

ON A SMITH NORMAL FORM CONJECTURE
OF LINDZEY

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ABSTRACT. This thesis examines the following conjecture of Nathan Lindzey. Consider the $n! \times n!$ matrix having rows and columns indexed by permutations in the symmetric group on n letters, whose (σ, τ) entry for $\sigma, \tau \in S_n$ is 1 if $\sigma^{-1}\tau$ is an n -cycle and 0 otherwise. Then this matrix has the same rank when taken modulo p for all odd primes p as its rank over the rationals $\mathbb{Q}$.

We explore the context of the conjecture further, by generalizing to real reflection groups and even to well-generated complex reflection groups. Here the role of n -cycles are played by the conjugacy class of what are called Coxeter elements. The analogous conjecture is shown to hold for certain sets of these reflection groups, but not for others.

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1. INTRODUCTION

In a 2019 Math Overflow post [4], Nathan Lindzey made the following conjecture. Consider the $n! \times n!$ matrix A_{S_n} having rows and columns indexed by permutations in the symmetric group on n letters, whose (σ, τ) entry for $\sigma, \tau \in S_n$ is 1 if $\sigma^{-1}\tau$ is

an n -cycle and 0 otherwise. For example, A_{S_3} is shown here:

$$A_{S_3} = \begin{matrix} & \begin{matrix} (1)(2)(3) & (12)(3) & (13)(2) & (23)(1) & (123) & (132) \end{matrix} \\ \begin{matrix} (1)(2)(3) \\ (12)(3) \\ (13)(2) \\ (23)(1) \\ (123) \\ (132) \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Conjecture 1.1 (Lindzey [4]). *The matrix A_{S_n} has the same rank when considered over the rationals $\mathbb{Q}$ as when considered over any field of characteristic p an odd prime.*

Equivalently, the integer Smith Normal Form (SNF) of A_{S_n} has all its non-zero entries being powers of 2.

Definition 1.2. Given a nonzero $m \times n$ matrix A over the a principal ideal domain R , there exist invertible $m \times m$ and $n \times n$ matrices with entries in R and determinants being units in R , which we call P and Q , such that PAQ is a diagonal matrix with entries $\alpha_1, \alpha_2, \dots, \alpha_{\min(m,n)}$, where α_i divides α_{i+1} .

Such a PAQ is unique, and is the Smith Normal Form of A . For this paper, we work with $m = n$ and $R = \mathbb{Z}$.

Example 1.3. The above matrix A_{S_3} has Smith Normal Form

$$\text{SNF}(A_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since there are four 1s and two 2s on the diagonal in S , we will abbreviate this by saying that $\text{SNF}(A_{S_3})$ is $1^4 2^2$.

As part of Lindzey's conjecture, labeled Conjecture 1.1, we claimed that the rank of a matrix being equivalent over both the rationals $\mathbb{Q}$ and over fields of characteristic p an odd prime was equivalent to all the non-zero entries of the Smith Normal Form of said matrix being powers of 2. Let us lay out why that is the case.

Proposition 1.4. *For a matrix $A \in \text{Mat}_n(\mathbb{Z})$, $\text{rank}_{\mathbb{F}}(A)$ for a field $\mathbb{F}$ of characteristic p is equal to the number of entries in the Smith Normal Form of A which are non-zero modulo p .*

Proof. As invertible matrices in $\text{Mat}_n(\mathbb{Z})$, the P and Q such that $S = PAQ$ have determinant ± 1 , and since the determinant is an operation based on addition and multiplication, the unique ring homomorphism from $\mathbb{Z}$ to $\mathbb{F}$ means that the determinants of P and Q remain ± 1 . But then, since they remain invertible, since ± 1 are always units, they do not affect the rank.

Thus, the rank of S over $\mathbb{F}$ is the same as the rank of A over $\mathbb{F}$, and since $\mathbb{F}$ has characteristic p , the number of non-zero entries and thus the rank of S over $\mathbb{F}$ is the number of entries of S over $\mathbb{Z}$ which are non-zero modulo p . $\square$

Lindzey checked Conjecture 1.1 up to $n = 7$:

n	SNF(A_{S_n})
2	1^2
3	$1^4 2^2$
4	$1^8 2^{12} 0^4$
5	$1^{16} 2^{52} 8^{20} 0^{50}$
6	$1^{32} 2^{200} 8^{20} 0^{468}$
7	$1^{64} 2^{728} 4^{28} 8^{128} 16^{20} 0^{4116}$

Note that A_{S_n} has the following re-interpretations:

- If one considers the set of all n -cycles in S_n , then since the inverse of an n -cycle is also an n -cycle, the *directed Cayley graph* for this generating set can be viewed as an undirected graph, and A_{S_n} is the $\{0, 1\}$ -incidence matrix for this *undirected Cayley graph*.
- If one considers within the group algebra $\mathbb{Z}[S_n]$ the element C which is the sum of all of the n -cycles, then A_{S_n} is the matrix that represents the $\mathbb{Z}$ -linear map $\mathbb{Z}[S_n] \rightarrow \mathbb{Z}[S_n]$ which multiplies on the right by C .

Here, we explore Conjecture 1.1 and analogues for *finite reflection groups* W . In particular, we have found similarly interesting patterns when examining the conjugacy class of the *Coxeter elements* in the finite *real reflection groups* and *well-generated complex reflection groups* W . Thus A_W here will refer to the matrix that represents the $\mathbb{Z}$ -linear map $\mathbb{Z}[W] \rightarrow \mathbb{Z}[W]$ which multiplies on the right by C , where C is the sum of the elements of the conjugacy class of the Coxeter elements. Considering the SNF of such A_W , we have a few results and conjectures.

- Although we were unable to prove Lindzey's initial conjecture, we were able to prove that the Smith normal form of A_{S_n} has a certain minimum number of 1s present (see Theorem 3.3).
- Additionally, we show that, for a particular family of complex finite reflection groups called $G(m, 1, n)$, the matrix $A_{G(m, 1, n)}$ has a block structure with each block replicating A_{S_n} . This results in their SNF essentially being multiples of the SNF for the corresponding A_{S_n} ; see Theorem 3.5.
- We were also able to find and prove a pattern for the Smith Normal Form of complex reflection groups in the dihedral group $G(m, m, 2)$ family (see Theorem 3.6), and find, but not prove, a similar pattern for the $G(m, m, 3)$ family (see Conjecture 3.8).

2. BACKGROUND

2.1. Coxeter elements for real reflection groups. This section gives background on reflection groups and Coxeter elements.

To start, we define the finite real reflection groups.

Definition 2.1. A finite real reflection group is a finite group generated by a set of reflections $s_\alpha \in GL(V)$, where V is a Euclidean space, and where each reflection s_α sends some non-zero vector $\alpha \in V$ to $-\alpha$ while fixing pointwise the hyperplane H_α orthogonal to α .

Each such finite real reflection group W acting on a Euclidean space V has a set of $n := \dim(V)$ reflections $S = \{s_1, \dots, s_n\}$ called *simple reflections* that give W a

Coxeter presentation, that is, one of the form

$$W = \langle S \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$$

for $m_{ij} \in \{2, 3, \dots\}$.

Definition 2.2. Given such a Coxeter presentation, the Coxeter elements are the length n products $s_{i_1} s_{i_2} \cdots s_{i_n}$ containing each generator s_i in S exactly once.

E.g., $s_1 s_2 \dots s_n$ is a Coxeter element, as is $s_2 s_1 \dots s_n$, and so on.

Example 2.3. For any n , S_n is a finite real reflection group, and the adjacent transpositions can be the simple reflections. Taking $s_i = (i \ i+1)$, the Coxeter elements are simply the n -cycles, as the product of $n-1$ distinct transpositions that are the edges of a spanning tree.

For S_4 , the simple reflections are $\{(12), (23), (34)\}$, and the Coxeter elements are $\{\sigma_{2341}, \sigma_{2413}, \sigma_{3421}, \sigma_{3142}, \sigma_{4312}, \sigma_{4123}\}$.

Furthermore, it is not obvious, but all such Coxeter elements are conjugate within W [1, §3.16]. These Coxeter elements, because they are conjugate, share their characteristic polynomial and their multiplicative order h , which is called the *Coxeter number* of W . They also share the same eigenvalues $(\zeta^{m_1}, \zeta^{m_2}, \dots, \zeta^{m_n})$ taking the form ζ^m for $\zeta := e^{\frac{2\pi i}{h}}$. The various values $m_1, \dots, m_n$ are called the *exponents* of W .

It turns out that these exponents have a (non-obvious) reformulation in terms of the *invariant theory* of W . One can let W act on the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$ via linear substitution of the variables $x_1, \dots, x_n$ which are the coordinate functions on V . A theorem of Shephard and Todd [7] implies that for a real reflection group W , the W -fixed subalgebra is again a polynomial algebra, that is, $\mathbb{R}[x_1, \dots, x_n]^W = \mathbb{R}[f_1, \dots, f_n]$ for some n algebraically independent polynomials $f_1, \dots, f_n$. One can choose these f_i to be homogenous of degree d_i , and then the $(d_1, \dots, d_n)$ are called the *degrees* of W , and one has $m_i = d_i - 1$ for $i = 1, 2, \dots, n$, while $h = \max\{d_1, \dots, d_n\}$. For more on invariant theory, exponents, degrees and construction and classification of finite real reflection groups, see [1, Chap. 1-3].

Definition 2.4. Given a finite real reflection group W , we let A_W be the $|W| \times |W|$ matrix with entries in $\{0, 1\}$ that represents the $\mathbb{Z}$ -linear map $\mathbb{Z}[W] \rightarrow \mathbb{Z}[W]$ which multiplies on the right by C , where C is the sum of the elements of the conjugacy class of the Coxeter elements.

Example 2.5. Taking $W = S_3$, $C = \sigma_{231} + \sigma_{312}$, which gives:

$$A_{S_3} = \begin{matrix} & \begin{matrix} (1)(2)(3) & (12)(3) & (13)(2) & (23)(1) & (123) & (132) \end{matrix} \\ \begin{matrix} (1)(2)(3) \\ (12)(3) \\ (13)(2) \\ (23)(1) \\ (123) \\ (132) \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix},$$

the same as our previous definition of A_{S_3} , as expected.

Although interesting patterns in the SNF of A_W did arise when examining the finite real reflection groups W (also called the finite Coxeter groups), generalizing to the complex finite reflection groups provides more material to work with, and also bypasses certain computational limitations.

2.2. Complex Finite Reflection Groups. Complex finite reflection groups are an extension from real finite reflection groups, and they are gone into in more depth in Lehrer and Taylor's book[3]. They are formally defined as follows:

Definition 2.6. Given some finite-dimensional complex vector space V , a finite complex reflection group is a finite subgroup of $GL(V)$ generated by complex reflections $s \in GL(V)$, where a complex reflection s is an element of $GL(V)$ whose fixed space $\ker(1_V - s)$ is a complex hyperplane.

Now, any real reflection group can be made complex by simply extending the scalars from $\mathbb{R}$ to $\mathbb{C}$. Beyond that, the finite complex reflection groups acting irreducibly on V were first classified by Shephard and Todd [7]. They showed that every irreducible finite complex reflection group either belonged to

- the infinite family $G(m, p, n)$ consisting of the $n \times n$ monomial matrices whose nonzero entries are m th roots of unity and whose product is an (m/p) th root of unity, or
- one of 34 exceptional cases, which they numbered $G_4, G_5, \dots, G_{37}$.

The bulk of our discussion will focus on the infinite family¹ $G(m, p, n)$. Note that

- $S_n = G(1, 1, n)$,
- $I_2(m) = G(m, m, 2)$, where $I_2(m)$ is the dihedral group of order $2m$, and
- the subfamily $G(m, 1, n)$ is also known as the generalized symmetric group, and can be understood as the wreath product of the symmetric group S_n by a cyclic group of order m , an understanding we use later.[3, §2.2]

2.3. Coxeter elements for the well-generated complex reflection groups.

To extend Lindzey's conjecture from real to complex reflection groups, we must also define the Coxeter elements in a slightly different manner than for the finite real reflection groups. In fact, we will only define Coxeter elements for complex reflection groups $W \subset GL_n(\mathbb{C})$ that are *well-generated*, so those which act irreducibly on $\mathbb{C}^n$ and can be generated by n reflections.²

Example 2.7. Real reflection groups extended from $\mathbb{R}$ to $\mathbb{C}$ are well-generated.

Example 2.8. Among the family $G(m, p, n)$, those with $p = 1$ or $p = m$ are well-generated.[3, §2.7]

The following definition of Coxeter elements in well-generated complex reflection groups has become fairly standard, and generalizes the definition for real reflection groups; see [6].

Definition 2.9. For a well-generated complex reflection group $W \subseteq GL(V)$, define its *Coxeter number* $h = \max\{d_1, \dots, d_n\}$ where d_i are the degrees of n homogeneous polynomials $f_1, \dots, f_n$ which generate the W -fixed polynomials

$$\mathbb{C}[x_1, \dots, x_n]^W = \mathbb{C}[f_1, \dots, f_n].$$

Define c in W to be a *Coxeter element* if c has an $e^{\frac{2\pi i}{h}}$ -eigenvector $\mathbf{v} \in V$, that is, $c(\mathbf{v}) = e^{\frac{2\pi i}{h}} \mathbf{v}$ such that $\mathbf{v}$ has a free W -orbit, meaning $|\{w(\mathbf{v}) \mid w \in W\}| = |W|$.

Remark 2.10. *There is also an alternate definition for Coxeter elements in the finite complex reflection groups, but in this paper we refer solely to the one above.*

¹Although we examined A_W in these exceptional cases (see Section 7 for some data), they were harder to analyze and no patterns were found.

²Some irreducible complex reflection groups on $\mathbb{C}^n$ actually require $n + 1$ reflection generators.

2.4. Matrix and Smith Normal Form. As mentioned above, all Coxeter elements are conjugate to one another in W .

Remark 2.11. It turns out (see [6]) that every well-generated (finite) complex reflection group W acting irreducibly on $\mathbb{C}^n$ has a set of n generating reflections $s_1, s_2, \dots, s_n$ for which their product $c = s_1 s_2 \cdots s_n$ is a Coxeter element. For some groups, we found one such Coxeter element via this method, and then took its W -conjugacy class C .

Using the conjugacy class C of these Coxeter elements, we can then create an adjacency matrix A_W of the Cayley digraph³ of the conjugacy class of the Coxeter elements within W . That is, we take a $|W| \times |W|$ matrix, associating an element of W to each $1 \leq i \leq |W|$, and have the ij th entry be 1 if $w_j^{-1} w_i \in C$, and 0 otherwise, creating our desired A_W . Technically, this is the transpose of the traditional definition of a matrix for a Cayley graph, but this does not affect the Smith Normal Form. Alternatively, A_W is the matrix that represents the $\mathbb{Z}$ -linear map $\mathbb{Z}[W] \rightarrow \mathbb{Z}[W]$ which multiplies on the right by the sum $\sum_{w \in C} w$.

From there, we can take the Smith Normal Form of A_W , to get $\text{SNF}(A_W)$.

3. THEOREMS AND CONJECTURE

We collect here a few results and conjectures.

3.1. The family $G(1, 1, n) = S_n$. Looking back at Lindzey's SNF data for A_{S_n} , which we verified independently[8], we have:

n	$\text{SNF}(A_{S_n})$
2	1^2
3	$1^4 2^2$
4	$1^8 2^{12} 0^4$
5	$1^{16} 2^{52} 8^2 0^{50}$
6	$1^{32} 2^{200} 8^{20} 0^{468}$
7	$1^{64} 2^{728} 4^{28} 8^{128} 16^2 0^{4116}$

There are two things that one notices, aside from his conjecture that all non-zero entries are powers of 2.

The first is something that he already observed in [4].

Proposition 3.1. *The rank of A_{S_n} over the rationals $\mathbb{Q}$ is*

$$\binom{2(n-1)}{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k}^2.$$

Consequently this is also the number of nonzero entries in the SNF, and so the number of zero entries is $n! - \binom{2(n-1)}{n-1}$.

Proof. To show this, we first note that $\text{rank}_{\mathbb{Q}}(A_{S_n})$ is equal to the number of non-zero eigenvalues of A_{S_n} , including multiplicities. If one indexes the irreducible representations of S_n in characteristic zero by partitions π of n , and denote by d_π the dimension of the irreducible, then, by [10, §5.5.2], one has a $\mathbb{Q}$ -algebra isomorphism

$$\mathbb{Q}[S_n] \rightarrow \bigoplus_{\pi} \text{Mat}_{d_\pi \times d_\pi}(\mathbb{Q}).$$

³The inverse of an element in C will not always lie in C , so this digraph is not undirected.

Since A_{S_n} represents multiplication in $\mathbb{Q}[S_n]$ by $\sum_{w \in C} w$, this can be used to show that the eigenvalues are

$$\lambda_\pi = \frac{1}{d_\pi} \sum_{w \in C} \chi_\pi(w)$$

with multiplicities of d_π^2 .

However, $\chi_\pi(w) = 0$ for $w \in C$ for many partitions π , with the exceptions being the exterior powers $\bigwedge^k V$, where V is the standard representation which correspond to the hook-shaped partitions $\pi = (n - k, 1, \dots, 1)$; this follows either from the *Murnaghan-Nakayama rule* or from [2, §A.2]. There is one such representation for each $0 \leq k \leq n - 1$, with dimension $\binom{n-1}{k}$. Thus, we get that

$$\text{rank}(A_{S_n}) = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \binom{2(n-1)}{n-1},$$

as we desired. $\square$

It also appears that the number of 1s in $\text{SNF}(A_{S_n})$ is 2^{n-1} , but we were only able to prove a lower bound.

Conjecture 3.2. *The number of 1s in $\text{SNF}(A_{S_n})$ is 2^{n-1} .*

Note that if this were true and we could prove that there was even at least one 2 in the Smith Normal Form of A_{S_n} , we would have the rank of A_{S_n} over any field of characteristic 2. Combined with Lindzey's conjecture, which covers all fields of characteristic p an odd prime by showing the rank in that case is equal to the rank over $\mathbb{Q}$, we would have the rank of A_{S_n} for any characteristic.

However, for now we only have a lower bound, which will be proven in Section 4:

Theorem 3.3. *The number of 1s in $\text{SNF}(A_{S_n})$ is at least 2^{n-1} .*

Finally, checking the number of 2s in the SNF of A_{S_n} for $n = 3, 4, 5, 6, 7$ through the Online Encyclopedia of Integer Sequences (OEIS), it matched with the sequence A094811[5], which has the following values: 1, 6, 26, 100, 364, 1288, 4488, 15504, $\dots$. That is, the number of 2s in $\text{SNF}(A_{S_n})$ is equal to 2 times the $n - 2$ nd entry in that sequence. This OEIS sequence counts the number of tuples $(s(0), s(1), \dots, s(2n+1))$ such that $0 < s(i) < 8$ and $|s(i) - s(i-1)| = 1$ for $i = 1, 2, \dots, 2n+1$, with $s(0) = 1$, $s(2n+1) = 6$.

Question 3.4. Does the number of 2s in $\text{SNF}(A_{S_n})$ continue to be equal to double the $n - 2$ nd entry of OEIS sequence A094811 for higher values of n , and if so, what is the connection?

3.2. The family $G(m, 1, n)$. Here is some data on the family $G(m, 1, n)$ of well-generated complex reflection groups[8].

$G(m, 1, n)$	$\text{SNF}(A_{G(m,1,n)})$
$G(2, 1, 3)$	$1^8 2^4 0^{36}$
$G(2, 1, 4)$	$1^{16} 2^{24} 0^{344}$
$G(2, 1, 5)$	$1^{32} 2^{104} 8^4 0^{3700}$
$G(3, 1, 3)$	$1^{12} 2^6 0^{144}$
$G(3, 1, 4)$	$1^{24} 2^{36} 0^{1884}$
$G(4, 1, 3)$	$1^{16} 2^8 0^{360}$

This suggested the following theorem, which will be proven in Section 5.

Theorem 3.5. *An integer $d \geq 1$ occurs in the Smith Normal Form of A_{S_n} for $S_n = G(1, 1, n)$ with multiplicity i if and only if it occurs with multiplicity mi in $\text{SNF}(A_{G(m, 1, n)})$.*

3.3. The families of the form $G(m, m, n)$. For the well-generated family $G(m, m, n)$, it is less clear what is happening in general. However, for $n = 2$, where $G(m, m, 2)$ is the dihedral group of order $2m$, the pattern is clear and provable, based on the calculated examples[8].

$G(m, m, 2)$	$\text{SNF}(A_{G(m, m, 2)})$
$G(2, 2, 2)$	1^4
$G(3, 3, 2)$	$1^4 2^2$
$G(4, 4, 2)$	$1^4 0^4$
$G(5, 5, 2)$	$1^8 2^2$
$G(6, 6, 2)$	$1^8 2^4$
$G(7, 7, 2)$	$1^{12} 2^2$
$\dots$	$\dots$
$G(35, 35, 2)$	$1^{68} 2^2$
$G(36, 36, 2)$	$1^{68} 0^4$
$G(37, 37, 2)$	$1^{72} 2^2$
$G(38, 38, 2)$	$1^{72} 2^4$
$G(39, 39, 2)$	$1^{76} 2^2$
$G(40, 40, 2)$	$1^{76} 0^4$

From there, we found the following theorem, which is proven in Section 6.

Theorem 3.6. *For $m > 2$, we have that:*

$$\text{SNF}(A_{G(m, m, 2)}) = \begin{cases} 1^{2m-4} 0^4, & m \equiv 0 \pmod{4} \\ 1^{2m-2} 2^2, & m \equiv 1, 3 \pmod{4} \\ 1^{2m-4} 2^4, & m \equiv 2 \pmod{4} \end{cases}$$

Corollary 3.7. *The rank of $A_{G(m, m, 2)}$ is $2m - 4$ if $m \equiv 0 \pmod{2}$ and $2m - 2$ if $m \equiv 1 \pmod{2}$ for fields of characteristic 2. For fields of any other characteristic, it is $2m - 4$ for $0 \pmod{4}$ and $2m$ otherwise.*

For $G(m, m, 3)$ with $m \geq 2$, we have this data[8].

$G(m, m, 3)$	$\text{SNF}(A_{G(m, m, 3)})$
$G(2, 2, 3)$	$1^0 1^8 2^{12} 0^4$
$G(3, 3, 3)$	$1^{18} 1^{23} 3^{16} 9^2 0^{16}$
$G(4, 4, 3)$	$1^{36} 2^8 4^{12} 0^{40}$
$G(5, 5, 3)$	$1^{54} 5^{20} 0^{76}$
$G(6, 6, 3)$	$1^{72} 1^2 3^6 6^{10} 18^2 0^{124}$
$G(7, 7, 3)$	$1^{90} 7^{20} 0^{184}$
$G(8, 8, 3)$	$1^{108} 4^8 8^{12} 0^{256}$
$G(9, 9, 3)$	$1^{126} 3^{29} 16^2 27^2 0^{340}$
$G(10, 10, 3)$	$1^{144} 5^8 10^{12} 0^{436}$
$G(11, 11, 3)$	$1^{162} 11^{20} 0^{544}$
$G(12, 12, 3)$	$1^{180} 2^{26} 6^{12} 10^3 36^2 0^{664}$
$G(13, 13, 3)$	$1^{198} 13^{20} 0^{796}$
$G(14, 14, 3)$	$1^{216} 7^8 14^{12} 0^{940}$
$G(15, 15, 3)$	$1^{234} 5^2 15^{16} 45^2 0^{1096}$
$G(16, 16, 3)$	$1^{252} 8^8 16^{12} 0^{1264}$
$G(17, 17, 3)$	$1^{270} 17^{20} 0^{1444}$
$G(18, 18, 3)$	$1^{288} 3^{29} 18^{10} 54^2 0^{1636}$
$G(19, 19, 3)$	$1^{306} 19^{20} 0^{1840}$
$G(20, 20, 3)$	$1^{324} 10^8 20^{12} 0^{2056}$
$G(21, 21, 3)$	$1^{342} 7^{22} 1^{16} 63^2 0^{2284}$
$G(22, 22, 3)$	$1^{360} 11^8 22^{12} 0^{2524}$
$G(23, 23, 3)$	$1^{378} 23^{20} 0^{2776}$
$G(24, 24, 3)$	$1^{396} 4^{22} 12^6 24^{10} 72^2 0^{3040}$
$G(25, 25, 3)$	$1^{414} 25^{20} 0^{3316}$
$G(26, 26, 3)$	$1^{432} 13^8 26^{12} 0^{3604}$
$G(27, 27, 3)$	$1^{450} 9^{22} 27^{16} 81^2 0^{3904}$
$G(28, 28, 3)$	$1^{468} 14^8 28^{12} 0^{4216}$
$G(29, 29, 3)$	$1^{486} 29^{20} 0^{4540}$
$G(30, 30, 3)$	$1^{504} 5^{22} 15^6 30^{10} 90^2 0^{4876}$

Note that, unlike in Lindzey's original Conjecture 1.1, and with Theorem 3.5 about $G(m, 1, n)$ and Theorem 3.6 about $G(m, m, 2)$, here for $G(m, m, 3)$ there are some nonzero SNF entries that are **not powers of 2**. Nevertheless, by examination, one has the following conjecture:

Conjecture 3.8. *For $m \geq 2$, it is the case that:*

$$\text{SNF}(A_{G(m, m, 3)}) = \begin{cases} 1^{18m-36} \left(\frac{m}{6}\right)^2 \left(\frac{m}{2}\right)^6 m^{10} (3m)^2 0^{z(m)}, & m = 0 \bmod 6 \\ 1^{18m-36} m^{20} 0^{z(m)}, & m = 1, 5 \bmod 6 \\ 1^{18m-36} \left(\frac{m}{2}\right)^8 m^{12} 0^{z(m)}, & m = 2, 4 \bmod 6 \\ 1^{18m-36} \left(\frac{m}{3}\right)^2 m^{16} (3m)^2 0^{z(m)}, & m = 3 \bmod 6 \end{cases}$$

where

$$z(m) := 6m^2 - (18m - 36) - 20 = 6m^2 - 18m + 16.$$

Proposition 3.9. *If Theorem 3.8 is true, then we would get the following table for the rank of $A_{G(m,m,3)}$ over a field of characteristic p :*

$m \bmod 6 \backslash p$	0	2	3	> 3
0	$18m - 16$	$18m - 36 + 8\delta_{4 \nmid m}$	$18m - 36 + 2\delta_{9 \nmid m}$	$18m - 36 + 20\delta_{p \nmid m}$
1, 5	$18m - 16$	$18m - 16$	$18m - 16$	$18m - 36 + 20\delta_{p \nmid m}$
2, 4	$18m - 16$	$18m - 36 + 8\delta_{4 \nmid m}$	$18m - 16$	$18m - 36 + 20\delta_{p \nmid m}$
3	$18m - 16$	$18m - 16$	$18m - 36 + 2\delta_{9 \nmid m}$	$18m - 36 + 20\delta_{p \nmid m}$

Going beyond $G(m, m, 3)$, for $G(m, m, n)$ with $n = 4, 5, \dots$, we have very little data on the Smith Normal Form, shown in the table below[8]. It is insufficient data to guess how the above patterns might extend.

$G(m, m, n)$	$SNF(A_{G(m,m,n)})$
$G(1, 1, 4)$	$1^8 2^{12} 0^4$
$G(2, 2, 4)$	$1^{28} 2^{40} 4^{10} 8^{20} 16^4 0^{90}$
$G(3, 3, 4)$	$1^{120} 3^{12} 6^{42} 18^{24} 0^{450}$
$G(4, 4, 4)$	$1^{156} 2^{36} 4^{68} 8^{10} 16^{20} 32^4 0^{1242}$
$G(5, 5, 4)$	$1^{260} 2^{28} 10^{102} 0^{2610}$
$G(6, 6, 4)$	$1^{284} 2^{124} 6^{44} 12^{10} 72^{20} 144^4 0^{4698}$
$G(1, 1, 5)$	$1^{16} 2^{52} 8^{20} 0^{50}$
$G(2, 2, 5)$	$1^{172} 2^{112} 4^{232} 8^{18} 24^{16} 72^{20} 0^{1368}$

4. PROOF OF THEOREM 3.3

This section is devoted to proving Theorem 3.3, the statement of which was as follows.

Theorem 3.3. *The number of 1s in $SNF(A_{S_n})$ is at least 2^{n-1} .*

It will be helpful to recall a characterization of the diagonal entries $\alpha_1, \alpha_2, \dots$ in the SNF of an integer matrix A .

Proposition 4.1. *If $d_i(A)$ denotes the greatest common divisor of the determinants of all $i \times i$ minors of A , with convention $d_0(A) = 1$, then $\alpha_i = \frac{d_i(A)}{d_{i-1}(A)} \cdot [9]$*

Thus to show the number of 1s in $SNF(A_{G(1,1,n)})$ is greater than or equal to 2^{n-1} , all we need to do is prove that there exists a $2^{n-1} \times 2^{n-1}$ minor of A_{S_n} with determinant 1 or -1 . Then $d_i(A_{G(1,1,n)}) = 1$ for $i \leq 2^{n-1}$, and so we have at least 2^{n-1} 1s in the Smith Normal Form.

Definition 4.2. For a subset $I = \{i_1 < i_2 < \dots < i_k\}$ of $\{1, 2, \dots, n-1\}$, with $0 \leq k \leq n-1$, define a permutation $\sigma_I := s_{i_1} s_{i_2} \dots s_{i_k}$ in S_n , with s_i the adjacent transposition $(i, i+1)$.

Then define the 2^{n-1} -element collection $B_n = \{\sigma_I\}_{I \subseteq \{1, 2, \dots, n-1\}}$ inside S_n .

Example 4.3. One has $\sigma_I = (12)(23)(56) = (123)(56)$ in B_6 (and in B_n for $n \geq 6$, as well).

Lemma 4.4. *Subsets $I, J \subseteq \{1, 2, \dots, n-1\}$, have product $\sigma_I^{-1} \sigma_J$ forming an n -cycle if and only if $J = \{1, 2, \dots, n-1\} \setminus I$.*

Corollary 4.5. *The submatrix of A_{S_n} indexed by $B_n \times B_n$ has determinant ± 1 .*

Proof of Corollary 4.5 assuming Lemma 4.4. In the row indexed by I , there exists only one column indexed by J such that the entry of the submatrix at IJ has value 1, and the same applies for each column J . Thus, the submatrix is a permutation matrix and has determinant ± 1 . $\square$

Before we prove Lemma 4.4, we illustrate the claim with an example.

Example 4.6. Inside $B_9 \subset S_9$, for $I = \{3, 4, 6\}$ and $J = \{1, 2, 5, 7, 8\}$, one has

$$\begin{aligned}\sigma_I &= (34)(45)(67) = (345)(67), \\ \sigma_J &= (12)(23)(56)(78)(89) = (123)(56)(789),\end{aligned}$$

and indeed

$$\sigma_I^{-1}\sigma_J = (76)(543) \cdot (123)(56)(789) = (125789643)$$

which is an n -cycle, as desired.

Proof of Lemma 4.4. For the backward implication, note that if

$$\begin{aligned}I &= \{i_1 < i_2 < \dots < i_k\}, \text{ and} \\ J &= \{1, 2, \dots, n-1\} \setminus I = \{j_1 < j_2 < \dots < j_{n-1-k}\},\end{aligned}$$

then $\sigma_i^{-1}\sigma_J = s_{i_k}s_{i_{k-1}} \dots s_{i_2}s_{i_1} \cdot s_{j_1}s_{j_2} \dots s_{j_{n-1-k}}$ is the product of *all* adjacent transpositions in S_n in some order, and thus a Coxeter element in S_n and an n -cycle.

For the forward implication, we claim $\sigma_I^{-1}\sigma_J$ being an n -cycle implies both

- (i) $|I \cup J| \geq n-1$, and
- (ii) $I \cap J = \emptyset$,

which would then imply $J = \{1, 2, \dots, n-1\} \setminus I$.

Claim (i) follows as n -cycles need at least $n-1$ transpositions in any factorization.

Claim (ii) follows from the observation that if $i \in I$, then $\sigma_I(i) = i+1$. Thus if $i \in I \cap J$, then $\sigma_I(i) = i+1 = \sigma_J(i)$, and hence $\sigma_I^{-1}\sigma_J(i) = \sigma_I^{-1}(i+1) = i$. But this means that $\sigma_I^{-1}\sigma_J$ has i as a fixed point, so it cannot be an n -cycle. $\square$

5. PROVING THEOREM 3.5

This section is devoted to proving Theorem 3.5, the statement of which was as follows.

Theorem 3.5. *An integer $d \geq 1$ occurs in the Smith Normal Form of A_{S_n} for $S_n = G(1, 1, n)$ with multiplicity i if and only if it occurs with multiplicity mi in $\text{SNF}(A_{G(m, 1, n)})$.*

5.1. Background. Before we can prove Theorem 3.5, we need to understand the group $G(m, 1, n)$ and the conjugacy class of its Coxeter elements.

Recall the group $G(m, 1, n)$ consists of the $n \times n$ monomial matrices g whose nonzero entries are m th roots of unity, so of the form ζ^k , where $\zeta = e^{\frac{2\pi i}{m}}$. In other words, if $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{C}^n$, then

$$g(e_j) = \zeta^{z_j} e_{\sigma(j)}$$

for a unique pair $(\sigma, \mathbf{z})$ of

- a permutation $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n)) \in S_n$, and
- a vector $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{Z}_m^n$, where $\mathbb{Z}_m$ here denotes $\mathbb{Z}/m\mathbb{Z}$.

We introduce a well-known alternate notation for such matrices here[3, §2.2].

Definition 5.1. Denote the element g in $G(m, 1, n)$ described above by $\sigma \times \mathbf{z}$. Then the product in $G(m, 1, n)$ corresponds to

$$(\sigma \times \mathbf{z})(\tau \times \mathbf{w}) = \sigma\tau \times (\mathbf{w} + \tau(\mathbf{z}))$$

where $\tau(\mathbf{z})$ is defined as $\tau(\mathbf{z}) = (z_{\tau(1)}, z_{\tau(2)}, \dots, z_{\tau(n)})$.

Corollary 5.2. In $G(m, 1, n)$, the inverse of an element $\sigma \times \mathbf{z}$ can be written as $\sigma^{-1} \times \mathbf{z}'$, where $z'_i = m - z_{\sigma^{-1}(i)}$, noting that since $\mathbf{z}' \in \mathbb{Z}_m^n$, the case where $z_{\sigma^{-1}(i)} = 0$ can be written equivalently as $z'_i = 0$.

Example 5.3. In $G(3, 1, 3)$, the element normally written in the matrix notation defined in Section 2.2 as

$$\begin{pmatrix} 0 & 0 & 1 \\ \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \end{pmatrix}$$

would instead be written as $\sigma_{231} \times (1, 2, 0)$, or, using two-line notation for the permutation, $\begin{pmatrix} 123 \\ 231 \end{pmatrix} \times (1, 2, 0)$. The element normally written as

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta & 0 \end{pmatrix}$$

would instead be written as $\sigma_{132} \times (1, 1, 0)$ or $\begin{pmatrix} 123 \\ 132 \end{pmatrix} \times (1, 1, 0)$. Their product, normally written

$$\begin{pmatrix} 0 & 0 & 1 \\ \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \end{pmatrix} \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \zeta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

would be instead written as $\sigma_{213} \times (2, 1, 2)$, precisely what the formula for multiplication in the $\sigma \times \mathbf{z}$ format gives.

Definition 5.4. For $\sigma \times \mathbf{z}$ in $G(m, 1, n)$, define $|\sigma \times \mathbf{z}| = \sum_{i=1}^n z_i \bmod m$, which we will call the *magnitude* of $\sigma \times \mathbf{z}$.

Corollary 5.5. A few results quickly follow:

1. The magnitude of the product of two elements is the sum in $\mathbb{Z}_m$ of the magnitudes of each element:

$$|(\sigma \times \mathbf{z}) \cdot (\tau \times \mathbf{w})| = |\sigma \times \mathbf{z}| + |\tau \times \mathbf{w}|$$

2. The magnitudes of an element and its inverse sum to zero:

$$|\pi^{-1} \times \mathbf{w}'| + |\pi \times \mathbf{w}| = 0$$

3. Conjugation of $\sigma \times \mathbf{z}$ by any element $\pi \times \mathbf{w}$ in $G(m, 1, n)$ keeps the magnitude fixed:

$$|(\pi^{-1} \times \mathbf{w}')(\sigma \times \mathbf{z})(\pi \times \mathbf{w})| = |\sigma \times \mathbf{z}|$$

With these tools, we can then prove our theorem, which begins with identifying the Coxeter elements and their conjugacy class.

Proposition 5.6. The element $c = \sigma_{23\dots n1} \times (1, 0, \dots, 0)$ is a Coxeter element of $G(m, 1, n)$, with eigenvector

$$\mathbf{v} = (\omega^{n-1}, \dots, \omega, \omega^0)^T,$$

where $\omega = e^{\frac{2\pi i}{mn}}$.

Proof. To start, we note that the Coxeter number for $G(m, 1, n)$ is mn , so to prove c is a Coxeter element with corresponding eigenvector $\mathbf{v}$, we need to prove both that $c(\mathbf{v}) = e^{\frac{2\pi i}{mn}} \mathbf{v}$ (which can be rewritten as $c(\mathbf{v}) = \omega \mathbf{v}$) and that $\mathbf{v}$ has a free W -orbit. To check the first, we simply plug in the values for c and $\mathbf{v}$, getting

$$c(\mathbf{v}) = (\omega^n, \omega^{n-1}, \dots, \omega) = \omega \mathbf{v},$$

which proves the first part of our definition of a Coxeter element.

Second, we need to show that $\mathbf{v}$ has a free W -orbit. To do this, we will construct an inverse function which retrieves the value of $w \in G(m, 1, n)$ from $w(\mathbf{v})$, that is, an inverse to $\cdot \mathbf{v} : w \mapsto w(\mathbf{v})$. By constructing such an inverse, we will show $\cdot \mathbf{v}$ is isomorphic, and thus that $\mathbf{v}$ has a free W -orbit.

First, we write the value of $w(\mathbf{v})$, where $w = \sigma \times \mathbf{z}$ and σ^{-1} is simply the inverse permutation of σ , by using the fact that $\zeta = e^{\frac{2\pi i}{m}} = \omega^n$:

$$w(\mathbf{v}) = (\omega^{n-\sigma^{-1}(1)+nz_{\sigma^{-1}(1)}}, \dots, \omega^{n-\sigma^{-1}(n)+nz_{\sigma^{-1}(n)}})$$

Now, defining $\log_\omega : \omega^{\mathbb{Z}/m\mathbb{Z}} \rightarrow \mathbb{Z}/m\mathbb{Z}$ so that $\log_\omega(\omega^k) = k$, we write:

$$n - \log_\omega w(\mathbf{v})_k \equiv \sigma^{-1}(k) \pmod{n},$$

which means we can retrieve σ . After that, we also see that

$$\lfloor \frac{\log_\omega w(\mathbf{v})_k}{n} \rfloor = \lfloor z_{\sigma^{-1}(k)} + 1 - \frac{\sigma^{-1}(k)}{n} \rfloor = z_{\sigma^{-1}(k)},$$

and since we know the value of σ^{-1} , we can find $\mathbf{z}$ in full. So, the function $\cdot \mathbf{v} : w \mapsto w(\mathbf{v})$ is invertible, so $\mathbf{v}$ has a free W -orbit, and thus c is a Coxeter element. $\square$

Next, we need the conjugacy class of the Coxeter element. To do this, we use the following lemmas.

Lemma 5.7. *Given some $\sigma \times \mathbf{z} \in G(m, 1, n)$, there exists some $\mathbf{w}$ such that $\tau \times \mathbf{w}$ is in the conjugacy class of $\sigma \times \mathbf{z}$ if and only if τ is conjugate to σ in S_n .*

Proof. Mapping $\sigma \times \mathbf{z} \mapsto \sigma$ is a surjective group morphism $G(m, 1, n) \rightarrow S_n$. $\square$

Lemma 5.8. *Given some $\sigma \times \mathbf{z} \in G(m, 1, n)$ with σ an n -cycle, any element $\sigma \times \mathbf{x}$ with $|\sigma \times \mathbf{x}| = |\sigma \times \mathbf{z}|$ is in the conjugacy class of $\sigma \times \mathbf{z}$.*

Proof. To show this, we start by conjugating $\sigma \times \mathbf{z}$ by $e \times \mathbf{w}$, noting that by the definition of the inverse, $(e \times \mathbf{w})^{-1} = e \times \mathbf{w}^{-1}$, where $\mathbf{w}^{-1}$ is the additive inverse of $\mathbf{w}$ in $\mathbb{Z}_m^n$. Then we get that

$$(e \times \mathbf{w}^{-1})(\sigma \times \mathbf{z})(e \times \mathbf{w}) = \sigma \times (\mathbf{w} + \mathbf{z} + \sigma(\mathbf{w}^{-1})).$$

Now, we need to show that for any $\mathbf{x}$ such that $|\sigma \times \mathbf{x}| = |\sigma \times \mathbf{z}|$, we can choose $\mathbf{w}$ such that $\mathbf{x} = \mathbf{w} + \mathbf{z} + \sigma(\mathbf{w}^{-1})$. We do this constructively, for an arbitrary $\mathbf{x}$.

First, we note that since $\mathbf{x}$ and $\mathbf{z}$ are fixed, we can write $\mathbf{y} = \mathbf{x} - \mathbf{z}$, and try to construct said $\mathbf{y}$, noting that since $|\sigma \times \mathbf{x}| = |\sigma \times \mathbf{z}|$, we have $\sum_{i=1}^n y_i = 0$. Second, we note that, working in $\mathbb{Z}_m$, we have $(\mathbf{w}^{-1})_i = -w_i$, so $\sigma(\mathbf{w}^{-1})_i = (\mathbf{w}^{-1})_{\sigma(i)} = -w_{\sigma(i)}$.

Then for any $1 \leq i \leq n$, we want to find w_i such that $y_i = w_i - w_{\sigma(i)}$, working within $\mathbb{Z}_m$. To do this, we start at y_1 , and choose $w_1 = y_1$, meaning $w_{\sigma(1)} = 0$. Since we want to also fulfill

$$y_{\sigma(1)} = w_{\sigma(1)} - w_{\sigma(\sigma(1))} = 0 - w_{\sigma^2(1)},$$

so we have $w_{\sigma^2(1)} = -y_{\sigma(1)}$. But then that means that

$$y_{\sigma^2(1)} = w_{\sigma^2(1)} - w_{\sigma(\sigma^2(1))} = -y_{\sigma(1)} - w_{\sigma^3(1)},$$

so $w_{\sigma^3(1)} = -y_{\sigma(1)} - y_{\sigma^2(1)}$.

Repeating this pattern, we get in general that $w_{\sigma^k(1)} = -\sum_{i=1}^{k-1} y_{\sigma^i(1)}$. Taking this all the way up to $k = n$, we get

$$w_{\sigma^n(1)} = -\sum_{i=1}^{n-1} y_{\sigma^i(1)}$$

$$w_1 = -\sum_{i=2}^n y_i$$

But since $\sum_{i=1}^n y_i = 0$, that gives $w_1 = y_1$, making the whole system self-consistent. Thus, for any $\mathbf{x}$ such that $|\sigma \times \mathbf{x}| = |\sigma \times \mathbf{z}|$, we can find a $\mathbf{w}$ such that conjugating $\sigma \times \mathbf{z}$ by $e \times \mathbf{w}$ results in $\sigma \times \mathbf{x}$, so we are done. $\square$

Proposition 5.9. *The conjugacy class C of the Coxeter elements of $G(m, 1, n)$ is*

$$C = \{\sigma \times \mathbf{z} : \sigma \text{ an } n\text{-cycle, and } |\sigma \times \mathbf{z}| = 1\}$$

Proof. Using Lemmas 5.7 and 5.8, any element $\sigma \times \mathbf{z}$ with σ an n -cycle and $|\sigma \times \mathbf{z}| = 1$ is conjugate to the element $c = \sigma_{23\dots n1} \times (1, 0, \dots, 0)$, and we know nothing else is in the conjugacy class thanks to Lemma 5.7 and Corollary 5.5.3. $\square$

We can additionally note that the size of C is $(n-1)!m^{n-1}$, as there are $(n-1)!$ n -cycles, and for each one, there are m^{n-1} ways to get the magnitude of the element to be 1.

5.2. Proof of Theorem 3.5. With the conjugacy class of the Coxeter elements established, we can then work on proving the actual statement of Theorem 3.5, whose statement is reproduced again below.

Theorem 3.5. *An integer $d \geq 1$ occurs in the Smith Normal Form of A_{S_n} for $S_n = G(1, 1, n)$ with multiplicity i if and only if it occurs with multiplicity mi in $\text{SNF}(A_{G(m, 1, n)})$.*

The theorem will result from Propositions 5.10 and 5.12, below, and their corollaries.

Proposition 5.10. *The group $W = G(m, 1, n)$ can be partitioned into sets which give a direct sum decomposition $\mathbb{Z}W = V_0 \oplus V_1 \oplus \dots \oplus V_{m-1}$, and said decomposition allows for the following block decomposition of $A_{G(m, 1, n)}$ for some matrix M :*

$$A_W = \begin{matrix} & \begin{matrix} V_0 & V_1 & V_2 & \dots & V_{m-2} & V_{m-1} \end{matrix} \\ \begin{matrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ \dots \\ V_{m-1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & M \\ M & 0 & 0 & \dots & 0 & 0 \\ 0 & M & 0 & \dots & 0 & 0 \\ 0 & 0 & M & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & M & 0 \end{pmatrix} \end{matrix}$$

Corollary 5.11. *With the matrix M defined as above, an integer $d \geq 0$ occurs in the Smith Normal Form of M with multiplicity i if and only if it occurs with multiplicity mi in the Smith Normal Form of $A_{G(m, 1, n)}$.*

Proof of Corollary 5.11 assuming Proposition 5.10. Permuting rows and columns in $A_{G(m,1,n)}$ gives the block-diagonal matrix

$$\begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M \end{pmatrix},$$

without changing the SNF. $\square$

Proposition 5.12. *The matrix M above has its own block decomposition, as follows:*

$$M = \begin{pmatrix} A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \end{pmatrix}$$

Corollary 5.13. *The SNF of M is the same as that of*

$$\begin{pmatrix} A_{S_n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

which is the same as the SNF of S_n with $n!(m^{n-1} - 1)$ additional zeroes.

Proof of Corollary 5.13, assuming Proposition 5.12. Use (block) row and column operations on M that are invertible over $\mathbb{Z}$: first use block row operations with the upper left A_{S_n} to zero out all of the rows below it, and then use block column operations with the same A_{S_n} in the upper left to zero out the other columns:

$$M = \begin{pmatrix} A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \end{pmatrix} \rightsquigarrow \begin{pmatrix} A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} A_{S_n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \quad \square$$

The number of additional zeroes follows from the relative sizes of A_{S_n} and M .

It therefore only remains to prove the key Propositions 5.10 and 5.12.

Proof of Proposition 5.10. Grouping the elements of $W = G(m, 1, n)$ by their magnitudes gives a direct sum decomposition

$$\mathbb{Z}W = V_0 \oplus \cdots \oplus V_{m-1}$$

where $V_i = \text{span}_{\mathbb{Z}}\{\sigma \times \mathbf{z} : |\mathbf{z}| = i\}$. Furthermore, when we multiply an element t in V_k by any element in C , Corollary 5.5.1 implies that the resulting product will

be in V_{k+1} . Thus, we get that $A_{G(m,1,n)}$ has the form

$$A_W = \begin{matrix} & \begin{matrix} V_0 & V_1 & V_2 & \dots & V_{m-2} & V_{m-1} \end{matrix} \\ \begin{matrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ \dots \\ V_{m-1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & M_m \\ M_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & M_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & M_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & M_{m-1} & 0 \end{pmatrix} \end{matrix},$$

where each M_i is the matrix representing the transformation

$$\begin{aligned} V_{i-1} &\longrightarrow V_i \\ x &\longmapsto x \cdot \sum_{w \in C} w. \end{aligned}$$

We now show that one can re-order the basis elements of each V_i so that the M_i are all the same matrix M . To do so, consider some element $\tau \times \mathbf{w}$, which is the t th basis element in our ordering of the basis of V_0 . Additionally, we let $\mathbf{e}_n$ be equal to $(0, \dots, 0, 1) \in \mathbb{Z}_m^n$, while e remains the identity permutation.

Then we choose the ordering of the basis elements of each V_k such that the t th basis element of V_k is $(e \times k\mathbf{e}_n)(\tau \times \mathbf{w}) = \tau \times (\mathbf{w} + \tau(k\mathbf{e}_n))$.

Now, consider some $c = \sigma \times \mathbf{z}$ such that $c \in C$. We see that multiplying $\tau \times \mathbf{w}$ by c gives

$$(\tau \times \mathbf{w})(\sigma \times \mathbf{z}) = \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w})),$$

and we claim that is the s th element of V_1 .

On the one hand, multiplying $\tau \times (\mathbf{w} + \tau(k\mathbf{e}_n))$, the t th element of V_k , by $c = \sigma \times \mathbf{z}$ results in

$$\begin{aligned} (\tau \times (\mathbf{w} + \tau(k\mathbf{e}_n)))(\sigma \times \mathbf{z}) &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w} + \tau(k\mathbf{e}_n))) \\ &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w} + k\mathbf{e}_{\tau^{-1}(n)})) \\ &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}) + \sigma(k\mathbf{e}_{\tau^{-1}(n)})) \\ &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}) + k\mathbf{e}_{\sigma^{-1}\tau^{-1}(n)}) \\ &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}) + k\mathbf{e}_{(\tau\sigma)^{-1}(n)}) \\ &= \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}) + \tau\sigma(k\mathbf{e}_n)). \end{aligned}$$

On the other hand, we know the s th element of V_{k+1} to be

$$(e \times k\mathbf{e}_n)(\tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}))) = \tau\sigma \times (\mathbf{z} + \sigma(\mathbf{w}) + \tau\sigma(k\mathbf{e}_n))$$

Generalizing this to every pair s, t of basis elements for each M_i , we see that the location of the non-zero entries in each matrix M_i is identical, and hence

$$M_0 = M_1 = \dots = M_{m-1} =: M. \quad \square$$

Proof of Proposition 5.12. Now, we claim that M can be decomposed into copies of the adjacency matrix of S_n .

To do this, we reorder⁴ the basis elements within V_0 . In particular, we choose to order them such that the first $n!$ are the basis elements $\sigma \times \mathbf{z}$ where the permutations vary but $\mathbf{z}$ stays fixed, and the same is true for each grouping of $n!$ basis elements, going through every value of $\mathbf{z}$ that keeps $|\sigma \times \mathbf{z}| = 1$ for V_0 .

⁴This is different than the order on the rows and columns of M used in the proof of Proposition 5.10.

Now, with this ordering, we can consider a basis element of V_0 , $\tau \times \mathbf{w}$. Then we claim that for all $\pi \times \mathbf{x}$ where $\pi = \tau\mu$ for some n -cycle μ , there exists some $\sigma \times \mathbf{z} \in C$ such that $(\tau \times \mathbf{w})(\sigma \times \mathbf{z}) = \pi \times \mathbf{x}$. To show this, we note that

$$(\tau \times \mathbf{w})(\sigma \times \mathbf{z}) = \tau\sigma \times (\sigma(\mathbf{w}) + \mathbf{z}).$$

Then to get any particular π , we simply choose $\sigma = \tau^{-1}\pi$, which is an n -cycle. We know that $|\tau \times \mathbf{w}| = 0$, so $|\tau\sigma \times (\sigma(\mathbf{w}) + \mathbf{z})|$, meaning that we can freely choose our $\mathbf{z}$ in order to get the desired $\mathbf{x}$.

On the other hand, there are $(n-1)!m^{n-1}$ possible values of $\pi \times \mathbf{x}$, and $(n-1)!m^{n-1}$ elements of C , so that means no other entries in the column indexed by $\tau \times \mathbf{w}$ will have a non-zero entry.

Now, if we examine each group of $n!$ basis elements which share a $\mathbf{z}$ value, we note that said $\mathbf{z}$ is in fact immaterial to where the 1s are. An entry of M indexed by row $\sigma \times \mathbf{z}$ and by column $\tau \times \mathbf{w}$ is 1 if and only if $\tau^{-1}\sigma$ is an n -cycle. Thus, each $n! \times n!$ submatrix where the columns share a single $\mathbf{z}$ and the rows share a single $\mathbf{w}$ is a copy of A_{S_n} , and we can write

$$M = \begin{pmatrix} A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{S_n} & A_{S_n} & \cdots & A_{S_n} \end{pmatrix},$$

where there are $(m^{n-1})^2$ copies of A_{S_n} , because there are m^{n-1} possible $\mathbf{z}$ vectors which give $\sigma \times \mathbf{z}$ a fixed magnitude. $\square$

Proof of Theorem 3.5. So, we get that if an integer $d \geq 1$ appears in $\text{SNF}(A_{G(1,1,n)})$ with multiplicity i , then it appears in the submatrix M with multiplicity i , and so appears in $\text{SNF}(A_{G(1,1,n)})$ with multiplicity mi . The same applies in reverse, so we are done. $\square$

6. PROVING THEOREM 3.6

This section is devoted to proving Theorem 3.6, the statement of which was as follows.

Theorem 3.6. *For $m > 2$, we have that:*

$$\text{SNF}(A_{G(m,m,2)}) = \begin{cases} 1^{2m-4}0^4, & m = 0 \bmod 4 \\ 1^{2m-2}2^2, & m = 1, 3 \bmod 4 \\ 1^{2m-4}2^4, & m = 2 \bmod 4 \end{cases}$$

6.1. Background. To prove this, we start with the Coxeter presentation for $G(m, m, 2)$:

$$\begin{aligned} G(m, m, 2) = I_2(m) &= \langle s_1, s_2 : s_1^2 = s_2^2 = (s_1 s_2)^m = 1 \rangle \\ &\cong \langle s_1, r : s_1^2 = 1 = r^m, s_1 r s_1 = r^{-1} \rangle \\ &= \{e, r, r^2, \dots, r^{m-1}, s_1, s_1 r, s_1 r^2, \dots, s_1 r^{m-1}\} \end{aligned}$$

where s_1, s_2 are two reflections with dihedral angle $\frac{2\pi}{2m}$ and $r = s_1 s_2$ is a rotation through $\frac{2\pi}{m}$. From the above Coxeter presentation and the fact that this is a real reflection group, the Coxeter elements are simply the products $s_1 s_2 = r$ and $s_2 s_1 = r^{-1}$, and because the conjugacy class of these elements is just those two elements, we have $C = \{r, r^{-1}\}$. That is, the only non-zero entries in a given

column are the two in the row corresponding to the column element multiplied by r and the row corresponding to the column element multiplied by r^{-1} . From this, we get that $A_{G(m,m,2)}$ is the following $2m \times 2m$ matrix:

$$\begin{array}{c} \begin{array}{c} e \\ r \\ \\ r^{m-2} \\ r^{m-1} \end{array} \end{array} \left(\begin{array}{ccc|ccc} e & r & & & r^{m-1} & \\ 0 & 1 & & & 1 & \\ 1 & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 & \end{array} \right) \begin{array}{c} s_1 \quad s_1 r \quad s_1 r^{m-1} \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{array}$$

$$\begin{array}{c} s_1 \\ s_1 r \\ \\ \\ s_1 r^{m-1} \end{array} \left(\begin{array}{ccc|ccc} 0 & 1 & & & 1 & \\ 1 & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 & \end{array} \right)$$

6.2. Proof of Theorem 3.6. Given this background, and the general Cayley matrix above, we now prove the theorem. To start, we need to prove the following lemma:

Lemma 6.1. *For $m \geq 8$, the nonzero entries in the SNF of $A_{I_2(m)}$ can be obtained from those of $A_{I_2(m-4)}$ by appending 8 copies of the entry 1 to the beginning.*

Proof. To prove this lemma, we can simply take the Smith Normal Form of $A_{G(m,m,2)}$ with $m > 8$. We first note that $A_{G(m,m,2)}$ is composed of four submatrices, with two of them being all zero, as shown above. But when we take the Smith Normal Form, we can simply treat each of the non-zero submatrices independently, because they won't affect one another. Since they are identical, we can simply operate on one matrix and then double the count of every element in the Smith Normal Form at the end.

With that in mind, we want to eliminate everything not on the diagonal from the following matrix using row and column operations. We start by moving the first row to be the last:

$$\begin{pmatrix} 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & \ddots & \\ 1 & & & & 1 & 0 \\ 0 & 1 & & & & 1 \end{pmatrix}$$

Then, we attempt to clear the first 4 rows and columns. To do this, we write the matrix out with more rows and columns filled in:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & & \ddots & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & & & & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

First, we subtract the first and second rows from the $(m-1)$ th and m th rows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & \ddots & & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

Then, we subtract the first and second columns from the third and fourth columns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & \ddots & & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & & & & 0 & 1 \end{pmatrix}$$

Then, we add the third and fourth rows to the $m-1$ th and m th rows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & 0 & 1 \end{pmatrix}$$

Finally, we subtract the third and fourth columns from the fifth and sixth columns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & & 0 & 1 \end{pmatrix}$$

But now looking at everything but the first 4 rows and columns, we have the submatrix that we would get for $I_2(m-4)$, since the only elements inside that submatrix that we changed were those in the bottom left corner, so the rest is of matching form to the initial submatrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & & & 0 & 1 \end{pmatrix}$$

Now, since there were two such submatrices we can do the same operations to, there are eight more 1s in $\text{SNF}(A_{I_2(m)})$ than in $\text{SNF}(A_{I_2(m-4)})$ for $m \geq 8$, so we have proven Lemma 6.1. $\square$

Proof of Theorem 3.6. From there, to prove Theorem 3.6, we just need to calculate $\text{SNF}(A_{G(4,4,2)})$, $\text{SNF}(A_{G(5,5,2)})$, $\text{SNF}(A_{G(6,6,2)})$, and $\text{SNF}(A_{G(7,7,2)})$. Additionally, we have to calculate $\text{SNF}(A_{G(3,3,2)})$ separately, since although it exists and fits the formula, it doesn't prove the case of $\text{SNF}(A_{G(7,7,2)})$. All of these indeed work, having been checked by the computer with the results being in the table at the beginning of Section 3.3 and reproduced here:

$G(m, m, 2)$	$\text{SNF}(A_{G(m,m,2)})$
$G(2, 2, 2)$	1^4
$G(3, 3, 2)$	$1^4 2^2$
$G(4, 4, 2)$	$1^4 0^4$
$G(5, 5, 2)$	$1^8 2^2$
$G(6, 6, 2)$	$1^8 2^4$
$G(7, 7, 2)$	$1^{12} 2^2$

From these, we can derive all other Smith Normal Forms of $A_{G(m,m,2)}$ by induction, so we are done. $\square$

7. EXCEPTIONAL GROUPS

For the exceptional irreducible complex reflection groups $G_4, G_5, \dots, G_{37}$, we have only limited data on the SNF for A_W , shown in the table below[8]. Note that only 26 of these 34 exceptional groups are well-generated. However, even for many of those, the groups were too large and the computation failed due to lack of memory. Nevertheless, one again sees at least one small case, G_5 , where the non-zero SNF entries of A_{G_5} are not just powers of 2. Beyond that, the pattern is not clear.

	SNF Diagonal
G_4	$1^{12}2^30^9$
G_5	$1^{27}3^90^{36}$
G_6	$1^{24}2^60^{18}$
G_7	Not Well Generated
G_8	$1^{40}2^84^{12}0^{36}$
G_9	$1^{80}2^{16}4^{24}0^{72}$
G_{10}	$1^{60}2^{48}4^{12}0^{168}$
G_{11}	Not Well Generated
G_{12}	Not Well Generated
G_{13}	Not Well Generated
G_{14}	$1^{84}0^{60}$
G_{15}	Not Well Generated
G_{16}	Failed to Compute: Lack of Memory
G_{17}	Failed to Compute: Lack of Memory
G_{18}	Failed to Compute: Lack of Memory
G_{19}	Not Well Generated
G_{20}	Failed to Compute: Lack of Memory
G_{21}	Failed to Compute: Lack of Memory
G_{22}	Not Well Generated
G_{23}	$1^{48}2^{20}4^20^{50}$
G_{24}	Failed to Compute: Lack of Memory
G_{25}	Failed to Compute: Lack of Memory
G_{26}	Failed to Compute: Lack of Memory
G_{27}	Failed to Compute: Lack of Memory
G_{28}	Failed to Compute: Lack of Memory
G_{29}	Failed to Compute: Lack of Memory
G_{30}	Failed to Compute: Lack of Memory
G_{31}	Not Well Generated
G_{32}	Failed to Compute: Lack of Memory
G_{33}	Failed to Compute: Lack of Memory
G_{34}	Failed to Compute: Lack of Memory
G_{35}	Failed to Compute: Lack of Memory
G_{36}	Failed to Compute: Lack of Memory
G_{37}	Failed to Compute: Lack of Memory

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