Face Numbers of Poset Associahedra
Son Nguyen
Advisor: Vic Reiner
University of Minnesota Twin Cities

Poset Associahedra
Given a poset $P$, a tube $\tau$ is a connected convex subposet of $P$ such that $1 < |\tau| < |P|$.

Given two tubes $\tau_1, \tau_2$, we say $\tau_1 < \tau_2$ if $\tau_1 \cap \tau_2 = \emptyset$, and there exists $v_1 \in \tau_1$ and $v_2 \in \tau_2$ such that $v_1 <_P v_2$.

A tubing $T$ of $P$ is a set of tubes such that
- any pair of tubes in $T$ is either nested or disjoint, and
- there is no $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 < \tau_2 < \ldots < \tau_k < \tau_1$.

For a finite poset $P$, there exists a simple, convex polytope $(P)$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. This polytope is called the poset associahedron of $P$.

Example: Permutohedron

Comparability Invariant
The comparability graph of a poset $P$ is a graph $C(P)$ whose vertices are the elements of $P$ and where $i$ and $j$ are connected by an edge if $i$ and $j$ are comparable.

Theorem. If $P$ and $P'$ have the same comparability graph, then $\mathcal{A}(P)$ and $\mathcal{A}(P')$ have the same face numbers.

Stack-sorting and Broom Posets
Stack-sorting
Given a permutation $\pi \in S_m$, the stack-sorting algorithm maps $\pi$ to $s(\pi)$ obtained through the following procedure. Iterate through the entries of $\pi$. In each iteration,
- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- else, pop the entry at the top of the stack to the end of the output permutation.

Broom Posets
Broom posets $A_{n,k} = C_{n+1} \oplus A_k$

Example: $A_{4,3}$

Cyclic Fence Posets
The (even) cyclic fence poset $CF_{2(n+1)}$ is the poset on the elements $\{1, 2, \ldots, 2n + 2\}$ with the covering relations $2k - 1, 2k + 1 \leq 2k$ for $1 \leq k \leq n$, and $1, 2n + 1 \leq 2n + 2$.

The (odd) cyclic fence poset $CF_{2n+1}$ is the poset on the elements $\{1, 2, \ldots, 2n + 1\}$ with the covering relations $2k - 1, 2k + 1 \leq 2k$ for $1 \leq k \leq n$, and $1 \leq 2n + 1$.

Example: $CF_6$ and $CF_7$

Conjectures and Questions
Conjecture. All poset associahedra are real-rooted and $\gamma$-nonnegative.

Question. Find a combinatorial interpretation for the face numbers of poset associahedra.

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