = 7 \cdot 5^{k-1}.$$

Preliminary: *Chebyshev polynomials of the first kind*

The *Chebyshev polynomials of the first kind* are denoted $\{T_k(x)\}_{k=0}^{\infty}$, with initial conditions

$$T_0(x) = 1, \, T_1(x) = x$$

and recurrence relation

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \text{ for } k \geq 2.$$

In particular, we will need their specialization at $x = 2$, satisfying:

$$\begin{aligned} T_0(2) &= 1 \\ T_1(2) &= 2 \\ T_k(2) &= 4T_{k-1}(2) - T_{k-2}(2) \quad \text{for } k \geq 2. \end{aligned}$$

The first 5 terms are $1, 2, 7, 26, 97$. The explicit formula and an asymptotic is

$$\left|\mathcal{O}_{(BW)^k}\right| = T_k(2) = \frac{1}{2}\left((2-\sqrt{3})^k + (2+\sqrt{3})^k\right) \sim (2+\sqrt{3})^k,$$

whose geometric ratio is $2 + \sqrt{3} \approx 3.732...$

Result: BW necklace

Theorem 4.2 (Pham 2022$^+$)

For each $k = 1, 2, 3, \ldots$, one has $\left|\mathcal{O}_{(BW)^k}\right| = T_k(2)$. Moreover, the generating functions for distance to the recurrent cycle $\mathcal{C}_{(BW)^k}$

$$\mathcal{D}_N(x) := \sum_{\lambda \in \mathcal{O}_N} x^{D_N(\lambda)} = \sum_{d=0}^{\infty} D^{-1}_{(BW)^k}(d)\, x^d$$

satisfies the following generalization of the recurrence of the Chebyshev polynomials evaluated at $x = 2$:

$$\mathcal{D}_{(BW)^0}(x) := 1 \quad \text{by convention,}$$

$$\mathcal{D}_{(BW)^1}(x) = 2,$$

$$\mathcal{D}_{(BW)^k}(x) = x(3x+1)\mathcal{D}_{(BW)^{k-1}}(x) - x^3\mathcal{D}_{(BW)^{k-2}}(x) + (x-1)^2(3x+2) \quad \text{for } k \geq 2.$$

Data: orbit sizes

N	$\lvert\mathcal{O}_N\rvert$
$BWWW$	15
$(BWWW)^2$	15^2
$(BWWW)^3$	15^3
$(BWWW)^4$	15^4

N	$\lvert\mathcal{O}_N\rvert$
$BBWW$	15
$(BBWW)^2$	$15 \cdot 10$
$(BBWW)^3$	$15 \cdot 10^2$
$(BBWW)^4$	$15 \cdot 10^3$

N	$\lvert\mathcal{O}_N\rvert$
$BBBW$	30
$(BBBW)^2$	$30 \cdot 15$
$(BBBW)^3$	$30 \cdot 15^2$
$(BBBW)^4$	$30 \cdot 15^3$

Table 2: Orbit sizes for orbits $\mathcal{O}_{P^k}$ of primitive necklaces P of length 4.

Data: orbit sizes

N	$\lvert\mathcal{O}_N\rvert$
$BBWWW$	45
$(BBWWW)^2$	$45 \cdot 27$
$(BBWWW)^3$	$45 \cdot 27^2$

N	$\lvert\mathcal{O}_N\rvert$
$BBBWW$	67
$(BBBWW)^2$	$67 \cdot 27$
$(BBBWW)^3$	$67 \cdot 27^2$

N	$\lvert\mathcal{O}_N\rvert$
$WBWBW$	32
$(WBWBW)^2$	$32 \cdot 17$
$(WBWBW)^3$	$32 \cdot 17^2$

N	$\lvert\mathcal{O}_N\rvert$
$BWBWB$	34
$(BWBWB)^2$	$34 \cdot 17$
$(BWBWB)^3$	$34 \cdot 17^2$

N	$\lvert\mathcal{O}_N\rvert$
$BWWWW$	56
$(BWWWW)^2$	$56 \cdot 44$

N	$\lvert\mathcal{O}_N\rvert$
$BBBBW$	135
$(BBBBW)^2$	$135 \cdot 44$

Table 3: Orbit sizes for orbits $\mathcal{O}_{P^k}$ of primitive necklaces P of length 5.

Conjecture: primitive necklaces of length 4 and 5

Conjecture 4.3

For each $k = 2, 3, 4, \ldots$, *one has*

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

where

$$c_P = \begin{cases} 15 \text{ for both } P = BWWW, BBBW \\ 10 \text{ for } \quad P = BBWW \\ 17 \text{ for both } P = WBWBW, BWBWB \\ 27 \text{ for both } P = BBWWW, WWBBB \\ 44 \text{ for both } P = BWWWW, WBBBB \end{cases}.$$

Remark 4.4

Together with the asymptotic for BW necklace, which is $2 + \sqrt{3} \approx 3.732\ldots$, *we expect that for primitive necklaces of length greater than* 1, *the geometric ratio is increasing as the length increases.*

Conjecture: general primitive necklaces of length at least 3

Conjecture 4.5

For any primitive necklace P with $|P| \geq 3$, there is an integer c_P such that for $k \geq 2$,

$$|\mathcal{O}_{P^k}| = (c_P)^{k-1}|\mathcal{O}_P|$$

for some constant c_P that depends only on P. Moreover, if P and P' are obtained from each other by swapping black beads to white beads and vice versa, then $c_P = c_{P'}$.

Data: BW level sizes

$d\backslash N$	BW	$(BW)^2$	$(BW)^3$	$(BW)^4$	$(BW)^5$	$(BW)^6$	$(BW)^7$	$(BW)^8$	$(BW)^9$	$(BW)^{10}$
0	2	2	2	2	2	2	2	2	2	2
1	0	1	1	1	1	1	1	1	1	1
2	0	2	3	3	3	3	3	3	3	3
3	0	2	6	7	7	7	7	7	7	7
4	0	0	8	14	15	15	15	15	15	15
5	0	0	6	24	32	33	33	33	33	33
6	0	0	0	28	60	70	71	71	71	71
7	0	0	0	18	92	142	154	155	155	155
8	0	0	0	0	96	248	320	334	335	335
9	0	0	0	0	54	344	614	712	728	729
10	0	0	0	0	0	324	996	1432	1560	1578

Table 4: $\left|D_N^{-1}(d)\right|$ Distribution by level sizes for necklaces $N = (BW)^k$ of alternating black-white beads.

Result: convergence of BW and BWW level sizes

Theorem 4.6 (Pham 2022$^+$)

There are power series $H_{BW}(x)$, $H_{BWW}(x)$ and $H_{BBW}(x)$ in $\mathbb{Z}[[x]]$ such that

$$\lim_{k\to\infty} \mathcal{D}_{(BW)^k} = H_{BW}(x) \text{ and } \lim_{k\to\infty} \mathcal{D}_{(BWW)^k} = H_{BWW}(x)$$

Moreover, $H_{BW}(x)$, $H_{BWW}(x)$ and $H_{BBW}(x)$ are rational functions, given by

$$H_{BW}(x) = \frac{(x-1)^2(3x+2)}{x^3-3x^2-x+1} = 2 + x + 3x^2 + 7x^3 + 15x^4 + 33x^5 + 71x^6 + \dots$$

$$H_{BWW}(x) = H_{BBW}(x) = (1-x)\frac{x^3-3x^2-4x-3}{2x^3+x^2-1} = 3 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 17x^7 + 25x^8 + \dots$$

Result: general convergence of level sizes

Theorem 4.7 (Pham 2022^{+})

For primitive necklaces P with $|P| \geq 3$, there is a power series H_P in $\mathbb{Z}[[x]]$ such that the sequence of generating functions $(\mathcal{D}_{P^k})_{k=0}^{\infty}$ converges to H_P. Moreover, H_P is a rational function having

- *denominator polynomial of degree at most $|P|$,*
- *numerator polynomial of degree at most $2 \cdot |P| - 1$.*

Example.

$$H_{BWWW}(x) = (1-x)\frac{x^5 + 8x^4 - 3x^3 - 8x^2 - 6x - 4}{6x^4 + 4x^3 + x^2 - 1},$$

$$H_{BBBW}(x) = (1-x)\frac{2x^5 + 8x^4 - 5x^3 - 10x^2 - 7x - 4}{6x^4 + 4x^3 + x^2 - 1},$$

$$H_{BBWW}(x) = (1-x)\frac{x^3 + x^2 + x + 1}{3x^4 + 2x^3 + x^2 - 1}.$$

Outline

Further study: more interesting properties

1. Generate more data to confirm the conjectures about the orbit sizes.
2. Characterize the class of partitions given by an orbit of Bulgarian Solitaire.
3. Improve the result for the denominator degree of the limit of generating functions by level sizes for general primitive necklaces.
4. An analogue of 5 in the recurrence of BBW and BWW cases?

List of References

[1] Jørgen Brandt. "Cycles of partitions". In: *Proceedings of the American Mathematical Society* (1982), pp. 483–486.

[2] Vesselin Drensky. "The Bulgarian solitaire and the mathematics around it". In: *arXiv preprint arXiv:1503.00885* (2015).

[3] Henrik Eriksson and Markus Jonsson. "Level Sizes of the Bulgarian Solitaire Game Tree". In: *The Fibonacci quarterly* 55.3 (2017), pp. 243–251.

Thank You So Much!