FACE NUMBERS OF POSET ASSOCIAHEDRA

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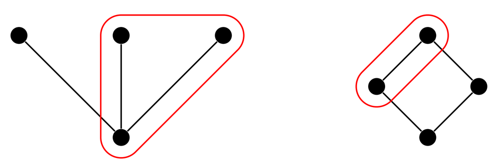
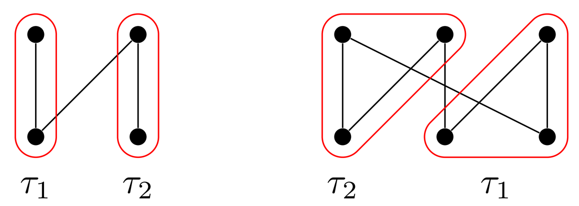
POSET ASSOCIAHEDRA

Given a poset P, a *tube* τ is a connected convex subposet of P such that $1 < |\tau| < |P|$.

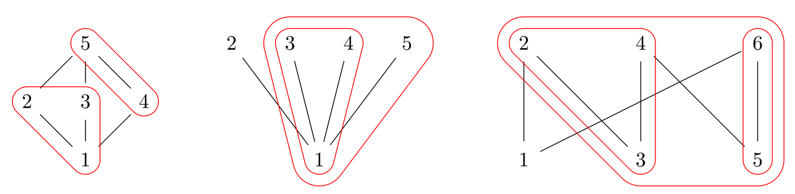


Given two tubes τ_1, τ_2, we say $\tau_1 \prec \tau_2$ if $\tau_1 \cap \tau_2 = \emptyset$, and there exists $v_1 \in \tau_1$ and $v_2 \in \tau_2$ such that $v_1 <_P v_2$.

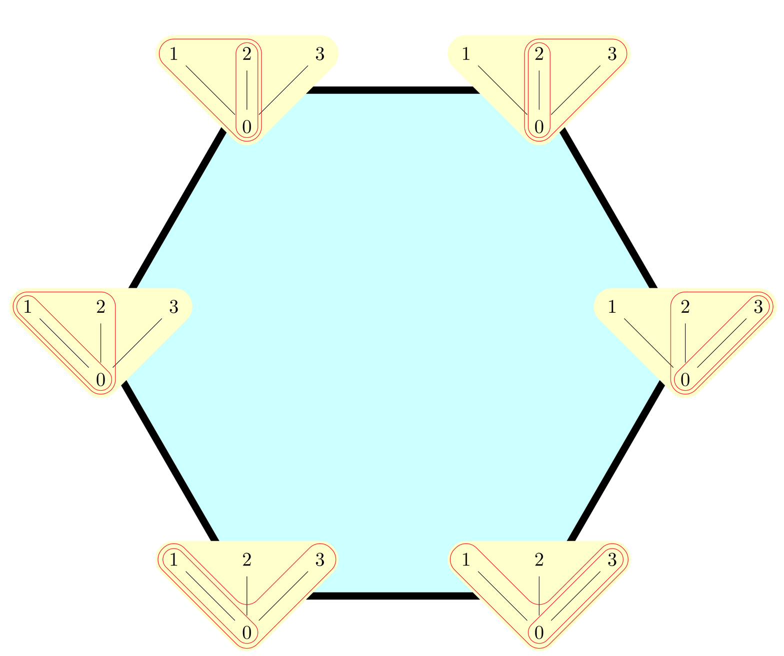


A *tubing* T of P is a set of tubes such that

- any pair of tubes in T is either nested or disjoint, and
- there is no $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.



For a finite poset P, there exists a simple, convex polytope (P) whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. This polytope is called the *poset associahedron* of P.



Example: Permutohedron

FACE NUMBERS

- f-polynomial: $\sum f_i t^i$ where $f_i = \#$ i-dimensional faces.
- h-polynomial: $h(t+1) = f(t)$.
- γ-polynomial: $(1+t)^d \gamma\left(\frac{t}{(1+t)^2}\right) = h(t)$.

Example: For the permutohedron above

- $f(t) = 6 + 6t + 6t^2$
- $h(t) = 1 + 4t + t^2$
- $\gamma(t) = 1 + 2t$

COMPARABILITY INVARIANT

The *comparability graph* of a poset P is a graph $C(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable.

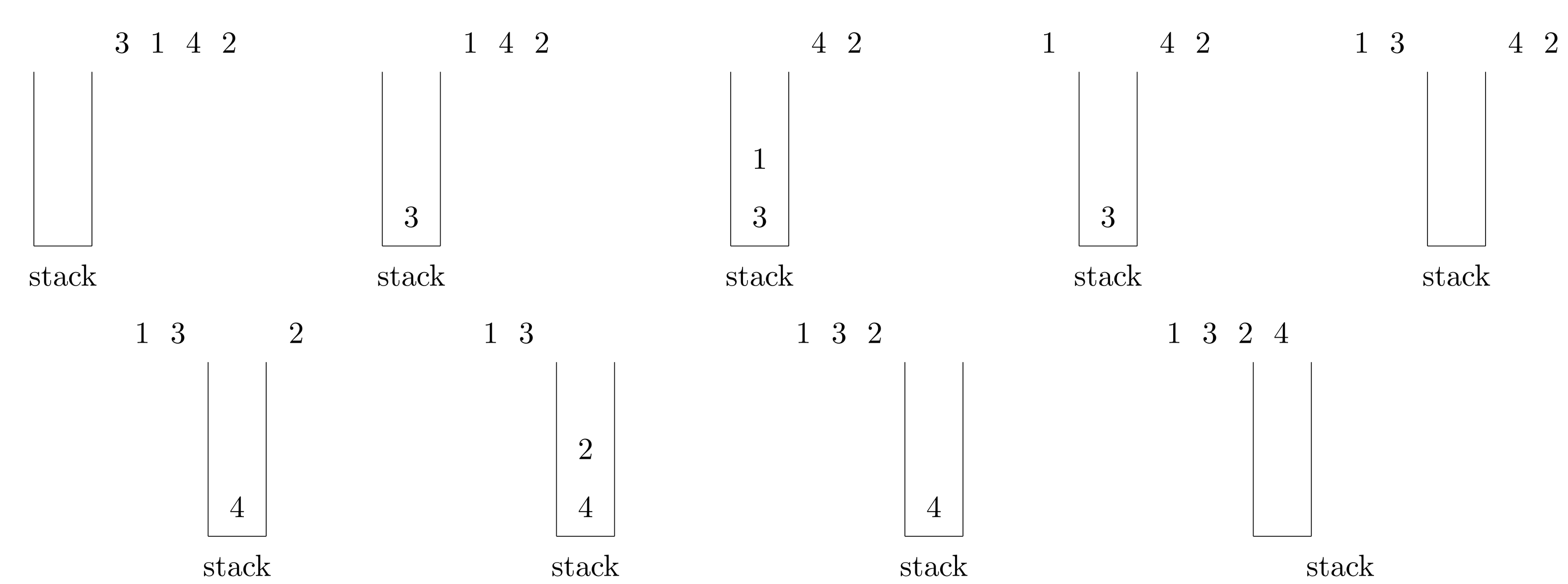
Theorem. *If P and P' have the same comparability graph, then $\mathscr{A}(P)$ and $\mathscr{A}(P')$ have the same face numbers.*

STACK-SORTING AND BROOM POSETS

Stack-sorting

Given a permutation $\pi \in \mathfrak{S}_n$, the *stack-sorting algorithm* maps π to $s(\pi)$ obtained through the following procedure. Iterate through the entries of π. In each iteration,

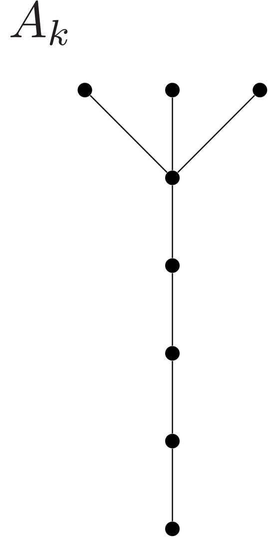
- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- else, pop the entry at the top of the stack to the end of the output permutation.



Example: $s(3142) = 1324$

Broom Posets

Broom posets $A_{n,k} = C_{n+1} \oplus A_k$



Example: $A_{4,3}$

Theorem. *Let $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i$ for all $i > k\}$ and $h(t) = \sum h_i t^i$ be the h-polynomial of $\mathscr{A}(A_{n,k})$, then*

$$h_i = |\{w \in s^{-1}(\mathfrak{S}_{n,k}) \mid \operatorname{des}(w) = i\}|.$$

Corollary. *$\mathscr{A}(A_{n,k})$ is γ-nonnegative.*

Theorem. *The h-polynomial of $\mathscr{A}(A_{n,2})$ is real-rooted.*

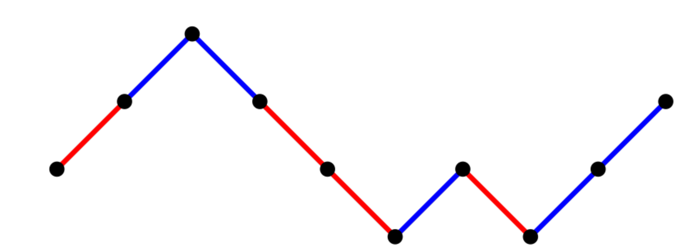
CYCLIC FENCE POSETS

The *(even) cyclic fence poset* $CF_{2(n+1)}$ is the poset on the elements $\{1, 2, \ldots, 2n+2\}$ with the covering relations $2k-1, 2k+1 \lessdot 2k$ for $1 \le k \le n$, and $1, 2n+1 \lessdot 2n+2$.

The *(odd) cyclic fence poset* CF_{2n+1} is the poset on the elements $\{1, 2, \ldots, 2n+1\}$ with the covering relations $2k-1, 2k+1 \lessdot 2k$ for $1 \le k \le n$, and $1 \lessdot 2n+1$.

Example: CF_6 and CF_7

A *colored* (m,n) *path* is a sequence of m upsteps and n downsteps where each step is colored red or blue. Let $CP_{m,n}$ denote the set of colored (m,n) path. A *peak* is an upstep followed by a downstep. A peak step is one of the two steps at some peak. The remaining steps are called *side steps*.



Example: A path in $CP_{5,4}$ with 4 peak steps and 5 side steps

Theorem. *For $CF_{2(n+1)}$, the h-vector is given by*

$$h_i = |\{w \in CP_{n,n} \mid \#\text{red peak steps} - \#\text{blue peak steps} = 2(i-n)\}|.$$

For CF_{2n+1}, the h-vector is given by

$$h_i = |\{w \in CP_{n-1,n} \mid \#\text{red side steps} - \#\text{blue side steps} = 2(i-n)+1\}|.$$

CONJECTURES AND QUESTIONS

Conjecture. *All poset associahedra are real-rooted and γ-nonnegative.*

Question. *Find a combinatorial interpretation for the face numbers of poset associahedra.*

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